Linear algebra I
Homework #1 solutions

1. Show that $A(5,3,4)$, $B(1,0,2)$ and $C(3,-4,4)$ are the vertices of a right triangle.

The given points are the vertices of a triangle whose sides are

$$\overrightarrow{AB} = \begin{bmatrix} -4 \\ -3 \\ -2 \end{bmatrix}, \qquad \overrightarrow{BC} = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix}, \qquad \overrightarrow{AC} = \begin{bmatrix} -2 \\ -7 \\ 0 \end{bmatrix}.$$

Since $\overrightarrow{AB} \cdot \overrightarrow{BC} = -8 + 12 - 4 = 0$, one of the angles is a right angle, indeed.

2. Find the equation of the plane that passes through the points

$$A(2,4,3), \qquad B(2,3,5), \qquad C(3,2,1).$$

The normal vector of the plane is given by the cross product

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \times \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}.$$

Since the plane passes through the point $A(2,4,3)$, its equation is then

$$6(x-2) + 2(y-4) + (z-3) = 0 \quad \Longrightarrow \quad 6x + 2y + z = 23.$$

3. Find the equation of the plane which contains both the point $(1,2,3)$ and the line

$$x = 3 + t, \qquad y = 1 + 2t, \qquad z = 2t.$$

Let $P(1,2,3)$ be the given point and pick any two points on the line, say

$$Q(3,1,0) \quad \text{and} \quad R(4,3,2),$$

obtained for $t = 0$ and $t = 1$, respectively. The normal vector of the plane is then

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} \times \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ 5 \end{bmatrix}$$

and the plane contains the point $P(1,2,3)$, so its equation is

$$4(x-1) - 7(y-2) + 5(z-3) = 0 \quad \Longrightarrow \quad 4x - 7y + 5z = 5.$$

4. Which point of the plane $x - 2y + 3z = 1$ is closest to the point $(1, 2, 6)$?

Consider the line through $(1, 2, 6)$ which is perpendicular to the plane. The direction of this line is the normal vector of the plane, so the equation of the line is

$$x = 1 + t, \qquad y = 2 - 2t, \qquad z = 6 + 3t.$$

We need to find the point P at which the line meets the plane. At that point, one has

$$x - 2y + 3z = 1 \quad \Longrightarrow \quad (1 + t) - 2(2 - 2t) + 3(6 + 3t) = 1 \quad \Longrightarrow \quad 14t = -14.$$

This gives $t = -1$, so the desired point P is the point whose coordinates are

$$(x, y, z) = (1 + t, 2 - 2t, 6 + 3t) = (0, 4, 3).$$

Linear algebra I
Homework #2 solutions

1. Solve the system of linear equations

$$\left\{\begin{aligned} x+4y-2z&=15\\ 2x-3y+5z&=11\\ 3x+6y-4z&=23 \end{aligned}\right\}.$$

Using row reduction of the associated matrix, we get

$$\begin{bmatrix} 1 & 4 & -2 & 15\\ 2 & -3 & 5 & 11\\ 3 & 6 & -4 & 23 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & -2 & 15\\ 0 & -11 & 9 & -19\\ 0 & -6 & 2 & -22 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & -2 & 15\\ 0 & 1 & 5 & 25\\ 0 & -6 & 2 & -22 \end{bmatrix}$$

and this can be further reduced to give

$$\begin{bmatrix} 1 & 0 & -22 & -85\\ 0 & 1 & 5 & 25\\ 0 & 0 & 32 & 128 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -22 & -85\\ 0 & 1 & 5 & 25\\ 0 & 0 & 1 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 3\\ 0 & 1 & 0 & 5\\ 0 & 0 & 1 & 4 \end{bmatrix}.$$

In particular, the system has a unique solution which is given by $(x, y, z) = (3, 5, 4)$.

2. Find a quadratic polynomial, say $f(x) = ax^2 + bx + c$, such that

$$f(1) = 1, \qquad f(2) = 9, \qquad f(3) = 27.$$

The given conditions hold if and only if

$$a+b+c=1, \qquad 4a+2b+c=9, \qquad 9a+3b+c=27.$$

Using row reduction of the associated matrix, we now get

$$\begin{bmatrix} 1 & 1 & 1 & 1\\ 4 & 2 & 1 & 9\\ 9 & 3 & 1 & 27 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1\\ 0 & -2 & -3 & 5\\ 0 & -6 & -8 & 18 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1\\ 0 & 1 & 3/2 & -5/2\\ 0 & 3 & 4 & -9 \end{bmatrix}$$

and this can be further reduced to give

$$\begin{bmatrix} 1 & 0 & -1/2 & 7/2\\ 0 & 1 & 3/2 & -5/2\\ 0 & 0 & -1/2 & -3/2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1/2 & 7/2\\ 0 & 1 & 3/2 & -5/2\\ 0 & 0 & 1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 5\\ 0 & 1 & 0 & -7\\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

In particular, the unique solution is $(a, b, c) = (5, -7, 3)$ and so $f(x) = 5x^2 - 7x + 3$.

