Chapter 4. Linear transformations

Lecture notes for MA1111

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Linear transformations

Definition 4.1 – Linear transformation

A linear transformation is a map $T\colon V \to W$ between vector spaces which preserves vector addition and scalar multiplication. It satisfies

1. $T(\boldsymbol{v}_1 + \boldsymbol{v}_2) = T(\boldsymbol{v}_1) + T(\boldsymbol{v}_2)$ for all $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$ and
2. $T(c\boldsymbol{v}) = cT(\boldsymbol{v})$ for all $\boldsymbol{v} \in V$ and all $c \in \mathbb{R}$.

- By definition, every linear transformation T is such that $T(0) = 0$.
- Two examples of linear transformations $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ are rotations around the origin and reflections along a line through the origin.
- An example of a linear transformation $T\colon P_n \to P_{n-1}$ is the derivative function that maps each polynomial $p(x)$ to its derivative $p'(x)$.
- As we are going to show, every linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is given by left multiplication with some $m \times n$ matrix.

Kernel and image

Definition 4.2 – Kernel and image

Suppose $T\colon V \to W$ is a linear transformation. Its kernel is the set of all elements $\boldsymbol{v} \in V$ such that $T(\boldsymbol{v}) = 0$ and its image is the set of all elements $\boldsymbol{w} \in W$ that have the form $\boldsymbol{w} = T(\boldsymbol{v})$ for some $\boldsymbol{v} \in V$.

- Some textbooks refer to the image of T as the range of T.
- When $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is left multiplication by the matrix A, the kernel is the null space of A and the image is the column space of A.

Theorem 4.3 – Dimension formula

Suppose $T\colon V \to W$ is a linear transformation. Then the kernel of T is a subspace of V, the image of T is a subspace of W and

$$\dim(\ker T) + \dim(\operatorname{im} T) = \dim V.$$

Injective and surjective

Definition 4.4 – Injective linear transformations

A linear transformation $T\colon V \to W$ is injective when $T(\boldsymbol{x}) = T(\boldsymbol{y})$ if and only if $\boldsymbol{x} = \boldsymbol{y}$. This is the case if and only if $\ker T = \{0\}$.

- Suppose $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is left multiplication by a matrix A. Then T is injective if and only if the columns of A are linearly independent.

Definition 4.5 – Surjective linear transformations

A linear transformation $T\colon V \to W$ is surjective when $\operatorname{im} T = W$.

- Suppose $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is left multiplication by a matrix A. Then T is surjective if and only if the columns of A form a complete set of $\mathbb{R}^m$.

Linear transformations and bases

Theorem 4.6 – Linear transformations and bases

Every linear transformation $T\colon V \to W$ is uniquely determined by the values $T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), \ldots, T(\boldsymbol{v}_n)$ when $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ form a basis of V.

- Indeed, each $\boldsymbol{v} \in V$ can be expressed as a linear combination

$$\boldsymbol{v} = x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \ldots + x_n\boldsymbol{v}_n$$

  for some unique coefficients $x_1, x_2, \ldots, x_n$ and this implies that

$$T(\boldsymbol{v}) = x_1 T(\boldsymbol{v}_1) + x_2 T(\boldsymbol{v}_2) + \ldots + x_n T(\boldsymbol{v}_n).$$

- Using the computation above, one may also conclude that

$$\operatorname{im} T = \operatorname{Span}\{T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), \ldots, T(\boldsymbol{v}_n)\}.$$

  This provides a simple formula for computing the image of T.

Linear transformations and bases: Example 1

- Suppose that we are given the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{w} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

and that we need to find $T(\boldsymbol{w})$ when $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is linear with

$$T(\boldsymbol{v}_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad T(\boldsymbol{v}_2) = \begin{bmatrix} 4 \\ 7 \end{bmatrix}.$$

- To express $\boldsymbol{w}$ in terms of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, we use the row reduction

$$\begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{w} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 1 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} \mathbf{1} & 0 & 9 \\ 0 & \mathbf{1} & -2 \end{bmatrix}.$$

This gives $\boldsymbol{w} = 9\boldsymbol{v}_1 - 2\boldsymbol{v}_2$, so we must also have

$$T(\boldsymbol{w}) = 9T(\boldsymbol{v}_1) - 2T(\boldsymbol{v}_2) = \begin{bmatrix} 9 - 8 \\ 18 - 14 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

Linear transformations and bases: Example 2

- Assume, as in the previous example, that $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is linear with

$$T(\boldsymbol{v}_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad T(\boldsymbol{v}_2) = \begin{bmatrix} 4 \\ 7 \end{bmatrix}, \qquad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

