MA121, 2008 Exam #1
Solutions

1. *Show that the set* $A = \{x \in \mathbb{R} : |x-2| < 1\}$ *is such that* $\sup A = 3$.

- First of all, note that the given inequality is equivalent to

$$|x-2| < 1 \quad \Longleftrightarrow \quad -1 < x-2 < 1 \quad \Longleftrightarrow \quad 1 < x < 3;$$

this makes 3 an upper bound of A. To show it is the least upper bound of A, suppose that $y < 3$ and consider two cases. If $y \leq 1$, then 2 is an element of A which is bigger than y. If $1 < y < 3$, on the other hand, then the average $\frac{y+3}{2}$ is an element of A which is bigger than y. This shows that no number $y < 3$ can be an upper bound of A.

2. *Let f be the function defined by*

$$f(x) = \left\{ \begin{array}{ll} 2x-1 & \text{if } x \leq 2 \\ 3x-2 & \text{if } x > 2 \end{array} \right\}.$$

Show that f is not continuous at $y = 2$.

- We will show that the ε-δ definition of continuity fails when $\varepsilon = 1$. Suppose it does not fail. Since $f(2) = 3$, there must then exist some $\delta > 0$ such that

$$|x-2| < \delta \quad \Longrightarrow \quad |f(x)-3| < 1. \qquad (*)$$

We now examine the last equation for the choice $x = 2 + \frac{\delta}{2}$. On one hand, we have

$$|x-2| = \frac{\delta}{2} < \delta,$$

so the assumption in equation $(*)$ holds. On the other hand, we also have

$$|f(x)-3| = |3x-5| = 1 + \frac{3\delta}{2} > 1$$

because $x = 2 + \frac{\delta}{2} > 2$ here. This actually violates the conclusion in equation $(*)$.

3. *Show that the polynomial* $f(x) = x^3 - 4x^2 - 3x + 1$ *has exactly one root in* $(0,2)$.

- Being a polynomial, f is continuous on the closed interval $[0,2]$ and we also have

$$f(0) = 1 > 0, \qquad f(2) = -13 < 0.$$

Thus, f has a root in $(0,2)$ by Bolzano's theorem. Suppose now that f has two roots in $(0,2)$. Then f' must also have a root in $(0,2)$ by Rolle's theorem. However,

$$f'(x) = 3x^2 - 8x - 3$$

and the roots of this quadratic are given by

$$x = \frac{8 \pm \sqrt{64 + 4 \cdot 3 \cdot 3}}{2 \cdot 3} = \frac{8 \pm 10}{6} \quad \Longrightarrow \quad x = 3, \quad x = -\frac{1}{3}.$$

Since neither of them lies in $(0,2)$, we conclude that f cannot have two roots in $(0,2)$.

4. *Find the maximum value of $f(x) = x(7 - x^2)^3$ over the closed interval $[0, 3]$.*

- Since f is continuous on a closed interval, it suffices to check the endpoints, the points at which f' does not exist and the points at which f' is equal to zero. In our case,

$$\begin{aligned} f'(x) &= 1 \cdot (7 - x^2)^3 + x \cdot 3(7 - x^2)^2 \cdot (7 - x^2)' \\ &= (7 - x^2)^2 \cdot (7 - x^2 - 6x^2) \\ &= (7 - x^2)^2 \cdot 7(1 - x^2). \end{aligned}$$

Keeping this in mind, the only points at which the maximum value may occur are

$$x = 0, \qquad x = 3, \qquad x = \pm 1, \qquad x = \pm\sqrt{7}.$$

Excluding the points that fail to lie in the given closed interval, we now compute

$$f(3) = -24, \qquad f(1) = 216, \qquad f(\sqrt{7}\,) = f(0) = 0.$$

Based on these facts, we may thus conclude that the maximum value is $f(1) = 216$.

5. *Show that $xe^x \geq e^x - 1$ for all $x \in \mathbb{R}$.*

- Letting $f(x) = xe^x - e^x + 1$ for convenience, one easily finds that

$$f'(x) = e^x + xe^x - e^x = xe^x.$$

Since exponentials are always positive, this implies $f'(x) > 0$ if and only if $x > 0$. In particular, f is decreasing when $x < 0$ and also increasing when $x > 0$, so

$$\min f(x) = f(0) = -e^0 + 1 = 0 \quad \Longrightarrow \quad f(x) \geq \min f(x) = 0.$$

6. *Suppose that $x > y > 0$. Using the mean value theorem or otherwise, show that*

$$1 - \frac{y}{x} < \log\frac{x}{y} < \frac{x}{y} - 1.$$

- Letting $f(x) = \log x$ for convenience, we use the mean value theorem to find that

$$f'(c) = \frac{f(x) - f(y)}{x - y} \quad \Longrightarrow \quad \frac{1}{c} = \frac{\log x - \log y}{x - y}$$

for some $y < c < x$. Inverting these positive numbers reverses the inequality, so

$$\begin{aligned} \frac{1}{x} < \frac{1}{c} < \frac{1}{y} \quad &\Longrightarrow \quad \frac{1}{x} < \frac{\log x - \log y}{x - y} < \frac{1}{y} \\ &\Longrightarrow \quad 1 - \frac{y}{x} < \log x - \log y < \frac{x}{y} - 1. \end{aligned}$$