**Linear algebra I**
**Tutorial solutions #1**

**1.** Find the equation of the line through $(1, 2, 4)$ which is perpendicular to the plane

$$x - 2y + 3z = 4.$$

The line passes through $(1, 2, 4)$ with direction $\boldsymbol{v} = \left[\begin{smallmatrix} 1 \\ -2 \\ 3 \end{smallmatrix}\right]$, so its equation is

$$x = 1 + t, \qquad y = 2 - 2t, \qquad z = 4 + 3t.$$

**2.** Find the equation of the plane that passes through the points

$$A(1, 0, 2), \qquad B(2, 3, 1), \qquad C(3, 2, 1).$$

The normal vector of the plane is given by the cross product

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -4 \end{bmatrix}.$$

Since the plane contains the point $A(1, 0, 2)$, its equation is then

$$-(x - 1) - (y - 0) - 4(z - 2) = 0 \quad \Longrightarrow \quad x + y + 4z = 9.$$

**3.** Consider the line that passes through $P(2, 4, 1)$ and $Q(4, 1, 5)$. At which point does this line intersect the plane $x - 2y + 3z = 37$?

The line passes through $P(2, 4, 1)$ with direction $\overrightarrow{PQ} = \left[\begin{smallmatrix} 2 \\ -3 \\ 4 \end{smallmatrix}\right]$, so its equation is

$$x = 2 + 2t, \qquad y = 4 - 3t, \qquad z = 1 + 4t.$$

The point at which the line intersects the plane is the point that satisfies both the equation of the line and that of the plane. This gives

$$\begin{aligned} x - 2y + 3z = 37 \quad &\Longrightarrow \quad (2 + 2t) - 2(4 - 3t) + 3(1 + 4t) = 37 \\ &\Longrightarrow \quad 20t = 40, \end{aligned}$$

so $t = 2$ and the point of intersection is the point

$$(x, y, z) = (2 + 2t, 4 - 3t, 1 + 4t) = (6, -2, 9).$$

**4.** Show that $||\boldsymbol{u}-\boldsymbol{v}||^2 = ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2$, if the vectors $\boldsymbol{u}, \boldsymbol{v}$ are perpendicular to one another. Which well-known theorem does that prove? Hint: one has $||\boldsymbol{x}||^2 = \boldsymbol{x} \cdot \boldsymbol{x}$.

Since the vectors $\boldsymbol{u}, \boldsymbol{v}$ are perpendicular, one has $\boldsymbol{u} \cdot \boldsymbol{v} = 0$ and $\boldsymbol{v} \cdot \boldsymbol{u} = 0$, so

$$\begin{aligned} ||\boldsymbol{u}-\boldsymbol{v}||^2 &= (\boldsymbol{u}-\boldsymbol{v}) \cdot (\boldsymbol{u}-\boldsymbol{v}) \\ &= \boldsymbol{u}\cdot\boldsymbol{u} - \boldsymbol{u}\cdot\boldsymbol{v} - \boldsymbol{v}\cdot\boldsymbol{u} + \boldsymbol{v}\cdot\boldsymbol{v} = ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2. \end{aligned}$$

Now, the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u}-\boldsymbol{v}$ may represent the three sides of a triangle. If the first two are perpendicular, then the triangle is a right triangle with the third vector as a hypotenuse, so the statement we just proved is Pythagoras' theorem.

**5.** Find the equation of the plane which contains both the point $(3, 2, 4)$ and the line

$$x = 1 + 3t, \qquad y = 1 + t, \qquad z = 4 - t.$$

Let $P(3, 2, 4)$ be the given point and pick any two points on the line, say

$$Q(1, 1, 4) \quad \text{and} \quad R(4, 2, 3),$$

obtained for $t = 0$ and $t = 1$, respectively. The normal vector of the plane is given by

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Since the plane contains the point $P(3, 2, 4)$, its equation is then

$$(x-3) - 2(y-2) + (z-4) = 0 \quad \Longrightarrow \quad x - 2y + z = 3.$$

**Linear algebra I**
**Tutorial solutions #2**

**1.** Let $a$ be some fixed parameter. Solve the system of linear equations

$$\left\{ \begin{array}{r} x + y = a \\ ax + y = 2 \end{array} \right\}.$$

Using row reduction of the associated matrix, one finds that

$$\begin{bmatrix} 1 & 1 & a \\ a & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & a \\ 0 & 1-a & 2-a^2 \end{bmatrix}.$$