- We now wish to determine $T(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^2$. Proceeding as before, we first express $\boldsymbol{x}$ as a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. This gives

$$\begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} 1 & 2 & x_1 \\ 1 & 1 & x_2 \end{bmatrix} \longrightarrow \begin{bmatrix} \mathbf{1} & 0 & 2x_2 - x_1 \\ 0 & \mathbf{1} & x_1 - x_2 \end{bmatrix}.$$

- In particular, $\boldsymbol{x} = (2x_2 - x_1)\boldsymbol{v}_1 + (x_1 - x_2)\boldsymbol{v}_2$, so we must also have

$$T(\boldsymbol{x}) = (2x_2 - x_1)T(\boldsymbol{v}_1) + (x_1 - x_2)T(\boldsymbol{v}_2) = \begin{bmatrix} 3x_1 - 2x_2 \\ 5x_1 - 3x_2 \end{bmatrix}.$$

Linear transformations $T\colon \mathbb{R}^n \to \mathbb{R}^m$

Theorem 4.7 – Linear transformations $T\colon \mathbb{R}^n \to \mathbb{R}^m$

Every linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is given by left multiplication with some $m \times n$ matrix A. To find this matrix explicitly, one uses the fact that its ith column is equal to $A\boldsymbol{e}_i = T(\boldsymbol{e}_i)$.

- In other words, T is given by left multiplication with the matrix

$$A = \begin{bmatrix} | & | & & | \\ T(\boldsymbol{e}_1) & T(\boldsymbol{e}_2) & \cdots & T(\boldsymbol{e}_n) \\ | & | & & | \end{bmatrix}.$$

- This theorem is easy to check when $T(\boldsymbol{x})$ is known explicitly, say

$$T(\boldsymbol{x}) = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} \quad \Longrightarrow \quad T(\boldsymbol{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\boldsymbol{x}.$$

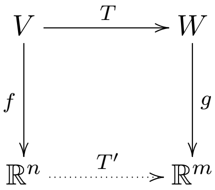
- We are mostly interested in the case that $T(\boldsymbol{x})$ is described implicitly as a rotation around the origin, a reflection along a line, and so on.

Examples of linear transformations $T\colon \mathbb{R}^2 \to \mathbb{R}^2$

- Reflection along the x-axis .... $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- Reflection along the y-axis .... $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
- Reflection along the line $y = x$ .... $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- Rotation by θ around the origin .... $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
- Scaling of the x-axis .... $A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
- Scaling of the y-axis .... $A = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

Linear transformations and coordinate vectors

- Suppose $T\colon V \to W$ is a linear transformation between vector spaces. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ be a basis of V and $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_m$ a basis of W.
- Associating each element $\boldsymbol{v} = \sum x_i \boldsymbol{v}_i$ with its coordinate vector $\boldsymbol{x}$, one may relate V with $\mathbb{R}^n$ and one may similarly relate W with $\mathbb{R}^m$.



- In particular, one may associate the linear transformation $T\colon V \to W$ with a linear transformation $T'\colon \mathbb{R}^n \to \mathbb{R}^m$. As we already know, the latter is given by left multiplication with some $m \times n$ matrix A.
- The matrix A obviously depends on the chosen bases. It is called the matrix of T with respect to the given bases.

Matrix of a linear transformation

Definition 4.8 – Matrix of a linear transformation

Suppose $T\colon V \to W$ is a linear transformation between vector spaces. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ be a basis of V and $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_m$ a basis of W. The matrix of T with respect to these bases is defined as the matrix whose ith column is equal to the coordinate vector of $T(\boldsymbol{v}_i)$.

- Finding the matrix of T is a routine computation. To obtain the first column, for instance, one needs to compute $T(\boldsymbol{v}_1)$, express it in terms of the vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_m$ and keep track of the coefficients. The other columns are obtained similarly by dealing with each $T(\boldsymbol{v}_i)$.
- Suppose $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is left multiplication with the matrix A. Then the matrix of T with respect to the standard bases is equal to A, but the matrix of T with respect to other bases might be different.
- A linear transformation $T\colon V \to V$ is also known as a linear operator. To study a linear operator, one usually introduces a single basis for V.

Matrix of a linear transformation: Example 1

- Consider the derivative map $T\colon P_2 \to P_1$ which is defined by

$$T(f(x)) = f'(x).$$

  We already know from analysis that T is a linear transformation.

- Let us use the basis $1, x, x^2$ for P_2 and the basis $1, x$ for P_1. To find the columns of the matrix of T, we compute $T(1), T(x), T(x^2)$ and then express each of those in terms of $1, x$. This gives

$$\begin{aligned} T(1) &= 0 = \underline{0} \cdot 1 + \underline{0} \cdot x, \\ T(x) &= 1 = \underline{1} \cdot 1 + \underline{0} \cdot x, \\ T(x^2) &= 2x = \underline{0} \cdot 1 + \underline{2} \cdot x. \end{aligned}$$

- The matrix of T with respect to the given bases is then

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Matrix of a linear transformation: Example 2

- Consider the function $T\colon P_2 \to P_2$ which is defined by

$$T(f(x)) = f(x+1).$$

  This function is easily seen to be a linear transformation.

