Linear algebra I
Tutorial problems #1

1. Find the equation of the line through $(1, 2, 4)$ which is perpendicular to the plane

$$x - 2y + 3z = 4.$$

2. Find the equation of the plane that passes through the points

$$A(1, 0, 2), \qquad B(2, 3, 1), \qquad C(3, 2, 1).$$

3. Consider the line that passes through $P(2, 4, 1)$ and $Q(4, 1, 5)$. At which point does this line intersect the plane $x - 2y + 3z = 37$?

4. Show that $||\boldsymbol{u} - \boldsymbol{v}||^2 = ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2$, if the vectors $\boldsymbol{u}, \boldsymbol{v}$ are perpendicular to one another. Which well-known theorem does that prove? Hint: one has $||\boldsymbol{x}||^2 = \boldsymbol{x} \cdot \boldsymbol{x}$.

5. Find the equation of the plane which contains both the point $(3, 2, 4)$ and the line

$$x = 1 + 3t, \qquad y = 1 + t, \qquad z = 4 - t.$$

Linear algebra I
Tutorial problems #2

1. Let a be some fixed parameter. Solve the system of linear equations

$$\left\{ \begin{array}{r} x + y = a \\ ax + y = 2 \end{array} \right\}.$$

2. For which value of a does the following system fail to have solutions?

$$\left\{ \begin{array}{r} x - 2y + 2z = 3 \\ 2x - 3y + 4z = 1 \\ x - 4y + az = 5 \end{array} \right\}.$$

3. Show that the reduced row echelon form of A is the identity matrix I_2 when

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad ad - bc \neq 0.$$

4. Compute the products AB, BC, CB and ABC in the case that

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 1 & 1 \end{bmatrix}.$$

5. A square matrix A is called upper triangular, if all the entries below its diagonal are zero, namely if $A_{ij} = 0$ whenever $i > j$. Suppose that both A and B are $n \times n$ upper triangular matrices. Show that their product AB is upper triangular as well.

Linear algebra I
Tutorial problems #3

1. Show that $A \cdot \operatorname{adj} A = (\det A) I_2$ for every 2×2 matrix A.

2. Compute the adjoint of A and also the product $A \cdot \operatorname{adj} A$ when

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & a & 2 \\ 4 & 2 & 1 \end{bmatrix}.$$

3. Let $x_1, x_2, x_3 \in \mathbb{R}$ be arbitrary. Use row reduction to show that

$$\det \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2).$$

4. Let A be a square matrix with integer entries. Show that $\det A$ is an integer as well.

5. Show that $(AB)^t = B^t A^t$ for all $n \times n$ matrices A, B. Hint: compare entries.

Linear algebra I
Tutorial problems #4

1. Express $\boldsymbol{w}$ as a linear combination of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ in the case that

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 4 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \qquad \boldsymbol{w} = \begin{bmatrix} 2 \\ 1 \\ 5 \\ 0 \end{bmatrix}.$$

2. Are the following vectors linearly independent? Do they form a complete set for $\mathbb{R}^4$?

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 3 \\ 5 \\ 2 \\ 7 \end{bmatrix}.$$

3. Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ form a basis of $\mathbb{R}^n$ and let $\boldsymbol{v} \in \mathbb{R}^n$ be arbitrary. Show that there is a unique way of expressing $\boldsymbol{v}$ as a linear combination of the vectors $\boldsymbol{v}_i$.

4. Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly independent vectors in $\mathbb{R}^3$ and let

$$\boldsymbol{w}_1 = \boldsymbol{v}_2 + \boldsymbol{v}_3, \qquad \boldsymbol{w}_2 = \boldsymbol{v}_1 + \boldsymbol{v}_3, \qquad \boldsymbol{w}_3 = \boldsymbol{v}_1 + \boldsymbol{v}_2.$$

Show that the vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ are linearly independent as well.

5. Find a subset of the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ that forms a basis of $\mathbb{R}^3$ when

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix}, \qquad \boldsymbol{v}_4 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.$$

Linear algebra I
Tutorial problems #5

1. Show that the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ form a basis of $\mathbb{R}^3$ and compute the coordinate vector of $\boldsymbol{v}$ with respect to this basis when

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \qquad \boldsymbol{v} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}.$$

2. Find a linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ such that

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

3. Suppose $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. Show that T is left multiplication by the matrix A whose columns are the vectors $T(\boldsymbol{e}_1)$ and $T(\boldsymbol{e}_2)$.

4. Suppose $T\colon \mathbb{R}^6 \to \mathbb{R}^4$ is left multiplication by a matrix A and T is surjective. How many pivots does the reduced row echelon form of A have? Can T be injective?

5. Consider the linear transformation $T\colon M_{22} \to M_{22}$ which is defined by

$$T(A) = A + A^t.$$

Find a basis for both the kernel and the image of this linear transformation.

Linear algebra I
Tutorial problems #6

1. Define a function $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by letting $T(x,y) = (9x+2y, 4x+y)$. Show that T is bijective and find the inverse of this function explicitly.

2. Consider the linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ and the basis B of $\mathbb{R}^2$, where

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ 2x+3y \end{bmatrix}, \qquad B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}.$$

Find the matrix of T with respect to the basis B.

3. Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ and the bases B_1, B_2 defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ 3z-y \end{bmatrix}, \qquad B_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}, \qquad B_2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}.$$

Find the matrix of T with respect to the given bases.

4. Find the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ whose matrix with respect to the bases of the previous problem is given by

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}.$$

5. Suppose $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ form a basis of $\mathbb{R}^n$ and let $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n$ be vectors in $\mathbb{R}^m$. Find a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ such that $T(\boldsymbol{v}_i) = \boldsymbol{w}_i$ for each $1 \le i \le n$.