When $a = 1$, the second equation reads $0y = 1$ and we have no solutions. When $a \neq 1$, on the other hand, one may divide through by $1 - a$ to conclude that

$$\begin{bmatrix} 1 & 1 & a \\ 0 & 1 & \frac{2-a^2}{1-a} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & a + \frac{a^2-2}{1-a} \\ 0 & 1 & \frac{2-a^2}{1-a} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{a-2}{1-a} \\ 0 & 1 & \frac{2-a^2}{1-a} \end{bmatrix}.$$

Thus, a unique solution exists whenever $a \neq 1$ and this solution is given by

$$x = \frac{a-2}{1-a}, \qquad y = \frac{2-a^2}{1-a}.$$

**2.** For which value of $a$ does the following system fail to have solutions?

$$\left\{ \begin{array}{r} x - 2y + 2z = 3 \\ 2x - 3y + 4z = 1 \\ x - 4y + az = 5 \end{array} \right\}.$$

In this case, row reduction of the associated matrix gives

$$\begin{bmatrix} 1 & -2 & 2 & 3 \\ 2 & -3 & 4 & 1 \\ 1 & -4 & a & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 2 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & -2 & a-2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & -7 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & a-2 & -8 \end{bmatrix}.$$

When $a \neq 2$, one may divide the last row by $a - 2$ and proceed to find a unique solution. When $a = 2$, on the other hand, the last equation reads $0z = -8$ and no solutions exist.

**3.** Show that the reduced row echelon form of $A$ is the identity matrix $I_2$ when

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad ad - bc \neq 0.$$

We consider two cases. When $a \neq 0$ and $ad - bc \neq 0$, we have

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & b/a \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & b/a \\ 0 & (ad-bc)/a \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

When $a = 0$ and $bc \neq 0$, on the other hand, we similarly get

$$A = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} c & d \\ 0 & b \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & d/c \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

**4.** Compute the products $AB$, $BC$, $CB$ and $ABC$ in the case that

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 1 & 1 \end{bmatrix}.$$

Using the definition of matrix multiplication, one easily finds that

$$AB = \begin{bmatrix} 5 & 1 & 8 \\ 4 & 1 & 5 \end{bmatrix}, \qquad BC = \begin{bmatrix} 8 & 8 \\ 4 & 5 \end{bmatrix}, \qquad CB = \begin{bmatrix} 5 & 1 & 8 \\ 9 & 3 & 6 \\ 4 & 1 & 5 \end{bmatrix}.$$

As for the last product $ABC$, this is given by

$$A \cdot BC = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 8 & 8 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 16 & 18 \\ 12 & 13 \end{bmatrix}.$$

**5.** A square matrix $A$ is called upper triangular, if all the entries below its diagonal are zero, namely if $A_{ij} = 0$ whenever $i > j$. Suppose that both $A$ and $B$ are $n \times n$ upper triangular matrices. Show that their product $AB$ is upper triangular as well.

Suppose that $i > j$ and consider the $(i, j)$th entry of $AB$, namely

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}.$$

If $i > k$, then $A_{ik} = 0$ and the summand is zero. Otherwise, $k \geq i > j$ so $B_{kj} = 0$ and the summand is still zero. This implies that $(AB)_{ij} = 0$, as needed.

**Linear algebra I**
**Tutorial solutions #3**

**1.** Show that $A \cdot \operatorname{adj} A = (\det A) I_2$ for every $2 \times 2$ matrix $A$.

Consider the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Its cofactors are easily found to be

$$C_{11} = d, \qquad C_{12} = -c, \qquad C_{21} = -b, \qquad C_{22} = a.$$

The adjoint matrix is the transpose of the cofactor matrix $C$, namely

$$C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \quad \Longrightarrow \quad \operatorname{adj} A = C^t = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

To establish the given identity, it thus remains to check that

$$A \cdot \operatorname{adj} A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = (\det A) I_2.$$

**2.** Compute the adjoint of $A$ and also the product $A \cdot \operatorname{adj} A$ when

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & a & 2 \\ 4 & 2 & 1 \end{bmatrix}.$$

