MA121, 2006 Final exam
Solutions

1. *Suppose that A is a nonempty subset of $\mathbb{R}$ that has an upper bound, and let B be the set of all upper bounds of A. Show that* $\inf B = \sup A$.

- Since $\sup A$ is the least upper bound of A, it is also the least element of B, namely

$$\sup A = \min B.$$

Since B has a minimum, however, it also has an infimum and the two are equal, so

$$\inf B = \min B = \sup A.$$

2. *Let $a \in \mathbb{R}$ be a given number and let f be the function defined by*

$$f(x) = \left\{ \begin{array}{cl} ax^2 + 2x & \text{if } x \neq 2 \\ 2a + 8 & \text{if } x = 2 \end{array} \right\}.$$

Find the value of a for which f is continuous at $y = 2$.

- To say that f is continuous at $y = 2$ is to say that

$$\lim_{x \to 2} f(x) = f(2).$$

In our case, the left hand side is equal to

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} (ax^2 + 2x) = a \cdot 2^2 + 2 \cdot 2 = 4a + 4,$$

while the right hand side is equal to $f(2) = 2a + 8$. In particular, we have

$$\lim_{x \to 2} f(x) = f(2) \quad \iff \quad 4a + 4 = 2a + 8 \quad \iff \quad a = 2.$$

3. *Find the minimum value of $f(x) = x^4 + 4x^3 - 8x^2 + 2$ over the whole real line.*

- The derivative of the given function is

$$f'(x) = 4x^3 + 12x^2 - 16x = 4x(x^2 + 3x - 4) = 4x(x-1)(x+4)$$

and we can determine the sign of f' using the table below.

x		-4		0		1	
$4x$	$-$		$-$		$+$		$+$
$x-1$	$-$		$-$		$-$		$+$
$x+4$	$-$		$+$		$+$		$+$
$f'(x)$	$-$		$+$		$-$		$+$
$f(x)$	$\searrow$		$\nearrow$		$\searrow$		$\nearrow$

According to the table, the minimum value of f can now be found by comparing

$$f(-4) = 4^4 - 4 \cdot 4^3 - 8 \cdot 4^2 + 2 = -126, \qquad f(1) = 1 + 4 - 8 + 2 = -1.$$

Since the former is smaller and also attained, this means that $\min f(x) = -126$.

4. *Let $a, b, c \in \mathbb{R}$ be some fixed constants such that $\frac{a}{3} + \frac{b}{2} + c = 0$. Show that*

$$ax^2 + bx + c = 0 \quad \textit{for some } x \in (0, 1).$$

As a hint, apply the mean value theorem to a function whose derivative is $ax^2 + bx + c$.

- Following the hint, let us consider the function

$$f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx.$$

Then f is differentiable on $[0, 1]$ with $f'(x) = ax^2 + bx + c$ for all x, and we also have

$$f(0) = 0, \qquad f(1) = \frac{a}{3} + \frac{b}{2} + c = 0.$$

Using the mean value theorem, we conclude that some $x \in (0, 1)$ exists such that

$$f'(x) = \frac{f(1) - f(0)}{1 - 0} = \frac{0 - 0}{1 - 0} = 0 \qquad \Longrightarrow \qquad ax^2 + bx + c = 0.$$

5. *Suppose that f is a function which satisfies the inequality*

$$|f(x) - f(y)| \leq |x - y|^2 \quad \textit{for all } x, y \in \mathbb{R}.$$

Show that f is actually constant.

- We need only show that $f'(y) = 0$ for all $y \in \mathbb{R}$. Using the given inequality, we get

$$0 \leq \frac{|f(x) - f(y)|}{|x - y|} \leq |x - y| \quad \text{whenever } x \neq y.$$

Since $|x - y|$ approaches zero as $x \to y$, the quotient above is thus squeezed between two functions which approach zero as $x \to y$. In view of the Squeeze Law, the quotient itself must approach zero as $x \to y$. This also implies that $f'(y) = 0$, as needed.

6. *Evaluate each of the following integrals:*

$$\int \frac{4x^2 - 15x + 12}{x^3 - 5x^2 + 6x}\, dx, \qquad \int \frac{x^3 - x + 1}{x + 1}\, dx.$$

As a hint for the first integral, you might want to factor the denominator.

- To evaluate the first integral, we use partial fractions to write

$$\frac{4x^2-15x+12}{x^3-5x^2+6x} = \frac{4x^2-15x+12}{x(x-2)(x-3)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-3}$$

for some constants A, B, C that need to be determined. Clearing denominators, we get

$$4x^2 - 15x + 12 = A(x-2)(x-3) + Bx(x-3) + Cx(x-2)$$

and we can now look at some suitable choices of x to find that

$$x = 0, 2, 3 \quad \Longrightarrow \quad 12 = 6A, \quad -2 = -2B, \quad 3 = 3C.$$

This gives $A = 2$ and $B = C = 1$, so the partial fractions decomposition reads

$$\frac{4x^2-15x+12}{x^3-5x^2+6x} = \frac{2}{x} + \frac{1}{x-2} + \frac{1}{x-3}\,.$$

Once we now integrate this equation term by term, we get

$$\int \frac{4x^2-15x+12}{x^3-5x^2+6x}\,dx = 2\log|x| + \log|x-2| + \log|x-3| + C.$$

- For the second integral, we use division of polynomials to write

$$\frac{x^3-x+1}{x+1} = x^2 - x + \frac{1}{x+1}\,.$$

Integrating this equation term by term, we then easily find that

$$\int \frac{x^3-x+1}{x+1}\,dx = \frac{x^3}{3} - \frac{x^2}{2} + \log|x+1| + C.$$

7. *Test each of the following series for convergence:*

$$\sum_{n=0}^{\infty} \frac{n!}{(2n)!}\,, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\,, \qquad \sum_{n=1}^{\infty} \frac{n^2+2}{n^3+n}\,.$$

- To test the first series for convergence, we use the ratio test. Since the limit

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{(2n)!}{(2n+2)!} = \lim_{n\to\infty} \frac{n+1}{(2n+1)(2n+2)} = 0$$

is strictly less than 1, the first series converges by the ratio test.

