MA121, 2006 Exam #2
Solutions

1. *Find the minimum value of* $f(x) = x^4 + 4x^3 - 8x^2 + 2$ *over the closed interval* $[0, 2]$.

Since we are dealing with a closed interval, it suffices to check the endpoints, the points at which f' does not exist and the points at which f' is equal to zero. In our case,

$$f'(x) = 4x^3 + 12x^2 - 16x = 4x(x^2 + 3x - 4) = 4x(x-1)(x+4)$$

for all $x \in \mathbb{R}$, so the only points at which the minimum value may occur are

$$x = 0, \qquad x = 2, \qquad x = 1, \qquad x = -4.$$

We exclude the rightmost point, which fails to lie in $[0, 2]$, and we now compute

$$f(0) = 2, \qquad f(2) = 16 + 32 - 32 + 2 = 18, \qquad f(1) = 1 + 4 - 8 + 2 = -1.$$

Based on these observations, we deduce that the minimum value is $f(1) = -1$.

2. *Determine the minimum and maximum values attained by* $f(x) = \frac{2x+1}{x^2+2}$.

According to the quotient rule, the derivative of the given function is

$$\begin{aligned} f'(x) &= \frac{2(x^2+2) - 2x \cdot (2x+1)}{(x^2+2)^2} = \frac{2x^2 + 4 - 4x^2 - 2x}{(x^2+2)^2} \\ &= \frac{-2(x^2 + x - 2)}{(x^2+2)^2} = \frac{-2(x+2)(x-1)}{(x^2+2)^2}. \end{aligned}$$

Using this fact, one can easily determine the sign of f' by means of a table.

x		-2		1	
$-2(x+2)$	$+$		$-$		$-$
$x-1$	$-$		$-$		$+$
$f'(x)$	$-$		$+$		$-$
$f(x)$	$\searrow$		$\nearrow$		$\searrow$

To find the minimum value attained, we need to compare

$$f(-2) = \frac{-4+1}{4+2} = -\frac{1}{2}, \qquad \lim_{x\to+\infty} \frac{2x+1}{x^2+2} = \lim_{x\to+\infty} \frac{2/x + 1/x^2}{1 + 2/x^2} = \frac{0+0}{1+0} = 0.$$

Since the former is smaller and also attained, we get $\min f(x) = f(-2) = -1/2$. To find the maximum value attained, we need to compare

$$f(1) = \frac{2+1}{1+2} = 1, \qquad \lim_{x\to-\infty} \frac{2x+1}{x^2+2} = \lim_{x\to-\infty} \frac{2/x + 1/x^2}{1 + 2/x^2} = \frac{0+0}{1+0} = 0.$$

Since the former is larger and also attained, we get $\max f(x) = f(1) = 1$.

3. *Compute the following limit:*

$$\lim_{x\to 1} \frac{x^3+4x^2+x-6}{x^3-x^2-4x+4}.$$

- Setting $x=1$ gives rise to a $0/0$ limit, so we can apply L'Hôpital's rule to get

$$\lim_{x\to 1} \frac{x^3+4x^2+x-6}{x^3-x^2-4x+4} = \lim_{x\to 1} \frac{3x^2+8x+1}{3x^2-2x-4} = \frac{3+8+1}{3-2-4} = \frac{12}{-3} = -4.$$

4. *Evaluate each of the following integrals:*

$$\int \frac{4x^2+x+2}{x^3+x}\,dx, \qquad \int x\cdot e^x\,dx.$$

- To compute the first integral, we factor the denominator and we write

$$\frac{4x^2+x+2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} \tag{$*$}$$

for some constants A, B, C that need to be determined. Clearing denominators gives

$$4x^2+x+2 = A(x^2+1) + Bx^2 + Cx$$

and we can now look at some suitable choices of x to find

$$\begin{aligned} x=0 \quad &\Longrightarrow \quad 2 = A \\ x=1 \quad &\Longrightarrow \quad 7 = 2A+B+C = 4+B+C \\ x=-1 \quad &\Longrightarrow \quad 5 = 2A+B-C = 4+B-C. \end{aligned}$$

Adding the last two equations, we get $12 = 8+2B$, and this implies

$$2B = 12-8 = 4 \quad \Longrightarrow \quad B = 2 \quad \Longrightarrow \quad C = 7-4-B = 1.$$