First of all, we compute the cofactors of $A$ which are given by

$$C_{11} = \det \begin{bmatrix} a & 2 \\ 2 & 1 \end{bmatrix} = a - 4, \quad C_{12} = -\det \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} = 5, \quad C_{13} = \det \begin{bmatrix} 3 & a \\ 4 & 2 \end{bmatrix} = 6 - 4a,$$
$$C_{21} = -\det \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} = 0, \quad C_{22} = \det \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} = -3, \quad C_{23} = -\det \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} = 6,$$
$$C_{31} = \det \begin{bmatrix} 2 & 1 \\ a & 2 \end{bmatrix} = 4 - a, \quad C_{32} = -\det \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} = 1, \quad C_{33} = \det \begin{bmatrix} 1 & 2 \\ 3 & a \end{bmatrix} = a - 6.$$

The adjoint matrix is the transpose of the cofactor matrix $C$, namely

$$C = \begin{bmatrix} a - 4 & 5 & 6 - 4a \\ 0 & -3 & 6 \\ 4 - a & 1 & a - 6 \end{bmatrix} \quad \Longrightarrow \quad \operatorname{adj} A = C^t = \begin{bmatrix} a - 4 & 0 & 4 - a \\ 5 & -3 & 1 \\ 6 - 4a & 6 & a - 6 \end{bmatrix}.$$

As for the product $A \cdot \operatorname{adj} A$, one can easily check that $A \cdot \operatorname{adj} A = (12 - 3a) I_3$.

**3.** Let $x_1, x_2, x_3 \in \mathbb{R}$ be arbitrary. Use row reduction to show that

$$\det \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2).$$

We let $A$ denote the given matrix and use row reduction to get

$$\det A = \det \begin{bmatrix} 1 & x_1 & x_1^2 \\ 0 & x_2 - x_1 & x_2^2 - x_1^2 \\ 0 & x_3 - x_1 & x_3^2 - x_1^2 \end{bmatrix}.$$

Taking out the common factors from the last two rows, we conclude that

$$\begin{aligned} \det A &= (x_2 - x_1)(x_3 - x_1) \cdot \det \begin{bmatrix} 1 & x_1 & x_1^2 \\ 0 & 1 & x_2 + x_1 \\ 0 & 1 & x_3 + x_1 \end{bmatrix} \\ &= (x_2 - x_1)(x_3 - x_1) \cdot \det \begin{bmatrix} 1 & x_1 & x_1^2 \\ 0 & 1 & x_2 + x_1 \\ 0 & 0 & x_3 - x_2 \end{bmatrix} \\ &= (x_2 - x_1)(x_3 - x_1)(x_3 - x_2). \end{aligned}$$

**4.** Let $A$ be a square matrix with integer entries. Show that $\det A$ is an integer as well.

The result holds for $1 \times 1$ matrices because $\det[a] = a$. Assume that it holds for $n \times n$ matrices and that $A$ is $(n+1) \times (n+1)$. Then expansion by minors gives

$$\det A = \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det M_{i1}$$

and each $\det M_{i1}$ is an integer by the induction hypothesis. Since the coefficients $a_{i1}$ are integers as well, we conclude that $\det A$ is itself an integer.

**5.** Show that $(AB)^t = B^t A^t$ for all $n \times n$ matrices $A, B$. Hint: compare entries.

The $(i, j)$th entry of the matrix $(AB)^t$ is given by

$$(AB)^t_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk} B_{ki}.$$

This is also the $(i, j)$th entry of the matrix $B^t A^t$ because

$$(B^t A^t)_{ij} = \sum_{k=1}^{n} (B^t)_{ik} (A^t)_{kj} = \sum_{k=1}^{n} A_{jk} B_{ki}.$$

**Linear algebra I**
**Tutorial solutions #4**

**1.** Express $\boldsymbol{w}$ as a linear combination of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ in the case that

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 4 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \qquad \boldsymbol{w} = \begin{bmatrix} 2 \\ 1 \\ 5 \\ 0 \end{bmatrix}.$$

Let $B$ denote the matrix whose columns are the vectors $\boldsymbol{v}_i$. In order to express $\boldsymbol{w}$ as a linear combination of these vectors, one has to solve the system $B\boldsymbol{x} = \boldsymbol{w}$. Since

$$\begin{bmatrix} 1 & 4 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 5 \\ 2 & 1 & 3 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

the unique solution is $(x_1, x_2, x_3) = (-5, 1, 3)$ and this means that $\boldsymbol{w} = -5\boldsymbol{v}_1 + \boldsymbol{v}_2 + 3\boldsymbol{v}_3$.

