Chapter 4

Integrals

4.1 The definition of an integral

Notation (Sigma notation). The Greek letter Σ is used to denote sums such as

$$a_1 + a_2 + a_3 + a_4 = \sum_{k=1}^{4} a_k.$$

Here, the index k ranges from $k = 1$ to $k = 4$ and it only takes integral values by convention. As for the letter Σ, this indicates that the corresponding terms a_1, a_2, a_3, a_4 are to be added.

Definition 4.1 (Partition). We say that $P = \{x_0, x_1, \ldots, x_n\}$ forms a partition of the closed interval $[a, b]$, if the elements of P are such that $a = x_0 < x_1 < \cdots < x_n = b$.

Definition 4.2 (Darboux sums). Suppose that f is bounded on $[a, b]$. Given a partition P of this interval, we may then define the lower Darboux sum

$$S^-(f, P) = \sum_{k=0}^{n-1} \inf_{[x_k, x_{k+1}]} f(x) \cdot (x_{k+1} - x_k)$$

and we may similarly define the upper Darboux sum

$$S^+(f, P) = \sum_{k=0}^{n-1} \sup_{[x_k, x_{k+1}]} f(x) \cdot (x_{k+1} - x_k).$$

Definition 4.3 (Refinement). We say that a partition Q is a refinement of the partition P, if the partition Q contains more points than P does, namely if $P \subset Q$.

Lemma 4.4 (Darboux sums and refinements). Suppose f is bounded on $[a, b]$. Given any partition P of the interval and any refinement Q of this partition, one must then have

$$S^-(f, P) \leq S^-(f, Q) \leq S^+(f, Q) \leq S^+(f, P).$$

Thus, more refined partitions give rise to larger lower sums but smaller upper sums.

Corollary 4.5 (Lower and upper sums). If f is bounded on $[a, b]$, then one has

$$S^-(f, P) \le S^+(f, Q)$$

for any two partitions P, Q of the interval. In particular, one has

$$\sup_P \{S^-(f, P)\} \le \inf_Q \{S^+(f, Q)\}.$$

Definition 4.6 (Integrability). Suppose f is bounded on $[a, b]$. If it happens that

$$\sup_P \{S^-(f, P)\} = \inf_Q \{S^+(f, Q)\},$$

then we say that f is integrable on $[a, b]$, and we also use the notation

$$\int_a^b f(x)\, dx = \sup_P \{S^-(f, P)\} = \inf_Q \{S^+(f, Q)\}.$$

Example 4.7 (An integrable function). Every constant function is integrable with

$$\int_a^b c\, dx = c \cdot (b - a).$$

Example 4.8 (A non-integrable function). Let f be the function defined by

$$f(x) = \left\{ \begin{array}{ll} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{array} \right\}.$$

Then f is bounded, yet it fails to be integrable on every closed interval $[a, b]$.

Theorem 4.9 (Integrability condition). Suppose f is bounded on $[a, b]$. To say that f is integrable on $[a, b]$ is to say that given any $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$S^+(f, P) - S^-(f, P) < \varepsilon.$$

In practice, this integrability condition is much easier to check than our previous condition

$$\sup_P \{S^-(f, P)\} = \inf_Q \{S^+(f, Q)\}.$$

4.2 Integration rules

Theorem 4.10 (Additivity). Let $a < b < c$ and suppose that f is bounded on $[a, c]$. If f is integrable on both $[a, b]$ and $[b, c]$, then f is also integrable on $[a, c]$, and we have

$$\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx.$$

Theorem 4.11 (Linearity). Suppose f, g are bounded and integrable on $[a, b]$. Let $c \in \mathbb{R}$.

(a) The sum $f + g$ is then integrable on $[a, b]$, and we have

$$\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$

(b) The scalar multiple cf is also integrable on $[a, b]$, and we have

$$\int_a^b cf(x)\, dx = c \int_a^b f(x)\, dx.$$

Definition 4.12. Our definition of integrability on $[a, b]$ implicitly assumes that $a < b$. It is quite convenient, however, to also introduce integrals for the remaining cases by setting

$$\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx, \qquad \int_a^a f(x)\, dx = 0.$$

As one can easily check then, the last two theorems apply for these integrals as well.

Theorem 4.13 (Integrals and inequalities). Suppose f, g are integrable on $[a, b]$ with

$$f(x) \leq g(x) \qquad \text{for all } x \in [a, b].$$

Then one may integrate both sides of the inequality to deduce that

$$\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx.$$

Theorem 4.14. Suppose f, g are differentiable with $f'(x) = g'(x)$ for all x. Then there exists some constant C such that $f(x) = g(x) + C$ for all x.

Theorem 4.15 (Continuous implies integrable). If a function is continuous on $[a, b]$, then it must also be integrable on $[a, b]$.

