**MA121, 2007 Final exam**
**Solutions**

**1**. *Suppose that $A$ is a nonempty subset of $\mathbb{R}$ that has a lower bound and let $\varepsilon > 0$ be given. Show that there exists an element $a \in A$ such that* $\inf A \le a < \inf A + \varepsilon$.

- Note that $\inf A + \varepsilon$ cannot be a lower bound of $A$ because it is larger than the greatest lower bound of $A$. This means that some $a \in A$ is such that $a < \inf A + \varepsilon$. On the other hand, we must also have $a \geq \inf A$ because $a \in A$ and $\inf A$ is a lower bound of $A$.

**2**. *Show that the polynomial $f(x) = x^3 - 7x^2 - 5x + 1$ has exactly one root in* $[0, 2]$.

- Being a polynomial, $f$ is continuous on the closed interval $[0, 2]$ and we also have

$$f(0) = 1 > 0, \qquad f(2) = 8 - 28 - 10 + 1 = -29 < 0.$$

Thus, $f$ has a root in $[0, 2]$ by Bolzano's theorem. Suppose it has two roots in $[0, 2]$. In view of Rolle's theorem, $f'$ must then have a root in $[0, 2]$ as well. On the other hand,

$$f'(x) = 3x^2 - 14x - 5$$

and the roots of this function are given by the quadratic formula

$$x = \frac{14 \pm \sqrt{14^2 + 4 \cdot 3 \cdot 5}}{2 \cdot 3} = \frac{14 \pm 16}{6} \qquad \Longrightarrow \qquad x = 5, \quad x = -1/3.$$

Since none of those lies in $[0, 2]$, we conclude that $f$ cannot have two roots in $[0, 2]$.

**3**. *Find the maximum value of $f(x) = \frac{x+1}{x^2+8}$ over the closed interval* $[0, 3]$.

- Since we are dealing with a closed interval, it suffices to check the endpoints, the points at which $f'$ does not exist and the points at which $f'$ is equal to zero. In this case,

$$f'(x) = \frac{x^2 + 8 - 2x \cdot (x+1)}{(x^2+8)^2} = -\frac{x^2 + 2x - 8}{(x^2+8)^2} = -\frac{(x+4)(x-2)}{(x^2+8)^2}$$

and so the only points at which the maximum value may occur are

$$x = -4, \qquad x = 2, \qquad x = 0, \qquad x = 3.$$

We exclude the leftmost point, which fails to lie in $[0, 3]$, and we now compute

$$f(2) = \frac{3}{12} = \frac{1}{4}, \qquad f(0) = \frac{1}{8}, \qquad f(3) = \frac{4}{17}.$$

Based on these observations, we deduce that the maximum value is $f(2) = 1/4$.

**4**. *Compute each of the following integrals:*

$$\int \frac{6x + 9}{x^3 + 3x^2}\, dx, \qquad \int 2x^3 e^{x^2}\, dx.$$

- To compute the first integral, we factor the denominator and we write

$$\frac{6x+9}{x^3+3x^2} = \frac{6x+9}{x^2(x+3)} = \frac{Ax+B}{x^2} + \frac{C}{x+3} \tag{$*$}$$

for some constants $A, B, C$ that need to be determined. Clearing denominators gives

$$6x+9 = (Ax+B)(x+3) + Cx^2$$

and we can now look at some suitable choices of $x$ to find

$$\begin{aligned} x &= -3 && \Longrightarrow & -9 &= 9C && \Longrightarrow & C &= -1, \\ x &= 0 && \Longrightarrow & 9 &= 3B && \Longrightarrow & B &= 3, \\ x &= -1 && \Longrightarrow & 3 &= -2A+2B+C && \Longrightarrow & A &= 1. \end{aligned}$$

Returning to equation $(*)$, we now get

$$\frac{6x+9}{x^3+3x^2} = \frac{x+3}{x^2} - \frac{1}{x+3} = \frac{1}{x} + \frac{3}{x^2} - \frac{1}{x+3}$$

and we may integrate this equation term by term to conclude that

$$\int \frac{6x+9}{x^3+3x^2}\,dx = \log|x| - 3x^{-1} - \log|x+3| + C.$$

- For the second integral, we use the substitution $u = x^2$. This gives $du = 2x\,dx$, hence

$$\int 2x^3 e^{x^2}\,dx = \int 2x \cdot x^2 e^{x^2}\,dx = \int ue^u\,du.$$

Focusing on the rightmost integral, we integrate by parts to find that

$$\int ue^u\,du = \int u\,(e^u)'\,du = ue^u - \int e^u\,du = ue^u - e^u + C.$$

Once we now combine the last two equations, we get

$$\int 2x^3 e^{x^2}\,dx = \int ue^u\,du = ue^u - e^u + C = x^2 e^{x^2} - e^{x^2} + C.$$

**5**. *Suppose $f$ is continuous on $[a,b]$. Show that there exists some $c \in (a,b)$ such that*

$$\int_a^b f(t)\,dt = (b-a)\cdot f(c).$$

*As a hint, apply the mean value theorem to the function $F(x) = \int_a^x f(t)\,dt$.*

- According to the mean value theorem, there exists some $c \in (a, b)$ such that
$$\frac{F(b) - F(a)}{b - a} = F'(c).$$
In addition, we have $F'(x) = f(x)$ for all $x$, and we also have
$$F(a) = \int_a^a f(t)\,dt = 0, \qquad F(b) = \int_a^b f(t)\,dt.$$
Once we now combine all these facts, we may conclude that
$$F(b) - F(a) = (b - a) \cdot F'(c) \quad \Longrightarrow \quad \int_a^b f(t)\,dt = (b - a) \cdot f(c).$$