**2.** Are the following vectors linearly independent? Do they form a complete set for $\mathbb{R}^4$?

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 3 \\ 5 \\ 2 \\ 7 \end{bmatrix}.$$

If $B$ is the matrix whose columns are the vectors $\boldsymbol{v}_i$, then row reduction gives

$$B = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 5 \\ 1 & 2 & 2 \\ 2 & 3 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since there is a pivot in every column, the given vectors are linearly independent. Since there is no pivot in the last row, they do not form a complete set for $\mathbb{R}^4$.

**3.** Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ form a basis of $\mathbb{R}^n$ and let $\boldsymbol{v} \in \mathbb{R}^n$ be arbitrary. Show that there is a unique way of expressing $\boldsymbol{v}$ as a linear combination of the vectors $\boldsymbol{v}_i$.

The vectors $\boldsymbol{v}_i$ form a complete set for $\mathbb{R}^n$, so one may certainly express $\boldsymbol{v}$ as a linear combination of them. Suppose that this can be done in two ways, say

$$\boldsymbol{v} = \sum_{i=1}^{n} x_i \boldsymbol{v}_i, \qquad \boldsymbol{v} = \sum_{i=1}^{n} y_i \boldsymbol{v}_i$$

for some scalar coefficients $x_i, y_i$. Subtracting these two equations, we then get

$$\sum_{i=1}^{n} (x_i - y_i)\boldsymbol{v}_i = 0.$$

Since the vectors $\boldsymbol{v}_i$ are linearly independent, this actually gives $x_i = y_i$ for all $i$. In other words, there is a unique way of expressing $\boldsymbol{v}$ as a linear combination of the vectors $\boldsymbol{v}_i$.

**4.** Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly independent vectors in $\mathbb{R}^3$ and let

$$\boldsymbol{w}_1 = \boldsymbol{v}_2 + \boldsymbol{v}_3, \qquad \boldsymbol{w}_2 = \boldsymbol{v}_1 + \boldsymbol{v}_3, \qquad \boldsymbol{w}_3 = \boldsymbol{v}_1 + \boldsymbol{v}_2.$$

Show that the vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ are linearly independent as well.

Suppose that some linear combination of the vectors $\boldsymbol{w}_i$ is zero, say

$$x_1\boldsymbol{w}_1 + x_2\boldsymbol{w}_2 + x_3\boldsymbol{w}_3 = 0 \quad \Longrightarrow \quad x_1(\boldsymbol{v}_2 + \boldsymbol{v}_3) + x_2(\boldsymbol{v}_1 + \boldsymbol{v}_3) + x_3(\boldsymbol{v}_1 + \boldsymbol{v}_2) = 0.$$

Rearranging terms, one may express the last equation in the form

$$(x_2 + x_3)\boldsymbol{v}_1 + (x_1 + x_3)\boldsymbol{v}_2 + (x_1 + x_2)\boldsymbol{v}_3 = 0.$$

Since the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly independent, it easily follows that

$$x_2 + x_3 = x_1 + x_3 = x_1 + x_2 = 0 \quad \Longrightarrow \quad x_1 = x_2 = x_3 = 0.$$

**5.** Find a subset of the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ that forms a basis of $\mathbb{R}^3$ when

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix}, \qquad \boldsymbol{v}_4 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.$$

If $B$ is the matrix whose columns are the vectors $\boldsymbol{v}_i$, then row reduction gives

$$B = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 3 & 6 & 3 \\ 3 & 5 & 8 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 6 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This includes a pivot in every row, so the vectors $\boldsymbol{v}_i$ form a complete set for $\mathbb{R}^3$. Since the third column is the only column without a pivot, a basis is formed by $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_4$.

**Linear algebra I**
**Tutorial solutions #5**

**1.** Show that the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ form a basis of $\mathbb{R}^3$ and compute the coordinate vector of $\boldsymbol{v}$ with respect to this basis when

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \qquad \boldsymbol{v} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}.$$

The first three vectors form a basis of $\mathbb{R}^3$ because

$$\det \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -5 \\ 0 & 0 & -1 \end{bmatrix} = -1.$$

The coordinate vector of $\boldsymbol{v}$ is merely the last column in the row reduction

$$\begin{bmatrix} | & | & | & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v} \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 & 4 \\ 2 & 3 & 1 & 1 \\ 1 & 1 & 2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