- For the second series, we use the alternating series test with
$$a_n = \frac{1}{n} = n^{-1}.$$
Note that a_n is non-negative for each $n \geq 1$ and that a_n is decreasing because
$$a_n' = -n^{-2} < 0.$$
Since $a_n = 1/n$ approaches zero as $n \to \infty$, we see that the second series converges.

- For the last series, we use the limit comparison test with
$$a_n = \frac{n^2+2}{n^3+n}, \qquad b_n = \frac{n^2}{n^3} = \frac{1}{n}.$$
Note that the limit comparison test is, in fact, applicable here because
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2+2}{n^3+n} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^2+2}{n^2+1} = 1.$$
Since the series $\sum_{n=1}^{\infty} b_n$ is a divergent p-series, the series $\sum_{n=1}^{\infty} a_n$ diverges as well.

8. *Evaluate each of the following sums:*
$$\sum_{n=0}^{\infty} \frac{2^{n+1}}{3^{n+2}}, \qquad \sum_{n=2}^{\infty} \frac{e^n}{n!}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n\, 9^{n+1}}{(2n)!}.$$

- The first sum is related to a geometric series, namely
$$\sum_{n=0}^{\infty} \frac{2^{n+1}}{3^{n+2}} = \frac{2}{9} \cdot \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{2}{9} \cdot \frac{1}{1-2/3} = \frac{6}{9} = \frac{2}{3}.$$

- Relating the second sum to the Taylor series for the exponential function, we get
$$\sum_{n=2}^{\infty} \frac{e^n}{n!} = \sum_{n=0}^{\infty} \frac{e^n}{n!} - 1 - e = e^e - 1 - e.$$

- Finally, the third sum is related to the Taylor series for the cosine function, namely
$$\sum_{n=1}^{\infty} \frac{(-1)^n\, 9^{n+1}}{(2n)!} = 9 \cdot \sum_{n=1}^{\infty} \frac{(-1)^n\, 3^{2n}}{(2n)!} = 9(\cos 3 - 1).$$

9. *Let f be the function defined by*
$$f(x) = \left\{ \begin{array}{ll} 1 & \textit{if } x \neq 0 \\ 0 & \textit{if } x = 0 \end{array} \right\}.$$
Show that f is integrable on $[0,1]$.

- Since $f(x) = 1$ at all points except for $x = 0$, it should be clear that

$$\begin{aligned} S^+(f, P) &= \sum_{k=0}^{n-1} \sup_{[x_k, x_{k+1}]} f(x) \cdot (x_{k+1} - x_k) \\ &= (x_1 - x_0) + (x_2 - x_1) + \ldots + (x_n - x_{n-1}) = x_n - x_0 = 1 \end{aligned}$$

for all partitions $P = \{x_0, x_1, \ldots, x_n\}$ of the closed interval $[0, 1]$.

- Since $[x_0, x_1]$ is the only subinterval that contains the point $x = 0$, we also have

$$\begin{aligned} S^-(f, P) &= \sum_{k=0}^{n-1} \inf_{[x_k, x_{k+1}]} f(x) \cdot (x_{k+1} - x_k) \\ &= (x_2 - x_1) + (x_3 - x_2) + \ldots + (x_n - x_{n-1}) = x_n - x_1 = 1 - x_1. \end{aligned}$$

Taking the supremum over all possible partitions, we may thus conclude that

$$\sup_P \{S^-(f, P)\} = \sup_{0<x_1<1} (1 - x_1) = 1 = \inf_P \{S^+(f, P)\}.$$

10. *Define a sequence* $\{a_n\}$ *by setting* $a_1 = 2$ *and*

$$a_{n+1} = \frac{1}{3 - a_n} \quad \textit{for each } n \geq 1.$$

Show that $0 < a_{n+1} \leq a_n \leq 2$ *for each* $n \geq 1$. *Use this fact to conclude that the sequence converges and then find its limit.*

- Since the first two terms are $a_1 = 2$ and $a_2 = 1$, the statement

$$0 < a_{n+1} \leq a_n \leq 2$$

does hold when $n = 1$. Suppose that it holds for some n, in which case

$$\begin{aligned} 0 > -a_{n+1} \geq -a_n \geq -2 \quad &\Longrightarrow \quad 3 > 3 - a_{n+1} \geq 3 - a_n \geq 1 \\ &\Longrightarrow \quad 1/3 < a_{n+2} \leq a_{n+1} \leq 1. \end{aligned}$$

Thus, the statement holds for $n + 1$ as well, so it must actually hold for all $n \in \mathbb{N}$. This shows that the given sequence is monotonic and bounded, hence also convergent; denote its limit by L. Using the definition of the sequence, we then find that

$$a_{n+1} = \frac{1}{3 - a_n} \quad \Longrightarrow \quad L = \frac{1}{3 - L} \quad \Longrightarrow \quad L^2 - 3L + 1 = 0.$$

Solving this quadratic equation now gives

$$L = \frac{3 \pm \sqrt{9 - 4}}{2} = \frac{3 \pm \sqrt{5}}{2}.$$

Since $0 < a_n \leq 2$ for each $n \in \mathbb{N}$, however, we must also have $0 \leq L \leq 2$, so

$$L = \frac{3 - \sqrt{5}}{2}.$$