- Using the standard basis $1, x, x^2$ for P_2, we find that

$$\begin{aligned} T(1) &= 1 = \underline{1} \cdot 1 + \underline{0} \cdot x + \underline{0} \cdot x^2, \\ T(x) &= x + 1 = \underline{1} \cdot 1 + \underline{1} \cdot x + \underline{0} \cdot x^2, \\ T(x^2) &= (x+1)^2 = \underline{1} \cdot 1 + \underline{2} \cdot x + \underline{1} \cdot x^2. \end{aligned}$$

- The matrix of T with respect to the standard basis is then

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Matrix of a linear transformation: Example 3

- Consider the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 17 & -20 \\ 12 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 17x_1 - 20x_2 \\ 12x_1 - 14x_2 \end{bmatrix}.$$

- Using the standard basis $\boldsymbol{e}_1, \boldsymbol{e}_2$ of $\mathbb{R}^2$, we find that

$$T(\boldsymbol{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 17 \\ 12 \end{bmatrix} = \underline{17} \cdot \boldsymbol{e}_1 + \underline{12} \cdot \boldsymbol{e}_2,$$

$$T(\boldsymbol{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -20 \\ -14 \end{bmatrix} = -\underline{20} \cdot \boldsymbol{e}_1 - \underline{14} \cdot \boldsymbol{e}_2.$$

- In particular, the matrix of T with respect to the standard basis is

$$A = \begin{bmatrix} 17 & -20 \\ 12 & -14 \end{bmatrix}.$$

Note that T itself is given by left multiplication with this matrix.

Matrix of a linear transformation: Example 4

- Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be as in the previous example, namely

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 17 & -20 \\ 12 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 17x_1 - 20x_2 \\ 12x_1 - 14x_2 \end{bmatrix}.$$

- We now compute the matrix of T with respect to the basis

$$\boldsymbol{v}_1 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

- Expressing $T(\boldsymbol{v}_1)$ and $T(\boldsymbol{v}_2)$ in terms of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, we get

$$T(\boldsymbol{v}_1) = \begin{bmatrix} 17 \cdot 5 - 20 \cdot 4 \\ 12 \cdot 5 - 14 \cdot 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} = 1\boldsymbol{v}_1 + 0\boldsymbol{v}_2,$$

$$T(\boldsymbol{v}_2) = \begin{bmatrix} 17 \cdot 4 - 20 \cdot 3 \\ 12 \cdot 4 - 14 \cdot 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix} = 0\boldsymbol{v}_1 + 2\boldsymbol{v}_2.$$

- Thus, the matrix of T with respect to this basis is $A' = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

Matrix of a linear transformation: Example 5

- Define the map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ and the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ by letting

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}, \qquad \boldsymbol{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

  Then T is a linear transformation and $\boldsymbol{v}_1, \boldsymbol{v}_2$ form a basis of $\mathbb{R}^2$.

- To find the matrix of T with respect to this basis, we need to express

$$T(\boldsymbol{v}_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad T(\boldsymbol{v}_2) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

  in terms of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. We can always do this using the row reduction

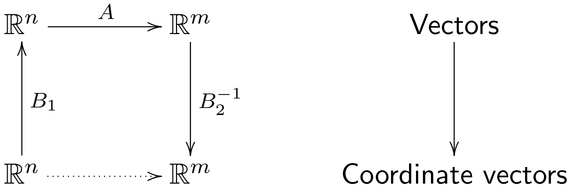
$$\begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & T(\boldsymbol{v}_1) & T(\boldsymbol{v}_2) \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 & 1 \\ 1 & 1 & 2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} \mathbf{1} & 0 & 5 & 8 \\ 0 & \mathbf{1} & -3 & -5 \end{bmatrix}.$$

- Thus, the matrix of T with respect to the given basis is

$$A = \begin{bmatrix} 5 & 8 \\ -3 & -5 \end{bmatrix}.$$

Matrix of a linear transformation: Explicit formulas

1. Suppose B_2 is an $m \times m$ matrix whose columns form a basis of $\mathbb{R}^m$. To find the coordinate vector of $\boldsymbol{w}$ with respect to this basis, one needs to solve the system $B_2\boldsymbol{x} = \boldsymbol{w}$ and the solution is $\boldsymbol{x} = B_2^{-1}\boldsymbol{w}$.
2. Suppose $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is left multiplication with the $m \times n$ matrix A. Let B_1 be a matrix whose columns form a basis of $\mathbb{R}^n$ and let B_2 be a matrix whose columns form a basis of $\mathbb{R}^m$. Then the matrix of T with respect to these bases is given by $B_2^{-1}AB_1$.
3. Suppose $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is left multiplication with the $m \times n$ matrix A. Then the matrix of T with respect to the standard bases is A.