**2.** Find a linear transformation $T \colon \mathbb{R}^3 \to \mathbb{R}^2$ such that

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

We are given $T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), T(\boldsymbol{v}_3)$ and we need to find $T(\boldsymbol{x})$ for any vector $\boldsymbol{x}$, so we need to express $\boldsymbol{x}$ as a linear combination of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$. Using row reduction, we now get

$$\begin{bmatrix} | & | & | & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{x} \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & x \\ 1 & 2 & 2 & y \\ 1 & 1 & 2 & z \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 2x - y \\ 0 & 1 & 0 & y - z \\ 0 & 0 & 1 & z - x \end{bmatrix}.$$

This implies that $\boldsymbol{x} = (2x - y)\boldsymbol{v}_1 + (y - z)\boldsymbol{v}_2 + (z - x)\boldsymbol{v}_3$, so we also have

$$T(\boldsymbol{x}) = (2x - y) \begin{bmatrix} 3 \\ 4 \end{bmatrix} + (y - z) \begin{bmatrix} 4 \\ 6 \end{bmatrix} + (z - x) \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 3x + y - z \\ x + 2y + z \end{bmatrix}.$$

**3.** Suppose $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. Show that $T$ is left multiplication by the matrix $A$ whose columns are the vectors $T(\boldsymbol{e}_1)$ and $T(\boldsymbol{e}_2)$.

Consider the images of the standard vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, say

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}a\\b\end{bmatrix}, \qquad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}c\\d\end{bmatrix}.$$

Since every vector in $\mathbb{R}^2$ is a linear combination of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, it easily follows that

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = T\left(x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\end{bmatrix}\right) = x_1\begin{bmatrix}a\\b\end{bmatrix} + x_2\begin{bmatrix}c\\d\end{bmatrix} = \begin{bmatrix}ax_1+cx_2\\bx_1+dx_2\end{bmatrix}.$$

In particular, $T$ is left multiplication by the matrix $A$ because

$$T(\boldsymbol{x}) = T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}ax_1+cx_2\\bx_1+dx_2\end{bmatrix} = \begin{bmatrix}a & c\\b & d\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = A\boldsymbol{x}.$$

**4.** Suppose $T\colon \mathbb{R}^6 \to \mathbb{R}^4$ is left multiplication by a matrix $A$ and $T$ is surjective. How many pivots does the reduced row echelon form of $A$ have? Can $T$ be injective?

Since $T$ is surjective, every vector in $\mathbb{R}^4$ is a linear combination of the columns of $A$, so these columns form a complete set. Thus, the reduced row echelon form must have a pivot in every row. Since $A$ is a $4\times 6$ matrix, there should be 4 pivots.

In order for $T$ to be injective, the reduced row echelon form of $A$ must contain a pivot in every column, hence 6 pivots. This is not the case, however, so $T$ is not injective.

**5.** Consider the linear transformation $T\colon M_{22} \to M_{22}$ which is defined by

$$T(A) = A + A^t.$$

Find a basis for both the kernel and the image of this linear transformation.

First of all, we express the given equation in terms of the entries of $A$ to get

$$T(A) = \begin{bmatrix}a & b\\c & d\end{bmatrix} + \begin{bmatrix}a & c\\b & d\end{bmatrix} = \begin{bmatrix}2a & b+c\\b+c & 2d\end{bmatrix}.$$

Since the entries $a, b, c, d$ are all arbitrary, it easily follows that the image of $T$ is

$$\operatorname{im} T = \operatorname{Span}\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}.$$

The kernel of $T$ consists of the matrices $A$ such that $T(A) = 0$ and this implies that

$$\ker T = \left\{\begin{bmatrix}0 & -c\\c & 0\end{bmatrix} : c\in\mathbb{R}\right\} = \operatorname{Span}\left\{\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\right\}.$$

# Linear algebra I
# Tutorial solutions #6

**1.** Define a function $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by letting $T(x,y) = (9x+2y, 4x+y)$. Show that $T$ is bijective and find the inverse of this function explicitly.

The given function is a linear transformation that can also be expressed in the form

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 9x+2y \\ 4x+y \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

This means that $T$ is left multiplication by a matrix $A$. Since $\det A = 1$, the matrix $A$ is invertible, so its reduced row echelon form is the identity matrix. This includes a pivot in every row, so $T$ is surjective, and also a pivot in every column, so $T$ is injective. As for the inverse of $T$, this is obviously left multiplication by $A^{-1}$, namely

$$T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 9 & 2 \\ 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-2y \\ 9y-4x \end{bmatrix}.$$

**2.** Consider the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ and the basis $B$ of $\mathbb{R}^2$, where

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ 2x+3y \end{bmatrix}, \qquad B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}.$$

Find the matrix of $T$ with respect to the basis $B$.

