MA121, Homework #1
Solutions

1. *Show that the set* $A = \{x \in \mathbb{R} : |2x - 3| < 4\}$ *has no minimum.*

- First of all, let us note that the given inequality is equivalent to

$$\begin{aligned} |2x-3| < 4 \qquad &\iff \qquad -4 < 2x - 3 < 4 \\ &\iff \qquad -1 < 2x < 7 \\ &\iff \qquad -1/2 < x < 7/2. \end{aligned}$$

To see that A has no minimum, suppose $-1/2 < x < 7/2$ is the minimum and let

$$y = \frac{-1/2 + x}{2}\,.$$

Being the average of $-1/2$ and x, this number must lie between them and so

$$-1/2 < y < x < 7/2.$$

This makes y an element of A which is smaller than $x = \min A$, a contradiction.

2. *Show that the set* $B = \{\frac{3n+1}{n+1} : n \in \mathbb{N}\}$ *has a minimum but no maximum.*

- First, we note that $\frac{3+1}{1+1} = 2$ is an element of B and that we also have

$$2 \leq \frac{3n+1}{n+1} \qquad \iff \qquad 2n + 2 \leq 3n + 1 \qquad \iff \qquad 1 \leq n.$$

Since the rightmost inequality is true, the leftmost one is true as well. This makes 2 an element of B which is at least as small as any other element, so $\min B = 2$.

- To see that B has no maximum, suppose $x = \frac{3n+1}{n+1}$ is the maximum and let

$$y = \frac{3(n+1)+1}{(n+1)+1} = \frac{3n+4}{n+2}\,.$$

Then y is an element of B, and it is actually larger than x because

$$\frac{3n+1}{n+1} < \frac{3n+4}{n+2} \qquad \iff \qquad 3n^2 + 7n + 2 < 3n^2 + 7n + 4 \qquad \iff \qquad 2 < 4.$$

As no element of B can be larger than the maximum of B, this is a contradiction.

3. *Show that the set* $B = \{\frac{3n+1}{n+1} : n \in \mathbb{N}\}$ *is such that* $\sup B = 3$.

- To show that 3 is an upper bound of the given set, we note that

$$\frac{3n+1}{n+1} \leq 3 \quad \iff \quad 3n+1 \leq 3n+3 \quad \iff \quad 1 \leq 3.$$

  Since the rightmost inequality is true, the leftmost one must also be true.

- To show that 3 is the least upper bound, we show that no number $x < 3$ is an upper bound. Let us then fix some $x < 3$ and try to find an element of B which is bigger than x. Since every element of B has the form $\frac{3n+1}{n+1}$, we need to make sure that

$$\frac{3n+1}{n+1} > x \quad \iff \quad 3n+1 > nx+x \quad \iff \quad (3-x)n > x-1.$$

  Since $3-x$ is positive by above, we thus need to make sure that

$$n > \frac{x-1}{3-x}.$$

  According to one of our theorems, we can always find an integer n that satisfies this inequality. Then our computation above shows that $\frac{3n+1}{n+1}$ is an element of B which is larger than x. This also means that x is not an upper bound of B, as needed.

4. *Determine* $\max C$ *when* $C = \{y \in \mathbb{R} : y = 1 + x - x^2$ *for some* $x \in \mathbb{R}\}$.

- To determine the maximum of the given set, we complete the square to get

$$\begin{aligned} 1 + x - x^2 &= -\left(x^2 - x - 1\right) = -\left(x^2 - x + \frac{1}{4} - \frac{1}{4} - 1\right) \\ &= -(x - 1/2)^2 + 5/4 \\ &\leq 5/4. \end{aligned}$$

  Note that equality holds in the last inequality when $x = 1/2$. This means that $5/4$ is an element of C which is at least as large as any other element, hence $\max C = 5/4$.