MA121, Homework #2
Solutions

1. *Make a table listing the min, inf, max and sup of each of the following sets; write DNE for all quantities which fail to exist. You need not justify any of your answers.*

(a) $A = \left\{ n \in \mathbb{N} : \frac{n}{n+1} < \frac{1}{2} \right\}$

(b) $B = \{x \in \mathbb{Z} : x > 2 \text{ and } 2x \leq 9\}$

(c) $C = \{x \in \mathbb{R} : x < y \text{ for all } y \in \mathbb{N}\}$

(d) $D = \{x \in \mathbb{R} : |x| < y \text{ for all } y > 0\}$

(e) $E = \{x \in \mathbb{R} : |x - 2| < 3\}$

- A complete list of answers is provided by the following table.

	min	inf	max	sup
A	DNE	DNE	DNE	DNE
B	3	3	4	4
C	DNE	DNE	DNE	1
D	0	0	0	0
E	DNE	-1	DNE	5

- The set A contains all $n \in \mathbb{N}$ with $2n < n + 1$; this gives $n < 1$ and so A is empty.
- The set B contains all integers x with $2 < x \leq 9/2$; this means that $B = \{3, 4\}$.
- The set C contains all real numbers x with $x < 1$, namely $C = (-\infty, 1)$.
- The set D contains the real numbers x with $-y < x < y$ for all $y > 0$. Since such a number is smaller than all positive reals and bigger than all negative ones, $D = \{0\}$.
- The set E contains the real numbers x with $-3 < x - 2 < 3$, namely $E = (-1, 5)$.

2. *Let f be a function such that $f(1) = 5$ and $f(n + 1) = 2f(n) + 1$ for all $n \in \mathbb{N}$. Use induction to show that we actually have $f(n) = 3 \cdot 2^n - 1$ for all $n \in \mathbb{N}$.*

- When $n = 1$, we have $f(1) = 5 = 3 \cdot 2 - 1$ and so the given formula holds. Suppose that it holds for some n, in which case $f(n) = 3 \cdot 2^n - 1$. Then we have

$$f(n + 1) = 2f(n) + 1 = 2(3 \cdot 2^n - 1) + 1 = 3 \cdot 2^{n+1} - 1$$

and so it holds for $n + 1$ as well. This shows that the formula holds for all $n \in \mathbb{N}$.

3. *Suppose f, g are functions with $f(x) \leq g(x)$ for all x. Show that* $\sup f(x) \leq \sup g(x)$.

- Since $\sup g(x)$ is an upper bound for the values $g(x)$, it is clear that

$$f(x) \leq g(x) \leq \sup g(x)$$

for all x. This makes $\sup g(x)$ an upper bound for the values $f(x)$. Since $\sup f(x)$ is the least upper bound for these values, we deduce that $\sup f(x) \leq \sup g(x)$.

4. *Evaluate the limit*

$$L = \lim_{x \to 1} \frac{x^3 + 3x^2 - 9x + 5}{(x-1)^2}.$$

- Using division of polynomials, one easily finds that

$$L = \lim_{x \to 1} \frac{x^3 + 3x^2 - 9x + 5}{x^2 - 2x + 1} = \lim_{x \to 1} (x + 5) = 1 + 5 = 6$$

because limits of polynomial functions can be computed by simple substitution.

5. *Show that the function f defined by*

$$f(x) = \left\{ \begin{array}{ll} 2x - 7 & \textit{if } x \leq 3 \\ 8 - 3x & \textit{if } x > 3 \end{array} \right\}$$

is continuous at $y = 3$.

- To prove that f is continuous at $y = 3$, let us first note that

$$|f(x) - f(3)| = |f(x) + 1| = \left\{ \begin{array}{ll} 2|x-3| & \text{if } x \leq 3 \\ 3|3-x| & \text{if } x > 3 \end{array} \right\}.$$

Now, let $\varepsilon > 0$ be given and set $\delta = \varepsilon/3$. Then $\delta > 0$ and we easily get

$$|x - 3| < \delta \quad \Longrightarrow \quad |f(x) - f(3)| \leq 3|x - 3| < 3\delta = \varepsilon.$$