MA121, 2008 Exam #2
Solutions

1. *Compute each of the following integrals:*

$$\int \frac{5x-1}{x^3-x}\,dx, \qquad \int (\log x)^2\,dx.$$

- To compute the first integral, we factor the denominator and we write

$$\frac{5x-1}{x^3-x} = \frac{5x-1}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} \qquad (*)$$

for some constants A, B, C that need to be determined. Clearing denominators gives

$$5x-1 = A(x+1)(x-1) + Bx(x-1) + Cx(x+1)$$

and we can now look at some suitable choices of x to find that

$$x=0, \quad x=-1, \quad x=1 \qquad \Longrightarrow \qquad -1=-A, \quad -6=2B, \quad 4=2C.$$

This gives $A=1$, $B=-3$ and $C=2$. In particular, equation $(*)$ reduces to

$$\frac{5x-1}{x^3-x} = \frac{1}{x} - \frac{3}{x+1} + \frac{2}{x-1}$$

and we may integrate this equation term by term to conclude that

$$\int \frac{5x-1}{x^3-x}\,dx = \log|x| - 3\log|x+1| + 2\log|x-1| + C.$$

- To compute the second integral, we integrate by parts to get

$$\begin{aligned}\int (\log x)^2\,dx &= x(\log x)^2 - \int x\cdot 2(\log x)(\log x)'\,dx \\ &= x(\log x)^2 - 2\int \log x\,dx.\end{aligned}$$

Using a similar computation for the rightmost integral, we also have

$$\int \log x\,dx = x\log x - \int x\cdot(\log x)'\,dx = x\log x - x + C$$

and we may now combine the last two equations to arrive at

$$\int (\log x)^2\,dx = x(\log x)^2 - 2x\log x + 2x + C.$$

2. *Let $a, b \in \mathbb{R}$ be some fixed numbers and consider the function f defined by*

$$f(x) = \left\{ \begin{array}{ll} a & \textit{if } x \in \mathbb{Q} \\ b & \textit{if } x \notin \mathbb{Q} \end{array} \right\}.$$

Show that f is integrable on $[0,1]$ if and only if $a = b$.

- Given any partition $P = \{x_0, x_1, \ldots, x_n\}$ of the closed interval $[0,1]$, we have

$$\begin{aligned} S^-(f,P) &= \sum_{k=0}^{n-1} \inf_{[x_k, x_{k+1}]} f(x) \cdot (x_{k+1} - x_k) \\ &= \sum_{k=0}^{n-1} \min\{a,b\} \cdot (x_{k+1} - x_k) = \min\{a,b\} \cdot (x_n - x_0) \end{aligned}$$

and a similar computation gives

$$S^+(f,P) = \sum_{k=0}^{n-1} \sup_{[x_k, x_{k+1}]} f(x) \cdot (x_{k+1} - x_k) = \max\{a,b\} \cdot (x_n - x_0).$$

Since $x_0 = 0$ and $x_n = 1$ by definition, this means f is integrable if and only if

$$\sup S^- = \inf S^+ \quad \Longleftrightarrow \quad \min\{a,b\} = \max\{a,b\} \quad \Longleftrightarrow \quad a = b.$$

3. *Define a sequence $\{a_n\}$ by setting $a_1 = 2$ and $a_{n+1} = \sqrt{4a_n + 3}$ for each $n \geq 1$. Show that $2 \leq a_n \leq a_{n+1} \leq 5$ for each $n \geq 1$, use this fact to conclude that the sequence converges and then find its limit.*

- Since the first two terms are $a_1 = 2$ and $a_2 = \sqrt{11}$, the statement $2 \leq a_n \leq a_{n+1} \leq 5$ does hold when $n = 1$. Suppose that it holds for some n, in which case

$$\begin{aligned} 8 \leq 4a_n \leq 4a_{n+1} \leq 20 \quad &\Longrightarrow \quad 11 \leq 4a_n + 3 \leq 4a_{n+1} + 3 \leq 23 \\ &\Longrightarrow \quad \sqrt{11} \leq a_{n+1} \leq a_{n+2} \leq \sqrt{23} \\ &\Longrightarrow \quad 2 \leq a_{n+1} \leq a_{n+2} \leq 5. \end{aligned}$$

