Chapter 3

Derivatives

3.1 The definition of a derivative

Definition 3.1 (Average rate of change). Given a function f that is defined on the closed interval $[x, y]$, its average rate of change over $[x, y]$ is defined as the ratio

$$\frac{f(y) - f(x)}{y - x} .$$

Definition 3.2 (Derivative). We say that f is differentiable at the point y, if the limit

$$\lim_{x \to y} \frac{f(y) - f(x)}{y - x}$$

exists. In the case that it does exist, we call it the derivative of f at y, and we also write

$$f'(y) = \lim_{x \to y} \frac{f(y) - f(x)}{y - x} .$$

Intuitively speaking then, $f'(y)$ is just the rate at which f changes around the point y.

Theorem 3.3 (Differentiable implies continuous). A function which is differentiable at a point must necessarily be continuous at that point.

Example 3.4 (Continuous but not differentiable). The absolute value function $f(x) = |x|$ is continuous but not differentiable at $y = 0$.

Proposition 3.5 (Basic derivatives). Each of the following statements is true.

(a) The derivative of a constant function is equal to zero; that is, $c' = 0$ for all $c \in \mathbb{R}$.

(b) The derivative of the identity function $f(x) = x$ is equal to 1; that is, $x' = 1$.

3.2 Differentiation rules

Proposition 3.6. The following rules of differentiation hold for each constant $c \in \mathbb{R}$ and all differentiable functions f, g.

(a) The derivative of a sum is given by $(f+g)' = f' + g'$.

(b) The derivative of a constant multiple is given by $(cf)' = cf'$.

(c) **Product Rule**: the derivative of a product is given by $(fg)' = f'g + fg'$.

(d) **Quotient Rule**: the derivative of a quotient is given by

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

at all points at which g is nonzero.

Corollary 3.7. Given any natural number n, one has the formula

$$(x^n)' = nx^{n-1},$$

where the power x^0 is defined by the rule $x^0 = 1$ for all $x \in \mathbb{R}$.

Corollary 3.8 (Integral powers). Given any integer n, one has the formula

$$(x^n)' = nx^{n-1},$$

where negative powers of x are defined by the rule $x^{-m} = 1/x^m$ for all $x \neq 0$.

Lemma 3.9. The derivative of $f(x) = \sqrt{x}$ is given by $f'(x) = \frac{1}{2\sqrt{x}}$ for all $x > 0$.

Theorem 3.10 (Chain rule). If f, g are both differentiable, then their composition $f \circ g$ is also differentiable and its derivative is given by $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$.

Example 3.11. Using the chain rule, one finds that

$$f(x) = (x^2+3)^5 \quad \Longrightarrow \quad f'(x) = 5(x^2+3)^4 \cdot (x^2+3)' = 10x(x^2+3)^4,$$

while a similar argument gives

$$f(x) = \sqrt{x^3+x} \quad \Longrightarrow \quad f'(x) = \frac{1}{2\sqrt{x^3+x}} \cdot (x^3+x)' = \frac{3x^2+1}{2\sqrt{x^3+x}}.$$

3.3 Applications of derivatives

Theorem 3.12 (Location of min/max). Suppose f is continuous on the closed interval $[a, b]$. Then f attains both a minimum and a maximum value on $[a, b]$. Moreover, these values can only be attained at

- one of the endpoints a, b;
- a point $x \in (a, b)$ where the derivative $f'(x)$ does not exist;
- a point $x \in (a, b)$ where the derivative $f'(x)$ is zero.

Example 3.13. We determine the minimum and maximum values of the function

$$f(x) = x^3 - 3x$$

over the closed interval $[0, 2]$. Note that f is differentiable on this interval and that

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x + 1)(x - 1).$$

Thus, the only points at which the minimum/maximum values may occur are the points

$$x = -1, \qquad x = 1, \qquad x = 0, \qquad x = 2.$$

We now exclude the leftmost point, as this fails to lie in the interval $[0, 2]$. Since

$$f(1) = 1^3 - 3 = -2, \qquad f(0) = 0, \qquad f(2) = 2^3 - 3 \cdot 2 = 2,$$

the minimum value is then $f(1) = -2$ and the maximum value is $f(2) = 2$.