4.3 The fundamental theorem of calculus

Theorem 4.16 (FTC, part 1). Suppose that f is a continuous function. Then

$$F(x) = \int_a^x f(t)\, dt$$

is a function whose derivative is $f(x)$. In other words, one has $F'(x) = f(x)$ for all x.

Theorem 4.17 (log and arctan). Each of the following statements is true.

(a) There exists a unique function $\log x$ which is defined for all $x > 0$ and satisfies

$$(\log x)' = \frac{1}{x}\,, \qquad \log 1 = 0.$$

(b) There exists a unique function $\arctan x$ which is defined for all $x \in \mathbb{R}$ and satisfies

$$(\arctan x)' = \frac{1}{x^2+1}\,, \qquad \arctan 0 = 0.$$

(c) Both $\log x$ and $\arctan x$ are strictly increasing wherever they are defined.

Theorem 4.18 (FTC, part 2). Let f be a continuous function. Given any function F whose derivative is f, one must then have

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Theorem 4.19 (Substitution). If the functions f, g' are both continuous, then

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Definition 4.20 (Antiderivative). When F is a function whose derivative is f, we say that F is an antiderivative of f. The most general antiderivative of f is usually denoted by

$$\int f(x)\,dx = F(x) + C,$$

where C is an arbitrary constant. This is merely another way of expressing the fact that

$$F'(x) = f(x).$$

Example 4.21. Start with any of the known facts about derivatives, say

$$(x^2)' = 2x, \qquad (e^x)' = e^x, \qquad (\arctan x)' = \frac{1}{x^2+1}\,.$$

Then one can readily express these facts in terms of antiderivatives, namely

$$\int 2x\,dx = x^2 + C, \qquad \int e^x\,dx = e^x + C, \qquad \int \frac{dx}{x^2+1} = \arctan x + C.$$

Theorem 4.22 (Properties of antiderivatives). If f, g are continuous and $c \in \mathbb{R}$, then

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx, \qquad \int cf(x)\,dx = c\int f(x)\,dx.$$

Theorem 4.23 (sin and cos). There exists a unique pair of functions $\sin x$, $\cos x$ such that

$$(\sin x)' = \cos x, \qquad (\cos x)' = -\sin x, \qquad \sin 0 = 0, \qquad \cos 0 = 1.$$

Moreover, these two functions are defined for all $x \in \mathbb{R}$, and they also satisfy the identity

$$\sin^2 x + \cos^2 x = 1 \quad \text{for all} \;\; x \in \mathbb{R}.$$

4.4 Techniques of integration

Formula 4.24 (Substitution). One method for computing integrals is given by the formula

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du, \qquad \text{where } u = g(x).$$

You should regard this formula as an intermediate step that helps us compute the integral on the left in terms of the integral on the right; once we have computed the latter, we can simply recall our choice of $u = g(x)$ to determine the former.

Example 4.25. We use the substitution formula to compute the integral

$$\int x^5 (x^6 + 10)^3\ dx.$$

Setting $u = x^6 + 10$, we find that $du = 6x^5\, dx$, hence also

$$\int x^5 (x^6 + 10)^3\ dx = \frac{1}{6} \int u^3\ du = \frac{u^4}{24} + C = \frac{(x^6 + 10)^4}{24} + C.$$

Example 4.26. We use the substitution formula to compute the integral

$$\int \frac{x+1}{(x+2)^2}\ dx.$$

In this case, we take $u = x + 2$ to simplify the denominator. Since $du = dx$, we find

$$\begin{aligned} \int \frac{x+1}{(x+2)^2}\ dx = \int \frac{u-1}{u^2}\ du &= \int u^{-1}\, du - \int u^{-2}\, du \\ &= \log|u| + u^{-1} + C = \log|x+2| + \frac{1}{x+2} + C. \end{aligned}$$

Theorem 4.27 (Integration by parts). If the functions f', g' are both continuous, then

$$\int f'(x) \cdot g(x)\, dx = f(x)\, g(x) - \int f(x) \cdot g'(x)\, dx.$$

It is quite common, and also convenient, to rewrite the last equation in the form

$$\int u\, dv = uv - \int v\, du, \qquad \text{where } u = g(x) \text{ and } v = f(x).$$

Example 4.28. We use integration by parts to compute the integral

$$\int x^2 \cdot \log x\, dx.$$

Setting $u = \log x$ and $dv = x^2\, dx$, we find that $du = \frac{1}{x}\, dx$ and $v = \frac{x^3}{3}$, hence also

$$\begin{aligned} \int x^2 \cdot \log x\, dx = \int u\, dv = uv - \int v\, du &= \frac{x^3 \log x}{3} - \int \frac{x^3}{3} \cdot \frac{1}{x}\ dx \\ &= \frac{x^3 \log x}{3} - \frac{1}{3} \int x^2\, dx = \frac{x^3 \log x}{3} - \frac{x^3}{9} + C. \end{aligned}$$

Trick 4.29 (Tabular integration). There is a standard trick for integrating a polynomial times any function f which is easy to integrate (like sines, cosines and exponentials are). One places the polynomial on the left side of a table and repeatedly differentiates it until a zero is produced; the function f is placed on the right side of the table and it is repeatedly integrated the same number of times. The result of this computation is then obtained by multiplying the entries of the table going diagonally down with signs alternating from each line to the next.