In particular, the statement holds for $n+1$ as well, so it actually holds for all $n \in \mathbb{N}$. This shows that the given sequence is monotonic and bounded, hence also convergent; denote its limit by L. Using the definition of the sequence, we then find that

$$a_{n+1} = \sqrt{4a_n + 3} \quad \Longrightarrow \quad L = \sqrt{4L + 3} \quad \Longrightarrow \quad L^2 - 4L - 3 = 0.$$

Solving this quadratic equation now gives

$$L = \frac{4 \pm \sqrt{4^2 + 4 \cdot 3}}{2} = \frac{4 \pm \sqrt{28}}{2} = 2 \pm \sqrt{7}.$$

Since $2 \leq a_n \leq 5$ for each $n \in \mathbb{N}$, however, we must also have $2 \leq L \leq 5$, so

$$L = 2 + \sqrt{7}.$$

4. *Test each of the following series for convergence:*

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}\,, \qquad \sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}\,, \qquad \sum_{n=1}^{\infty} \frac{n^e}{e^n}\,.$$

- To test the first series for convergence, we use the limit comparison test with

  $$a_n = \frac{n}{n^2+1}\,, \qquad b_n = \frac{n}{n^2} = \frac{1}{n}\,.$$

  Note that the limit comparison test is, in fact, applicable here because

  $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1.$$

  Since the series $\sum_{n=1}^{\infty} b_n$ is a divergent p-series, the series $\sum_{n=1}^{\infty} a_n$ must also diverge.

- To test the second series for convergence, we use the comparison test. Since

  $$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2} \leq \sum_{n=1}^{\infty} \frac{e}{n^2}\,,$$

  the given series is smaller than a convergent p-series, so it must also converge.

- To test the last series for convergence, we use the ratio test to get

  $$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^e \cdot \frac{e^n}{e^{n+1}} = 1^e \cdot \frac{1}{e} = \frac{1}{e}\,.$$

  Since this limit is strictly less than 1, the ratio test implies convergence.

5. *Find the radius of convergence of the power series*

$$f(x) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \cdot x^n.$$

- To find the radius of convergence, one always uses the ratio test. In our case,

  $$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} \cdot \frac{(n+1)!}{n!} \cdot \frac{(2n)!}{(2n+2)!} \cdot \frac{x^{n+1}}{x^n} = \frac{(n+1)^2 \cdot x}{(2n+1)(2n+2)}$$

  and this implies that

  $$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n^2+2n+1)\,|x|}{4n^2+6n+2} = \frac{|x|}{4}\,.$$

  In particular, the power series converges when $|x| < 4$ and it diverges when $|x| > 4$, so its radius of convergence is $R = 4$.

6. *Let f be the function defined by the formula*

$$f(x) = \int_0^x e^{t^2}\, dt.$$

Find the Taylor series of f (around the point $x = 0$). For which values of x does this series converge? As a hint, you need only integrate the Taylor series for e^{t^2}.

- The idea is to integrate the Taylor series for e^{t^2} term by term, namely

$$e^{t^2} = \sum_{n=0}^{\infty} \frac{(t^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{t^{2n}}{n!} \quad \Longrightarrow \quad f(x) = \int_0^x e^{t^2}\, dt = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!}\,.$$

Although this approach is perfectly fine, you did not really know that a Taylor series can be integrated term by term, so some further justification was needed here. From that point of view, it is safer to work backwards and start by defining

$$g(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!}\,.$$

Using the ratio test, it is then easy to check that this series converges for all x. Since power series can be differentiated term by term, we find that

$$g'(x) = \sum_{n=0}^{\infty} \frac{(2n+1)x^{2n}}{(2n+1)n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = e^{x^2}.$$

Integrating this equation between 0 and x, we may thus conclude that

$$\int_0^x g'(t)\, dt = \int_0^x e^{t^2}\, dt \quad \Longrightarrow \quad g(x) - g(0) = f(x) \quad \Longrightarrow \quad g(x) = f(x).$$