Theorem 3.14 (ROLLE'S THEOREM). Suppose f is differentiable on the closed interval $[a, b]$ and suppose $f(a) = f(b)$. Then there exists some $c \in (a, b)$ such that $f'(c) = 0$.

Application 3.15 (Number of roots). Combining Bolzano's theorem with Rolle's theorem, we will show that the polynomial

$$f(x) = x^3 + 3x + 2$$

has exactly one real root. First of all, f is continuous on the closed interval $[-1, 0]$ and

$$f(-1) = -1 - 3 + 2 = -2 < 0, \qquad f(0) = 2 > 0.$$

Thus, f has a root in $(-1, 0)$ by Bolzano's theorem. Suppose f has two roots, say $a < b$. Then

$$f(a) = f(b) = 0$$

and we can apply Rolle's theorem to find that $f'(c) = 0$ for some $c \in (a, b)$. On the other hand,

$$f'(c) = 3c^2 + 3 \geq 3$$

cannot possibly be zero, so this is a contradiction. In particular, f has exactly one root.

Theorem 3.16 (MEAN VALUE THEOREM). Suppose f is differentiable on the closed interval $[a, b]$. Then there exists some $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}\,.$$

Namely, the average rate of change is equal to the actual rate of change at some point.

Definition 3.17 (Increasing and decreasing). Suppose f is defined on some interval I. We say that f is

- increasing on I, if $x \leq y \implies f(x) \leq f(y)$ for all $x, y \in I$.
- decreasing on I, if $x \leq y \implies f(x) \geq f(y)$ for all $x, y \in I$.
- strictly increasing on I, if $x < y \implies f(x) < f(y)$ for all $x, y \in I$.
- strictly decreasing on I, if $x < y \implies f(x) > f(y)$ for all $x, y \in I$.

Plainly stated, (strictly) increasing functions preserve an inequality when applied to both sides of an inequality, whereas (strictly) decreasing functions reverse it.

Warning. A constant function is considered to be increasing, yet not strictly increasing.

Theorem 3.18 (Up or down). Suppose that f is differentiable on some interval I.

(a) If $f'(x) > 0$ for all $x \in I$, then f is strictly increasing throughout the interval.

(b) If $f'(x) < 0$ for all $x \in I$, then f is strictly decreasing throughout the interval.

(c) If $f'(x) = 0$ for all $x \in I$, then f is constant throughout the interval.

Example 3.19. To compute the maximum value of $f(x) = -3x^2 + 12x - 5$, we note that

$$f'(x) = -6x + 12 = -6(x - 2).$$

As for the sign of the derivative f', this can be determined using the table below.

x	2	
$x - 2$	$-$	$+$
$f'(x)$	$+$	$-$
$f(x)$	$\nearrow$	$\searrow$

According to the table, f is strictly increasing on $(-\infty, 2)$ and strictly decreasing on $(2, +\infty)$. In particular, $f(2) = -3 \cdot 4 + 12 \cdot 2 - 5 = 7$ is the maximum value attained by f.

Remark. The only reason that we had to construct a table in our previous example was that we did not really know that a maximum value exists; this piece of information is provided by the table itself. If we were interested in the maximum value over a closed interval, instead, then we could simply apply Theorem 3.12 and thus avoid the table.

3.4 Logarithmic and exponential functions

Theorem 3.20 (Definition of log). There exists a unique function $\ell(x)$ which is defined for all $x > 0$ and satisfies the equation

$$\ell'(x) = \frac{1}{x}, \qquad \ell(1) = 0.$$

It is usually denoted by $\ell(x) = \log x$, and it is known as the logarithmic function. Moreover,

- $\log x^m = m \log x$ for all $x > 0$ and each $m \in \mathbb{Z}$;
- $\log(xy) = \log x + \log y$ for all $x, y > 0$; and
- $\log(x/y) = \log x - \log y$ for all $x, y > 0$.

Theorem 3.21 (Definition of exp). There exists a unique function $e(x)$ which is defined for all $x \in \mathbb{R}$ and satisfies the equation

$$e'(x) = e(x), \qquad e(0) = 1.$$

This function is known as the exponential function, and it also has the following properties:

- $e(x) \cdot e(-x) = 1$ for all $x \in \mathbb{R}$;
- $e(x) > 0$ for all $x \in \mathbb{R}$;
- $e(x + y) = e(x) \cdot e(y)$ for all $x, y \in \mathbb{R}$.