Example 4.30. Using tabular integration, one can easily compute the integral

$$\int x^2 \cdot \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C.$$

Namely, the terms on the right hand side are merely those obtained from the following table by multiplying entries going diagonally down and by changing the sign of the second term.

Differentiating	Integrating
x^2	$\sin x$
$2x$	$-\cos x$
2	$-\sin x$
0	$\cos x$

4.5 Integration of rational functions

Definition 4.31 (Degree of a polynomial). The degree of a nonzero polynomial $p(x)$ is the highest power of x that appears in the polynomial.

Definition 4.32 (Proper rational function). A rational function is said to be proper, if the degree of its numerator is strictly smaller than that of its denominator.

Lemma 4.33 (Division with remainder). Every rational function can be expressed as the sum of a polynomial and a proper rational function.

Example 4.34. To apply the previous lemma for the rational function

$$\frac{x^2 - x + 4}{x + 2},$$

one simply divides the two polynomials to get

$$\frac{x^2 - x + 4}{x + 2} = x - 3 + \frac{10}{x + 2}.$$

In fact, this computation is quite representative of the general case, which reads

$$\frac{f(x)}{g(x)} = Q(x) + \frac{R(x)}{g(x)}.$$

In that case, $Q(x)$ is the quotient and $R(x)$ the remainder one obtains upon dividing f by g.

Theorem 4.35 (Partial fractions). Let f be a proper rational function whose denominator is the product of m polynomials having no non-constant factor in common. Then f itself is the sum of m proper rational functions whose denominators are the m polynomials.

Example 4.36. Some typical partial fractions decompositions are given by the equations

$$\begin{aligned} \frac{4x+2}{(x-2)(x+3)} &= \frac{2}{x-2} + \frac{2}{x+3}\,; \\ \frac{3x^2+1}{(x^2+1)(x+1)} &= \frac{x-1}{x^2+1} + \frac{2}{x+1}\,; \\ \frac{3x^2+3x+2}{x^3(x+2)} &= \frac{x^2+x+1}{x^3} - \frac{1}{x+2}\,. \end{aligned}$$

In practice, one is given the function on the left hand side and has to come up with the partial fractions decomposition on the right hand side. The only information provided by the previous theorem is that the rational functions on the right hand side must all be proper.

Example 4.37. We determine the partial fractions decomposition of the function

$$f(x) = \frac{x+3}{x^2-1} = \frac{x+3}{(x-1)(x+1)}\,.$$

Since the denominator is the product of two polynomials, f can be written as the sum of two fractions whose denominators are these two polynomials. In other words, we can write

$$\frac{x+3}{(x-1)(x+1)} = \frac{???}{x-1} + \frac{???}{x+1}\,.$$

Note that the unknown fractions must both be proper by the theorem above. Since each of the denominators has degree 1, each of the numerators must have degree zero. This means that

$$\frac{x+3}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} \tag{4.1}$$

for some constants A, B that need to be determined. We clear denominators to get

$$x+3 = A(x+1) + B(x-1)$$

and then we look at some suitable choices of x. Setting $x=1$ gives

$$4 = 2A \quad\Longrightarrow\quad A = 2,$$

while setting $x=-1$ gives

$$2 = -2B \quad\Longrightarrow\quad B = -1.$$

In view of the last two equations, the desired partial fractions decomposition (4.1) is thus

$$\frac{x+3}{(x-1)(x+1)} = \frac{2}{x-1} - \frac{1}{x+1}\,.$$

Example 4.38. We use partial fractions to compute the integral

$$\int \frac{x^2+x-2}{(x^2+1)(x+1)}\,dx.$$

Proceeding as before, we know that we can write

$$\frac{x^2+x-2}{(x^2+1)(x+1)} = \frac{???}{x^2+1} + \frac{???}{x+1}$$

for a suitable choice of numerators. Since each of these fractions is proper, the first numerator has degree 1 or less, while the second numerator has degree zero. In other words,

$$\frac{x^2+x-2}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1} \tag{4.2}$$

for some constants A, B, C that need to be determined. Clearing denominators gives

$$x^2+x-2 = (Ax+B)(x+1) + C(x^2+1)$$

and we can now look at some suitable choices of x. We set $x = -1$ to get

$$1-1-2 = 2C \quad\Longrightarrow\quad 2C = -2 \quad\Longrightarrow\quad C = -1,$$

we set $x = 0$ to similarly get

$$-2 = B + C = B - 1 \quad\Longrightarrow\quad B = -1,$$

and we set $x = 1$ to get

$$1+1-2 = 2(A+B) + 2C \quad\Longrightarrow\quad A = -B - C = 2.$$

Returning to equation (4.2), we may finally conclude that

$$\int \frac{x^2+x-2}{(x^2+1)(x+1)}\,dx = \int \frac{2x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx - \int \frac{1}{x+1}\,dx.$$

