MA121, 2007 Exam #2
Solutions

1. *Compute each of the following integrals:*

$$\int \frac{3x-1}{x^3-x}\,dx, \qquad \int x\log x\,dx.$$

- To compute the first integral, we factor the denominator and we write

$$\frac{3x-1}{x^3-x} = \frac{3x-1}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} \qquad (*)$$

for some constants A, B, C that need to be determined. Clearing denominators gives

$$3x-1 = A(x+1)(x-1) + Bx(x-1) + Cx(x+1)$$

and we can now look at some suitable choices of x to find

$$x=0, \quad x=-1, \quad x=1 \qquad \Longrightarrow \qquad -1=-A, \quad -4=2B, \quad 2=2C.$$

This means that $A = C = 1$ and $B = -2$. In particular, equation $(*)$ reduces to

$$\frac{3x-1}{x^3-x} = \frac{1}{x} - \frac{2}{x+1} + \frac{1}{x-1}$$

and we may integrate this equation term by term to conclude that

$$\int \frac{3x-1}{x^3-x}\,dx = \log|x| - 2\log|x+1| + \log|x-1| + C.$$

- To compute the second integral, we integrate by parts to find that

$$\begin{aligned}\int x\log x\,dx &= \int \left(\frac{x^2}{2}\right)' \log x\,dx = \frac{x^2\log x}{2} - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx \\ &= \frac{x^2\log x}{2} - \int \frac{x}{2}\,dx = \frac{x^2\log x}{2} - \frac{x^2}{4} + C.\end{aligned}$$

2. *Suppose f, g are integrable on $[a,b]$ with $f(x) \leq g(x)$ for all $x \in [a,b]$. Show that*

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

- Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. Starting with the inequality

$$f(x) \leq g(x) \quad \text{for all } x \in [x_k, x_{k+1}],$$

we take the infimum of both sides to get

$$\inf_{[x_k,x_{k+1}]} f(x) \leq \inf_{[x_k,x_{k+1}]} g(x).$$

Multiplying by the positive quantity $x_{k+1} - x_k$ and then adding, we conclude that

$$\sum_{k=0}^{n-1} \inf_{[x_k,x_{k+1}]} f(x) \cdot (x_{k+1} - x_k) \leq \sum_{k=0}^{n-1} \inf_{[x_k,x_{k+1}]} g(x) \cdot (x_{k+1} - x_k).$$

Since the last inequality holds for all partitions P by above, we must thus have

$$S^-(f,P) \leq S^-(g,P)$$

for all partitions P. Taking the supremum of both sides, we finally deduce that

$$\int_a^b f(x)\,dx = \sup_P\{S^-(f,P)\} \;\leq\; \sup_P\{S^-(g,P)\} = \int_a^b g(x)\,dx.$$

3. *Define a sequence* $\{a_n\}$ *by setting* $a_1 = 1$ *and*

$$a_{n+1} = \sqrt{3a_n - 1} \quad \text{for each } n \geq 1.$$

Show that $1 \leq a_n \leq a_{n+1} \leq 3$ *for each* $n \geq 1$, *use this fact to conclude that the sequence converges and then find its limit.*

- Since the first two terms are $a_1 = 1$ and $a_2 = \sqrt{2}$, the statement

$$1 \leq a_n \leq a_{n+1} \leq 3$$

does hold when $n = 1$. Suppose that it holds for some n, in which case

$$\begin{aligned} 3 - 1 \leq 3a_n - 1 \leq 3a_{n+1} - 1 \leq 9 - 1 \quad &\Longrightarrow \quad \sqrt{2} \leq a_{n+1} \leq a_{n+2} \leq \sqrt{8} \\ &\Longrightarrow \quad 1 \leq a_{n+1} \leq a_{n+2} \leq 3. \end{aligned}$$

In particular, the statement holds for $n+1$ as well, so it actually holds for all $n \in \mathbb{N}$. This shows that the given sequence is monotonic and bounded, hence also convergent; denote its limit by L. Using the definition of the sequence, we then find that

$$a_{n+1} = \sqrt{3a_n - 1} \quad \Longrightarrow \quad L = \sqrt{3L - 1} \quad \Longrightarrow \quad L^2 - 3L + 1 = 0.$$

Solving this quadratic equation now gives

$$L = \frac{3 \pm \sqrt{3^2 - 4}}{2} = \frac{3 \pm \sqrt{5}}{2}\,.$$

Since $1 \leq a_n \leq 3$ for each $n \in \mathbb{N}$, however, we must also have $1 \leq L \leq 3$, hence

$$L = \frac{3 + \sqrt{5}}{2}\,.$$

4. *Compute each of the following limits:*

$$\lim_{x\to 1} \frac{x^3 - 5x^2 + 7x - 3}{x^3 - 4x^2 + 5x - 2}, \qquad \lim_{x\to\infty} x\sin(1/x).$$

- Since the first limit is a 0/0 limit, we may apply L'Hôpital's rule to find that

$$L = \lim_{x\to 1} \frac{x^3 - 5x^2 + 7x - 3}{x^3 - 4x^2 + 5x - 2} = \lim_{x\to 1} \frac{3x^2 - 10x + 7}{3x^2 - 8x + 5}.$$