Proposition 3.22 (Properties of log and exp). Each of the following statements is true:

- both $e(x)$ and $\log x$ are increasing functions;
- $\log e(x) = x$ for all $x \in \mathbb{R}$;
- $e(\log x) = x$ for all $x > 0$.

Lemma 3.23 (Arbitrary powers). Given any integer m, one has the formula

$$x^m = e(m \log x) \qquad \text{for all } x > 0.$$

Given any real number k, we can thus define the power x^k using the formula

$$x^k = e(k \log x) \qquad \text{for all } x > 0.$$

It is then easy to show that $e(x) = e(1)^x$. Once we now introduce the notation

$$e = e(1),$$

we may deduce the standard formula $e(x) = e^x$, where e is some constant.

Corollary 3.24. The formula $\log x^k = k \log x$ holds for all $x > 0$ and each $k \in \mathbb{R}$.

Corollary 3.25. The formula $(x^k)' = kx^{k-1}$ holds for all $x > 0$ and each $k \in \mathbb{R}$.

Example 3.26. We use logarithms to compute the derivative of the function

$$f(x) = \frac{x^4 \cdot (x^2+3)^3 \cdot e^{2x}}{(x^4+1)^5}.$$

This complicated function involves products, quotients and exponents that are not very easy to differentiate directly. In order to overcome this difficulty, we begin by writing

$$f(x) = x^4 \cdot (x^2+3)^3 \cdot e^{2x} \cdot (x^4+1)^{-5},$$

thus getting rid of the quotient. Applying the logarithmic function, we then find

$$\begin{aligned} \log f(x) &= \log x^4 + \log(x^2+3)^3 + \log e^{2x} + \log(x^4+1)^{-5} \\ &= 4\log x + 3\log(x^2+3) + 2x - 5\log(x^4+1) \end{aligned}$$

using the main properties of log. Once we now differentiate both sides, we arrive at

$$\frac{1}{f(x)} \cdot f'(x) = \frac{4}{x} + \frac{3}{x^2+3} \cdot 2x + 2 - \frac{5}{x^4+1} \cdot 4x^3.$$

In particular, the derivative of f is given by

$$f'(x) = f(x) \cdot \left(\frac{4}{x} + \frac{6x}{x^2+3} + 2 - \frac{20x^3}{x^4+1} \right).$$

Example 3.27. We'll compute the maximum value of $f(x) = x^4 e^{-x}$ over the interval $(0, \infty)$. Using both the product rule and the chain rule, one finds that

$$f'(x) = 4x^3 \cdot e^{-x} + x^4 \cdot e^{-x} \cdot (-1) = 4x^3e^{-x} - x^4e^{-x} = x^3e^{-x}(4-x).$$

According to the table below then, the maximum value is $f(4) = 4^4e^{-4}$.

x	0		4	
$4-x$		$+$		$-$
$f'(x)$		$+$		$-$
$f(x)$		$\nearrow$		$\searrow$

3.5 Limits revisited

Theorem 3.28 (Squeeze law). Suppose that f, g, h are functions such that

$$f(x) \leq g(x) \leq h(x) \quad \text{for all } x \in \mathbb{R}.$$

Suppose also that $\lim_{x \to y} f(x) = \lim_{x \to y} h(x) = L$. Then it must be the case that $\lim_{x \to y} g(x) = L$.

Lemma 3.29 (Limits and continuity). Suppose that f is a continuous function. Then

$$\lim_{x\to y} f(g(x)) = f\left(\lim_{x\to y} g(x)\right)$$

for all functions g for which the limit on the right hand side exists. In particular, the limit of a logarithm is the logarithm of the limit and so on.

Theorem 3.30 (L'Hôpital's rule). If f, g are differentiable with $f(y) = g(y) = 0$, then

$$\lim_{x\to y} \frac{f(x)}{g(x)} = \lim_{x\to y} \frac{f'(x)}{g'(x)}\,,$$

as long as the limit on the right hand side exists. This rule allows us to compute limits of the form $0/0$, and the exact same rule applies for limits of the form ∞/∞.