First of all, we compute the images of the elements of $B$, namely

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 8 \end{bmatrix}.$$

To express these vectors in terms of the basis $B$, we use the row reduction

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 5 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -5 & -10 \\ 0 & 1 & 5 & 9 \end{bmatrix}.$$

This means that the matrix of $T$ with respect to the given basis is

$$A = \begin{bmatrix} -5 & -10 \\ 5 & 9 \end{bmatrix}.$$

**3.** Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ and the bases $B_1, B_2$ defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ 3z-y \end{bmatrix}, \qquad B_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}, \qquad B_2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}.$$

Find the matrix of $T$ with respect to the given bases.

First of all, we compute the images of the elements of $B_1$, namely

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

To express these vectors in terms of the basis $B_2$, we use the row reduction

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 3 \\ 1 & 1 & 3 & 2 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 5 & 1 & -5 \\ 0 & 1 & -2 & 1 & 4 \end{bmatrix}.$$

This means that the matrix of $T$ with respect to the given bases is

$$A = \begin{bmatrix} 5 & 1 & -5 \\ -2 & 1 & 4 \end{bmatrix}.$$

**4.** Find the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ whose matrix with respect to the bases of the previous problem is given by

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}.$$

Write $B_1 = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ and $B_2 = \{\boldsymbol{w}_1, \boldsymbol{w}_2\}$ for convenience. The $i$th column of $A$ is the coordinate vector of $T(\boldsymbol{v}_i)$ with respect to the basis $B_2$ and this means that

$$T(\boldsymbol{v}_1) = \boldsymbol{w}_1 + 2\boldsymbol{w}_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \quad T(\boldsymbol{v}_2) = \boldsymbol{w}_1 + \boldsymbol{w}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad T(\boldsymbol{v}_3) = \boldsymbol{w}_1 + 3\boldsymbol{w}_2 = \begin{bmatrix} 7 \\ 4 \end{bmatrix}.$$

To find $T(\boldsymbol{x})$ for any vector $\boldsymbol{x}$, we need to express $\boldsymbol{x}$ as a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$. Using row reduction, it is easy to check that

$$\begin{bmatrix} | & | & | & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{x} \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & x \\ 0 & 1 & 1 & y \\ 1 & 1 & 0 & z \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & x-y \\ 0 & 1 & 0 & y+z-x \\ 0 & 0 & 1 & x-z \end{bmatrix}.$$

This implies that $\boldsymbol{x} = (x-y)\boldsymbol{v}_1 + (y+z-x)\boldsymbol{v}_2 + (x-z)\boldsymbol{v}_3$, so we also have

$$T(\boldsymbol{x}) = (x-y)\begin{bmatrix} 5 \\ 3 \end{bmatrix} + (y+z-x)\begin{bmatrix} 3 \\ 2 \end{bmatrix} + (x-z)\begin{bmatrix} 7 \\ 4 \end{bmatrix} = \begin{bmatrix} 9x-2y-4z \\ 5x-y-2z \end{bmatrix}.$$

**5.** Suppose $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ form a basis of $\mathbb{R}^n$ and let $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ be vectors in $\mathbb{R}^m$. Find a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ such that $T(\boldsymbol{v}_i) = \boldsymbol{w}_i$ for each $1 \leq i \leq n$.

Since $T$ is linear, it must be left multiplication by some $m \times n$ matrix $A$. Consider the matrix $B_1$ whose columns are the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ and the matrix $B_2$ whose columns are the vectors $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$. In view of the given assumption, we must then have

$$T(\boldsymbol{v}_i) = \boldsymbol{w}_i \quad \Longrightarrow \quad T(B_1\boldsymbol{e}_i) = B_2\boldsymbol{e}_i \quad \Longrightarrow \quad AB_1\boldsymbol{e}_i = B_2\boldsymbol{e}_i \text{ for each } i.$$

This gives $AB_1 = B_2$ and thus $A = B_2B_1^{-1}$. In fact, the matrix $B_1$ is invertible because its columns form a basis of $\mathbb{R}^n$, while the function $T(\boldsymbol{x}) = B_2B_1^{-1}\boldsymbol{x}$ is linear with

$$T(\boldsymbol{v}_i) = T(B_1\boldsymbol{e}_i) = B_2B_1^{-1} \cdot B_1\boldsymbol{e}_i = B_2 \cdot \boldsymbol{e}_i = \boldsymbol{w}_i \text{ for each } i.$$