**6**. *Test each of the following series for convergence:*
$$\sum_{n=1}^{\infty} \frac{n^2 + 2}{n^3 + n}, \qquad \sum_{n=1}^{\infty} \frac{n!}{n^n}.$$

- To test the first series for convergence, we use the limit comparison test with
$$a_n = \frac{n^2 + 2}{n^3 + n}, \qquad b_n = \frac{n^2}{n^3} = \frac{1}{n}.$$
Note that the limit comparison test is, in fact, applicable here because
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2 + 2}{n^3 + n} \cdot n = \lim_{n\to\infty} \frac{n^2 + 2}{n^2 + 1} = 1.$$
Since the series $\sum_{n=1}^{\infty} b_n$ is a divergent $p$-series, the series $\sum_{n=1}^{\infty} a_n$ must also diverge.
- To test the second series for convergence, we use the ratio test. In this case, we have
$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n$$
and this implies that
$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e}.$$
Since $e > 1$, this limit is strictly less than 1 and so the given series converges.

**7**. *Let $f$ be the function defined by*
$$f(x) = \left\{ \begin{array}{ll} 1 & \textit{if } x \in \mathbb{Q} \\ 0 & \textit{if } x \notin \mathbb{Q} \end{array} \right\}.$$
*Show that $f$ is <u>not</u> integrable on any closed interval $[a, b]$.*

- Given a partition $P = \{x_0, x_1, \ldots, x_n\}$ of the interval $[a, b]$, one easily finds that

$$\begin{aligned} S^-(f, P) &= \sum_{k=0}^{n-1} \inf_{[x_k, x_{k+1}]} f(x) \cdot (x_{k+1} - x_k) \\ &= 0(x_1 - x_0) + 0(x_2 - x_1) + 0(x_3 - x_2) + \ldots + 0(x_n - x_{n-1}) = 0, \end{aligned}$$

hence $\sup S^-(f, P) = 0$ as well. On the other hand, one also has

$$\begin{aligned} S^+(f, P) &= \sum_{k=0}^{n-1} \sup_{[x_k, x_{k+1}]} f(x) \cdot (x_{k+1} - x_k) \\ &= 1(x_1 - x_0) + 1(x_2 - x_1) + 1(x_3 - x_2) + \ldots + 1(x_n - x_{n-1}) \\ &= x_n - x_0 = b - a \end{aligned}$$

so that $\inf S^+(f, P) = b - a$ as well. This gives $\sup S^- \neq \inf S^+$ because $b - a \neq 0$.

8. *Suppose that $z = z(r, s, t)$, where $r = u - v$, $s = v - w$ and $t = w - u$. Assuming that all partial derivatives exist, show that $z_u + z_v + z_w = 0$.*

- Using the definitions of $r, s, t$ together with the chain rule, we get

$$\begin{aligned} z_u &= z_r r_u + z_s s_u + z_t t_u = z_r - z_t \\ z_v &= z_r r_v + z_s s_v + z_t t_v = -z_r + z_s \\ z_w &= z_r r_w + z_s s_w + z_t t_w = -z_s + z_t. \end{aligned}$$

Adding these three equations, one now finds that $z_u + z_v + z_w = 0$, indeed.

9. *Classify the critical points of the function defined by $f(x, y) = 3xy - x^3 - y^3$.*

- To find the critical points, we need to solve the equations

$$\begin{aligned} 0 &= f_x(x, y) = 3y - 3x^2 = 3(y - x^2), \\ 0 &= f_y(x, y) = 3x - 3y^2 = 3(x - y^2). \end{aligned}$$

These give $y = x^2$ and also $x = y^2$, so we easily get

$$x = y^2 = x^4 \quad \Longrightarrow \quad x^4 - x = 0 \quad \Longrightarrow \quad x(x^3 - 1) = 0 \quad \Longrightarrow \quad x = 0, 1.$$

In particular, the only critical points are $(0, 0)$ and $(1, 1)$.

- In order to classify the critical points, we compute the Hessian matrix

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} -6x & 3 \\ 3 & -6y \end{bmatrix}.$$

When it comes to the critical point $(0, 0)$, this gives

$$H = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} \quad \Longrightarrow \quad \det H = -9 < 0$$

so the origin is a saddle point. When it comes to the critical point $(1, 1)$, we have

$$H = \begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix} \quad \Longrightarrow \quad \det H = 36 - 9 > 0$$

and also $f_{xx} = -6 < 0$, so this critical point is a local maximum.

**10**. *Compute the double integral*

$$\int_0^1 \int_y^1 e^{x^2}\, dx\, dy.$$

- To compute the given integral, we switch the order of integration to get

$$\int_0^1 \int_y^1 e^{x^2}\, dx\, dy = \int_0^1 \int_0^x e^{x^2}\, dy\, dx = \int_0^1 x e^{x^2}\, dx.$$

Using the substitution $u = x^2$, we now get $du = 2x\, dx$, and this implies that

$$\int_0^1 \int_y^1 e^{x^2}\, dx\, dy = \frac{1}{2} \int_0^1 e^u\, du = \left[\frac{e^u}{2}\right]_0^1 = \frac{e-1}{2}.$$