Example 3.31. When one tries to compute the limit

$$L = \lim_{x\to 1} \frac{x^3 - 4x^2 + 7x - 4}{2x^3 - 3x^2 + 3x - 2}$$

using simple substitution, one ends up with $0/0$. In view of L'Hôpital's rule, this implies

$$L = \lim_{x\to 1} \frac{x^3 - 4x^2 + 7x - 4}{2x^3 - 3x^2 + 3x - 2} = \lim_{x\to 1} \frac{3x^2 - 8x + 7}{6x^2 - 6x + 3}\,.$$

Using simple substitution for the rightmost limit, we then finally arrive at

$$L = \lim_{x\to 1} \frac{3x^2 - 8x + 7}{6x^2 - 6x + 3} = \frac{3 - 8 + 7}{6 - 6 + 3} = \frac{2}{3}\,.$$

Remark. To compute limits of the form $\infty \cdot 0$, one simply expresses them in such a way that L'Hôpital's rule becomes applicable. As a simple example, one can write

$$\lim_{x\to\infty} x \log\left(1 + \frac{1}{x}\right) = \lim_{x\to\infty} \frac{\log(1 + 1/x)}{1/x}\,,$$

where the limit on the left gives $\infty \cdot 0$ and the limit on the right gives $0/0$.

Lemma 3.32. One has $e^x \geq x + 1$ for all $x \in \mathbb{R}$. Moreover, one has

$$\lim_{x\to\infty} e^x = \infty, \qquad \lim_{x\to\infty} \log x = \infty$$

in the sense that both e^x and $\log x$ can be made arbitrarily large for large enough x.

Lemma 3.33 (A useful limit). Given any real number a, one has

$$\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x = e^a.$$

3.6 Convexity and concavity

Definition 3.34 (Convex and concave). Suppose f is defined on some interval I. We say that f is convex on I, if the inequality

$$f(tz + (1-t)x) \leq tf(z) + (1-t)f(x)$$

holds for all $x, z \in I$ and each $0 < t < 1$. Similarly, we say that f is concave on I, if the exact opposite inequality holds for all $x, z \in I$ and each $0 < t < 1$.

Lemma 3.35 (Equivalent formulation). To say that f is convex is to say that

$$f(y) \leq \frac{y-x}{z-x} \cdot f(z) + \frac{z-y}{z-x} \cdot f(x) \qquad \text{for all } x < y < z.$$

In particular, any line that joins two points on the graph of a convex function must always lie above the graph of the function, so the graph of a convex function looks like a smile $\cup$. In a similar fashion, the graph of a concave function must look like a frown $\cap$.

Lemma 3.36 (Third formulation). To say that f is convex is to say that

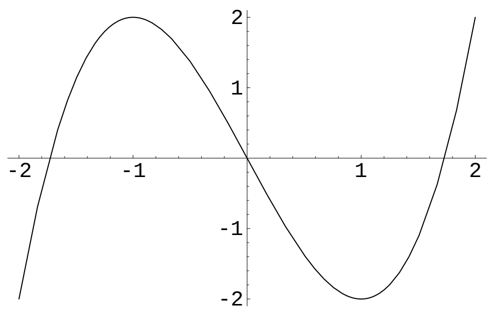
$$\frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(y)}{z-y} \qquad \text{for all } x < y < z.$$

Theorem 3.37 (Smile or frown). Suppose that both f' and f'' exist on some interval I.

(a) If $f''(x) \geq 0$ for all $x \in I$, then f is convex on I and its graph looks like a smile $\cup$.

(b) If $f''(x) \leq 0$ for all $x \in I$, then f is concave on I and its graph looks like a frown $\cap$.

Example 3.38. Let $f(x) = x^3 - 3x$. Then $f'(x) = 3x^2 - 3$ and $f''(x) = 6x$. Thus, f is convex for all $x \geq 0$ and concave for all $x \leq 0$. The graph of this function is depicted below.



Theorem 3.39 (Second derivative test). Suppose f is a twice differentiable function.

(a) If $f'(y) = 0$ and $f''(y) > 0$ at some point y, then this point is a local minimum in the sense that f can only attain larger values at nearby points.

(b) If $f'(y) = 0$ and $f''(y) < 0$ at some point y, then this point is a local maximum in the sense that f can only attain smaller values at nearby points.