MA121, Homework #5
Solutions

1. *Compute each of the following integrals:*

$$\int \frac{8x-5}{2x^2-3x+1}\,dx, \qquad \int \log(x^2-1)\,dx.$$

- To compute the first integral, we factor the denominator and we write

$$\frac{8x-5}{2x^2-3x+1} = \frac{8x-5}{(2x-1)(x-1)} = \frac{A}{2x-1} + \frac{B}{x-1}$$

for some constants A, B that need to be determined. Clearing denominators gives

$$8x-5 = A(x-1) + B(2x-1)$$

and we can now look at some suitable choices of x to find

$$\begin{aligned} x = 1/2 \quad &\Longrightarrow \quad -1 = -A/2 \quad \Longrightarrow \quad A = 2, \\ x = 1 \quad &\Longrightarrow \quad 3 = B. \end{aligned}$$

In particular, the partial fractions decomposition takes the form

$$\frac{8x-5}{2x^2-3x+1} = \frac{2}{2x-1} + \frac{3}{x-1}$$

and we can now integrate this equation term by term to get

$$\int \frac{8x-5}{2x^2-3x+1}\,dx = \log|2x-1| + 3\log|x-1| + C.$$

- To compute the second integral, we use integration by parts to find that

$$\int \log(x^2-1)\,dx = x\log(x^2-1) - \int \frac{2x^2}{x^2-1}\,dx.$$

Using division of polynomials to simplify the rational function, we get

$$\frac{2x^2}{x^2-1} = 2 + \frac{2}{x^2-1} \quad \Longrightarrow \quad \int \frac{2x^2}{x^2-1}\,dx = 2x + \int \frac{2}{x^2-1}\,dx$$

and we can now use partial fractions as before, namely

$$\frac{2}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1} \quad \Longrightarrow \quad 2 = A(x+1) + B(x-1).$$

Setting $x = 1$ and $x = -1$ gives $2 = 2A$ and $2 = -2B$, respectively, hence

$$\begin{aligned}\int \frac{2x^2}{x^2-1}\, dx &= 2x + \int \frac{1}{x-1} - \frac{1}{x+1}\, dx \\ &= 2x + \log|x-1| - \log|x+1| + C.\end{aligned}$$

In particular, the original integral is given by

$$\int \log(x^2-1)\, dx = x\log(x^2-1) - 2x - \log|x-1| + \log|x+1| + C.$$

2. *Define a sequence $\{a_n\}$ by setting $a_1 = 1$ and $a_{n+1} = \sqrt{2a_n+1}$ for each $n \geq 1$. Show that $1 \leq a_n \leq a_{n+1} \leq 3$ for each $n \geq 1$, use this fact to conclude that the sequence converges and then find its limit.*

- Since the first two terms are $a_1 = 1$ and $a_2 = \sqrt{3}$, the statement

$$1 \leq a_n \leq a_{n+1} \leq 3$$

does hold when $n = 1$. Suppose that it holds for some n, in which case

$$\begin{aligned} 2 \leq 2a_n \leq 2a_{n+1} \leq 6 \quad &\implies \quad 3 \leq 1 + 2a_n \leq 1 + 2a_{n+1} \leq 7 \\ &\implies \quad \sqrt{3} \leq a_{n+1} \leq a_{n+2} \leq \sqrt{7} \\ &\implies \quad 1 \leq a_{n+1} \leq a_{n+2} \leq 3. \end{aligned}$$

Thus, the statement holds for $n+1$ as well, so it holds for all $n \in \mathbb{N}$. This shows that the given sequence is monotonic and bounded, hence also convergent; denote its limit by L. Using the definition of the sequence, we then find that

$$\begin{aligned} a_{n+1} = \sqrt{2a_n+1} \quad &\implies \quad L = \sqrt{2L+1} \quad &&\implies \quad L^2 = 2L+1 \\ &\implies \quad L^2 - 2L - 1 = 0 \quad &&\implies \quad L = 1 \pm \sqrt{2}. \end{aligned}$$

Since $1 \leq a_n \leq 3$ for all n, however, we also have $1 \leq L \leq 3$, hence $L = 1 + \sqrt{2}$.

3. *Compute each of the following integrals:*

$$\int x^2 \sin x\, dx, \qquad \int \sin(\log x)\, dx.$$

- To compute the first integral, we shall use tabular integration.

Differentiating	Integrating
x^2	$\sin x$
$2x$	$-\cos x$
2	$-\sin x$
0	$\cos x$

Consulting the table above, we find that

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C.$$

- To compute the second integral, we integrate by parts to get

$$\begin{aligned}\int \sin(\log x) \, dx &= x \sin(\log x) - \int x \cdot \cos(\log x) \cdot x^{-1} \, dx \\ &= x \sin(\log x) - \int \cos(\log x) \, dx.\end{aligned}$$

Using another integration by parts, one similarly finds that

$$\begin{aligned}\int \cos(\log x) \, dx &= x \cos(\log x) + \int x \cdot \sin(\log x) \cdot x^{-1} \, dx \\ &= x \cos(\log x) + \int \sin(\log x) \, dx.\end{aligned}$$

Once we now combine these observations, we arrive at

$$\int \sin(\log x) \, dx = x \sin(\log x) - x \cos(\log x) - \int \sin(\log x) \, dx.$$

Bringing the rightmost integral to the left hand side, we conclude that

$$\int \sin(\log x) \, dx = \frac{x \sin(\log x) - x \cos(\log x)}{2} + C.$$