The two rightmost integrals are rather easy to compute, and so is the integral

$$\int \frac{2x}{x^2+1}\,dx = \int \frac{du}{u} = \log|u| + C = \log(x^2+1) + C,$$

if one uses the substitution $u = x^2 + 1$. Once we now combine the last two equations, we find

$$\int \frac{x^2+x-2}{(x^2+1)(x+1)}\,dx = \log(x^2+1) - \arctan x - \log|x+1| + C.$$

Example 4.39 (Improper rational functions). Consider a rational function such as

$$f(x) = \frac{x^3 - x^2 + x}{x^2 - 1} = \frac{x^3 - x^2 + x}{(x-1)(x+1)}\,.$$

Although the denominator is a product of two factors, we cannot use partial fractions directly since the given rational function is not proper. We first use division of polynomials to write

$$f(x) = \frac{x^3 - x^2 + x}{x^2 - 1} = x - 1 + \frac{2x - 1}{x^2 - 1}$$

and then we integrate both sides of this equation to get

$$\int f(x)\,dx = \frac{x^2}{2} - x + \int \frac{2x-1}{x^2-1}\,dx. \tag{4.3}$$

The rightmost integral does involve a proper rational function, so we can now write

$$\frac{2x-1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1} \tag{4.4}$$

using partial fractions. Proceeding as usual, we clear denominators to get

$$2x - 1 = A(x+1) + B(x-1)$$

and we look at some suitable choices of x to find

$$x = 1, \quad x = -1 \quad \Longrightarrow \quad 1 = 2A, \quad -3 = -2B \quad \Longrightarrow \quad A = 1/2, \quad B = 3/2.$$

This determines the missing coefficients in (4.4), hence that equation actually reads

$$\frac{2x-1}{x^2-1} = \frac{1/2}{x-1} + \frac{3/2}{x+1}\,.$$

Inserting this fact in equation (4.3), we may finally conclude that

$$\begin{aligned}\int f(x)\,dx &= \frac{x^2}{2} - x + \int \frac{1/2}{x-1}\,dx + \int \frac{3/2}{x+1}\,dx \\ &= \frac{x^2}{2} - x + \frac{1}{2}\cdot \log|x-1| + \frac{3}{2}\cdot\log|x+1| + C.\end{aligned}$$

Example 4.40 (Integrals involving arctan). Given any fixed $a \neq 0$, it is easy to show that

$$\int \frac{dx}{x^2 + a^2} = \frac{\arctan(x/a)}{a} + C. \tag{4.5}$$

Namely, one uses the substitution $x = au$ to get $dx = a\,du$, hence also

$$\int \frac{dx}{x^2+a^2} = \int \frac{a\,du}{a^2u^2 + a^2} = \frac{1}{a}\int \frac{du}{u^2+1} = \frac{\arctan u}{a} + C = \frac{\arctan(x/a)}{a} + C.$$

This settles the simplest case in which the denominator of the given rational function cannot be factored; a more general case is described in our next example.

Example 4.41 (Completing the square). Consider the rational function

$$f(x) = \frac{2x+4}{x^2-4x+8}.$$

Its denominator cannot be factored since the discriminant $(-4)^2 - 4 \cdot 8 = -16$ is negative. In particular, one cannot use partial fractions to integrate this function, either. Let us write

$$x^2 - 4x + 8 = x^2 - 4x + 4 + 4 = (x-2)^2 + 4 = u^2 + 4 \tag{4.6}$$

by completing the square and by setting $u = x - 2$ for convenience. This actually gives

$$f(x) = \frac{2x+4}{x^2-4x+8} = \frac{2(u+2)+4}{u^2+4} = \frac{2u+8}{u^2+4}$$

and the new denominator resembles that of the previous example. Since $du = dx$, we get

$$\begin{aligned}\int f(x)\,dx &= \int \frac{2u}{u^2+4}\,du + \int \frac{8}{u^2+4}\,du \\ &= \log(u^2+4) + 8 \cdot \frac{\arctan(u/2)}{2} + C \\ &= \log(x^2-4x+8) + 4\arctan(x/2-1) + C\end{aligned}$$

by using equation (4.5) in the second equality and equation (4.6) in the third.