MA121, Homework #4
Solutions

1. *Let f be the function defined by*

$$f(x) = \left\{ \begin{array}{ll} 0 & \textit{if } x \neq 0 \\ 1 & \textit{if } x = 0 \end{array} \right\}.$$

Show that f is integrable on $[0,1]$.

- Given any partition $P = \{x_0, x_1, \ldots, x_n\}$ of the interval $[0,1]$, one clearly has

$$S^-(f,P) = \sum_{k=0}^{n-1} \inf_{[x_k,x_{k+1}]} f(x) \cdot (x_{k+1} - x_k) = 0$$

because all the summands are zero. This also implies that $\sup S^-(f,P) = 0$ as well. Using a similar computation for the upper sums, one finds

$$S^+(f,P) = \sum_{k=0}^{n-1} \sup_{[x_k,x_{k+1}]} f(x) \cdot (x_{k+1} - x_k) = 1 \cdot (x_1 - x_0) = x_1.$$

This gives $\inf S^+(f,P) = \inf x_1 = 0$ and thus f is integrable on $[0,1]$, indeed.

2. *Suppose f, g are both integrable on $[a,b]$ and $f(x) \leq g(x)$ for all $x \in [a,b]$. Show that*

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

- Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. Starting with the inequality

$$f(x) \leq g(x) \quad \text{for all } x \in [x_k, x_{k+1}],$$

we may take the infimum of both sides to get

$$\inf_{[x_k,x_{k+1}]} f(x) \leq \inf_{[x_k,x_{k+1}]} g(x).$$

Multiplying by the positive quantity $x_{k+1} - x_k$ and then adding, we conclude that

$$\sum_{k=0}^{n-1} \inf_{[x_k,x_{k+1}]} f(x) \cdot (x_{k+1} - x_k) \leq \sum_{k=0}^{n-1} \inf_{[x_k,x_{k+1}]} g(x) \cdot (x_{k+1} - x_k).$$

Since the last inequality holds for all partitions P, we must thus have

$$S^-(f,P) \leq S^-(g,P)$$

for all partitions P. Taking the supremum of both sides, we finally deduce that

$$\int_a^b f(x)\,dx = \sup_P S^-(f,P) \;\leq\; \sup_P S^-(g,P) = \int_a^b g(x)\,dx.$$

3. *Show that there exists a unique function f which is defined for all $x \in \mathbb{R}$ and satisfies*

$$f'(x) = e^{-x^2}, \qquad f(0) = 0.$$

- Being continuous, the function $g(x) = e^{-x^2}$ is integrable, and we also have

$$f(x) = \int_0^x e^{-t^2}\,dt \quad \Longrightarrow \quad f'(x) = e^{-x^2}$$

by the fundamental theorem of calculus. Since $f(0) = 0$ by above, we conclude that f has the desired properties. Suppose that g also does, namely suppose

$$g'(x) = e^{-x^2} = f'(x), \qquad g(0) = 0 = f(0).$$

Then $g(x) - f(x)$ must be constant and so $g(x) - f(x) = g(0) - f(0) = 0$ for all x.

4. *Let f be the function of the previous exercise. Show that f is increasing and that*

$$0 \leq f(x) \leq x \quad \text{for all } x \geq 0.$$

- Since $f'(x) = e^{-x^2} > 0$, it is clear that f is increasing. In addition, we have

$$0 \leq e^{-x^2} \leq e^0 = 1$$

and we know that constant functions are integrable. Using exercise 2, we now get

$$0 \leq e^{-t^2} \leq 1 \quad \Longrightarrow \quad \int_0^x 0\,dt \leq \int_0^x e^{-t^2}\,dt \leq \int_0^x 1\,dt.$$

Since the integral in the middle is merely $f(x)$ by definition, this actually gives

$$0(x-0) \leq f(x) \leq 1(x-0) \quad \Longrightarrow \quad 0 \leq f(x) \leq x.$$