Once we now return to equation $(*)$, we may conclude that

$$\frac{4x^2+x+2}{x(x^2+1)} = \frac{2}{x} + \frac{2x}{x^2+1} + \frac{1}{x^2+1}.$$

Integrating term by term, we may thus conclude that

$$\int \frac{4x^2+x+2}{x(x^2+1)}\,dx = 2\log|x| + \log(x^2+1) + \arctan x + C.$$

- To compute the second integral, we integrate by parts to find that

$$\int xe^x\,dx = \int x\,(e^x)'\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C.$$

5. *Suppose that f is a differentiable function such that*

$$f'(x) = \cos x \cdot f(x) \quad \textit{for all } x \in \mathbb{R}.$$

Show that there exists some constant C such that $f(x) = Ce^{\sin x}$ for all $x \in \mathbb{R}$.

- Setting $g(x) = f(x) \cdot e^{-\sin x}$ for convenience, we use the product rule to get

$$\begin{aligned} g'(x) &= f'(x) \cdot e^{-\sin x} + f(x) \cdot e^{-\sin x} \cdot (-\sin x)' \\ &= \cos x \cdot f(x) \cdot e^{-\sin x} - f(x) \cdot e^{-\sin x} \cdot \cos x = 0. \end{aligned}$$

This shows that $g(x)$ is actually constant, say $g(x) = C$, and it also implies that

$$g(x) = C \quad \implies \quad f(x) \cdot e^{-\sin x} = C \quad \implies \quad f(x) = Ce^{\sin x}.$$

6. *Assuming that f is continuous on $[a, b]$ with $\int_a^b f(t)\, dt = 0$, show that*

$$f(c) = 0 \quad \textit{for some } c \in (a, b).$$

As a hint, apply the mean value theorem to the function $F(x) = \int_a^x f(t)\, dt$.

- According to the mean value theorem, there exists some $c \in (a, b)$ such that

$$F'(c) = \frac{F(b) - F(a)}{b - a}.$$

In addition, we have $F'(x) = f(x)$ for all x, and we also have

$$F(a) = \int_a^a f(t)\, dt = 0, \qquad F(b) = \int_a^b f(t)\, dt = 0.$$

Once we now combine all these facts, we may conclude that

$$f(c) = F'(c) = \frac{F(b) - F(a)}{b - a} = 0.$$

7. *Let f be a non-negative function which is integrable on $[0, 1]$ with $f(x) = 0$ for all $x \in \mathbb{Q}$. Show that $\int_0^1 f(x)\, dx = 0$.*

- Suppose that $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[0, 1]$. Then we must clearly have

$$\inf_{[x_k, x_{k+1}]} f(x) = 0 \quad \text{for each } 0 \leq k \leq n - 1$$

because f is non-negative and since every subinterval contains a rational. Thus,

$$S^-(f, P) = \sum_{k=0}^{n-1} \inf_{[x_k, x_{k+1}]} f(x) \cdot (x_{k+1} - x_k) = 0$$

as well. Taking the supremum of both sides, we conclude that

$$\int_0^1 f(x)\, dx = \sup_P \{S^-(f, P)\} = \sup_P \{0\} = 0.$$

8. *Let f be the function defined by*

$$f(x) = \left\{ \begin{array}{cc} x^3 \cdot \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{array} \right\}.$$

Using the limit definition of a derivative, show that $f'(0) = 0$. You may use the fact that

$$-1 \leq \sin(1/x) \leq 1 \quad \text{for all } x \neq 0.$$

- According to the limit definition of the derivative, we have

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}.$$

Since $x \neq 0$ whenever $x \to 0$, this actually gives

$$f'(0) = \lim_{x \to 0} \frac{x^3 \sin(1/x)}{x} = \lim_{x \to 0} x^2 \sin(1/x).$$

To compute the rightmost limit, we shall now use the fact that

$$-1 \leq \sin(1/x) \leq 1 \quad \text{for all } x \neq 0.$$

Multiplying by the positive quantity x^2 preserves the inequality, so we get

$$-x^2 \leq x^2 \sin(1/x) \leq x^2 \quad \text{for all } x \neq 0.$$

Once we now note that

$$\lim_{x \to 0} (-x^2) = 0 \quad \text{and} \quad \lim_{x \to 0} x^2 = 0,$$

we may apply the Squeeze Law to finally conclude that

$$f'(0) = \lim_{x \to 0} x^2 \sin(1/x) = 0.$$