Since this is still a 0/0 limit, L'Hôpital's rule is still applicable and we get

$$L = \lim_{x\to 1} \frac{3x^2 - 10x + 7}{3x^2 - 8x + 5} = \lim_{x\to 1} \frac{6x - 10}{6x - 8} = \frac{-4}{-2} = 2.$$

- When it comes to the second limit, we can express it in the form

$$M = \lim_{x\to\infty} x\sin(1/x) = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x}.$$

This is now a 0/0 limit, so L'Hôpital's rule becomes applicable and we get

$$M = \lim_{x\to\infty} \frac{\cos(1/x)\cdot(1/x)'}{(1/x)'} = \lim_{x\to\infty} \cos(1/x) = \cos 0 = 1.$$

5. *Test each of the following series for convergence:*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, e^{1/n}}{n}, \qquad \sum_{n=1}^{\infty} \log\left(1 + \frac{1}{n}\right).$$

- To test the first series for convergence, we use the alternating series test with

$$a_n = \frac{e^{1/n}}{n}.$$

Note that a_n is certainly non-negative for each $n \geq 1$, and that we also have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^{1/n}}{n} = \lim_{n\to\infty} \frac{e^0}{n} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Moreover, a_n is decreasing for each $n \geq 1$ because

$$\left(\frac{e^{1/n}}{n}\right)' = \frac{e^{1/n}\cdot(-n^{-2})\cdot n - e^{1/n}}{n^2} = -\frac{e^{1/n}}{n^2}\cdot(n^{-1} + 1) < 0$$

for each $n \geq 1$. Thus, the given series converges by the alternating series test.

- To test the second series for convergence, we use the limit comparison test with

$$a_n = \log\left(1 + \frac{1}{n}\right), \qquad b_n = \frac{1}{n}.$$

Note that the limit comparison test is, in fact, applicable here because

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} n \log\left(1 + \frac{1}{n}\right) = \lim_{n\to\infty} \log\left(1 + \frac{1}{n}\right)^n = \log e = 1.$$

Since the series $\sum_{n=1}^\infty b_n$ is a divergent p-series, the series $\sum_{n=1}^\infty a_n$ must also diverge.

6. *Find the radius of convergence of the power series*

$$f(x) = \sum_{n=0}^\infty \frac{(n!)^2}{(2n)!} \cdot x^n.$$

- To find the radius of convergence, one always uses the ratio test. In our case,

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} \cdot \frac{(n+1)!}{n!} \cdot \frac{(2n)!}{(2n+2)!} \cdot \frac{x^{n+1}}{x^n} = \frac{(n+1)^2 \cdot x}{(2n+1)(2n+2)}$$

and this implies that

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n^2+2n+1)\,|x|}{4n^2+6n+2} = \frac{|x|}{4}.$$

Thus, the power series converges when $|x|/4 < 1$ and diverges when $|x|/4 > 1$. In other words, it converges when $|x| < 4$ and diverges when $|x| > 4$. This also means that $R = 4$.

7. *Suppose f is a differentiable function such that $f'(x) = f(x) + e^x$ for all $x \in \mathbb{R}$. Show that there exists some constant C such that $f(x) = xe^x + Ce^x$ for all $x \in \mathbb{R}$.*

- Letting $g(x) = f(x)e^{-x} - x$ for convenience, one easily finds that

$$\begin{aligned} g'(x) &= f'(x)e^{-x} - f(x)e^{-x} - 1 \\ &= e^{-x}\left[f'(x) - f(x)\right] - 1 \\ &= e^{-x}e^x - 1 = 0. \end{aligned}$$

In particular, $g(x)$ is actually constant, say $g(x) = C$ for all $x \in \mathbb{R}$, and this implies

$$g(x) = C \quad\Longrightarrow\quad f(x)e^{-x} = x + C \quad\Longrightarrow\quad f(x) = xe^x + Ce^x.$$

8. *Use the formula for a geometric series to show that*

$$\sum_{n=0}^\infty n^2 x^n = \frac{x(1+x)}{(1-x)^3} \quad \textit{whenever } |x| < 1.$$

- Since $|x| < 1$ by assumption, the formula for a geometric series is applicable and so

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = (1-x)^{-1}.$$

We differentiate both sides of this equation and we multiply by x to get

$$\sum_{n=0}^{\infty} nx^{n-1} = (1-x)^{-2} \quad \Longrightarrow \quad \sum_{n=0}^{\infty} nx^n = x(1-x)^{-2} = \frac{x}{(1-x)^2}\,.$$

Using the quotient rule to differentiate once again, we arrive at

$$\sum_{n=0}^{\infty} n^2 x^{n-1} = \frac{(1-x)^2 + 2(1-x)\cdot x}{(1-x)^4} = \frac{(1-x)(1-x+2x)}{(1-x)^4}\,.$$

Multiplying by x and simplifying, we may finally conclude that

$$\sum_{n=0}^{\infty} n^2 x^n = \frac{x(1-x)(1-x+2x)}{(1-x)^4} = \frac{x(1+x)}{(1-x)^3}\,.$$