Reflection along a line through the origin

- Suppose $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is reflection along a line through the origin. To study this linear transformation, we choose the basis $\boldsymbol{v}_1, \boldsymbol{v}_2$ so that $\boldsymbol{v}_1$ lies on the line of reflection and $\boldsymbol{v}_2$ is perpendicular to this line.
- The two chosen vectors are such that $T(\boldsymbol{v}_1) = \boldsymbol{v}_1$ and $T(\boldsymbol{v}_2) = -\boldsymbol{v}_2$. Thus, the matrix of T with respect to the given basis is

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

- If one uses the standard basis, instead, then the matrix of T becomes

$$A' = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix},$$

  where θ is the angle between the line of reflection and the x-axis.
- This example illustrates that the matrix of a linear transformation may turn out to be very simple, if the basis is suitably chosen. In fact, we ended up with the exact same matrix for any reflection whatsoever.

Similar matrices

Definition 4.9 – Similar matrices

Two $n \times n$ matrices A, C are called similar, if there exists an invertible matrix B such that $C = B^{-1}AB$. Similar matrices represent the same linear transformation with respect to different bases.

- Suppose that A is similar to C and that $A^2 = 0$. Then we also have

$$C^2 = B^{-1}AB \cdot B^{-1}AB = B^{-1}A^2B = 0.$$

- Suppose that A is similar to C and that $A^2 = I_n$. Then we also have

$$C^2 = B^{-1}AB \cdot B^{-1}AB = B^{-1}A^2B = B^{-1}B = I_n.$$

- Similar matrices have many properties in common. For instance, they have the same determinant, and their column/null spaces have the same dimension. We shall obtain a more thorough list of common properties when we study similar matrices in MA1212.

Powers of square matrices

- If two square matrices A, C are similar, then one easily finds that

$$C = B^{-1}AB \quad \Longrightarrow \quad C^k = B^{-1}A^kB.$$

Thus, powers of one matrix may be related to powers of the other.

- Suppose now that C is a diagonal matrix. Then powers of C are easy to compute because an induction argument gives

$$C = \begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{bmatrix} \quad \Longrightarrow \quad C^k = \begin{bmatrix} a_1^k & & & \\ & a_2^k & & \\ & & \ddots & \\ & & & a_n^k \end{bmatrix}.$$

One may thus compute powers of A by noting that $A^k = BC^kB^{-1}$.

- More generally, one may compute powers of a matrix A, if one can find a similar matrix C whose powers are easy to determine. We are going to develop this approach in much more detail in MA1212.

Recursive relations involving two terms

- A recursive relation expresses each term of a sequence $x_1, x_2, \dots$ as a function of the previous terms. One generally prefers to have a closed formula, namely one that expresses x_n as a function of n alone.
- For instance, suppose that the sequences x_n, y_n are such that

$$\left\{ \begin{array}{l} x_n = 3x_{n-1} + 2y_{n-1} \\ y_n = 2x_{n-1} + 3y_{n-1} \end{array} \right\}.$$

- Letting $\boldsymbol{u}_n$ denote the vector of unknowns, one can then write

$$\boldsymbol{u}_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix} \quad \Longrightarrow \quad \boldsymbol{u}_n = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix} = A\boldsymbol{u}_{n-1}$$

  for some 2×2 matrix A. This equation is easily seen to imply that

$$\boldsymbol{u}_n = A\boldsymbol{u}_{n-1} = A^2\boldsymbol{u}_{n-2} = \dots = A^n\boldsymbol{u}_0.$$

- One may thus obtain a closed formula by computing the powers A^n of the matrix A. We shall learn how to do this in MA1212.

Recursive relations involving three or more terms

- A recursive relation may involve several consecutive terms such as

$$x_{n+3} = x_{n+2} + 5x_{n+1} + 3x_n.$$

  This relation can be handled exactly as before, although one needs to keep track of three terms in order to determine the next one.

- Letting $\boldsymbol{u}_n$ consist of three consecutive terms, we can now write

$$\boldsymbol{u}_n = \begin{bmatrix} x_{n+1} \\ x_{n+2} \\ x_{n+3} \end{bmatrix} \quad \Longrightarrow \quad \boldsymbol{u}_n = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n+1} \\ x_{n+2} \end{bmatrix} = A\boldsymbol{u}_{n-1}$$

  for some 3×3 matrix A. This equation is easily seen to imply that

$$\boldsymbol{u}_n = A\boldsymbol{u}_{n-1} = A^2\boldsymbol{u}_{n-2} = \ldots = A^n\boldsymbol{u}_0.$$

- Once we are able to compute powers of the matrix A, we can then obtain a closed formula for $\boldsymbol{u}_n$ and also one for the sequence x_n.