3. The intersection of the following planes is a line. Find the equation of this line.

$$x + 2y + 3z = 1, \qquad 2x + 3y + 5z = 2, \qquad 2x + y + 3z = 2.$$

Using row reduction of the associated matrix, we get

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 5 & 2 \\ 2 & 1 & 3 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -3 & -3 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This gives $x + z = 1$ and $y + z = 0$, so the variable z is free and x, y are given by

$$x = 1 - z, \qquad y = -z.$$

Regarding z as a parameter, say $z = t$, we obtain the parametric equation of a line

$$x = 1 - t, \qquad y = -t, \qquad z = t.$$

4. Solve the system of linear equations

$$\left\{ \begin{aligned} x_1 + 2x_2 + 4x_3 + 6x_4 - 2x_5 &= 9 \\ 2x_1 + 2x_2 + 3x_3 + 8x_4 - x_5 \;\; &= 8 \\ 3x_1 + 3x_2 + 2x_3 + 7x_4 - 4x_5 &= 7 \end{aligned} \right\}.$$

In this case, row reduction leads to the reduced row echelon form

$$\begin{bmatrix} 1 & 2 & 4 & 6 & -2 & 9 \\ 2 & 2 & 3 & 8 & -1 & 8 \\ 3 & 3 & 2 & 7 & -4 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 4 & 2 & 1 \\ 0 & 1 & 0 & -3 & -4 & 0 \\ 0 & 0 & 1 & 2 & 1 & 2 \end{bmatrix}.$$

Note that the corresponding system of equations reads

$$x_1 + 4x_4 + 2x_5 = 1, \qquad x_2 - 3x_4 - 4x_5 = 0, \qquad x_3 + 2x_4 + x_5 = 2.$$

Thus, the variables x_4, x_5 are free and the remaining variables are given by

$$x_1 = 1 - 4x_4 - 2x_5, \qquad x_2 = 3x_4 + 4x_5, \qquad x_3 = 2 - 2x_4 - x_5.$$

Linear algebra I
Homework #3 solutions

1. Let a, b be some fixed parameters. Solve the system of linear equations

$$\left\{ \begin{aligned} x + ay &= 2 \\ bx + 2y &= 3 \end{aligned} \right\}.$$

Using row reduction of the associated matrix, one finds that

$$\begin{bmatrix} 1 & a & 2 \\ b & 2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & a & 2 \\ 0 & 2-ab & 3-2b \end{bmatrix}.$$

Case 1. When $ab = 2$ and $b \neq 3/2$, the second equation gives $0y = 3 - 2b \neq 0$ and this is obviously a contradiction. Thus, the system has no solutions in this case.
Case 2. When $ab = 2$ and $b = 3/2$, the second equation reads $0y = 0$, so y is a free variable. In particular, the system has infinitely many solutions of the form $(x, y) = (2 - ay, y)$.
Case 3. When $ab \neq 2$, one may proceed using row reduction to get

$$\begin{bmatrix} 1 & a & 2 \\ 0 & 1 & \frac{3-2b}{2-ab} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 - \frac{3a-2ab}{2-ab} \\ 0 & 1 & \frac{3-2b}{2-ab} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{4-3a}{2-ab} \\ 0 & 1 & \frac{3-2b}{2-ab} \end{bmatrix}.$$

In other words, a unique solution exists and this solution is given by

$$x = \frac{4-3a}{2-ab}, \qquad y = \frac{3-2b}{2-ab}.$$

2. For which value of a does the following system fail to have solutions?

$$\left\{ \begin{aligned} x + 2y + 3z &= 4 \\ 2x + 3y + 4z &= 3 \\ 2x + 4y + az &= 2 \end{aligned} \right\}.$$

Using row reduction of the associated matrix, one finds that

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 3 \\ 2 & 4 & a & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -5 \\ 0 & 0 & a-6 & -6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & -6 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & a-6 & -6 \end{bmatrix}.$$

When $a \neq 6$, one may divide the last row by $a - 6$ and proceed to find a unique solution. When $a = 6$, on the other hand, the last equation reads $0z = -6$ and no solutions exist.

3. Compute the products AB, AC, BC and $ACAB$ in the case that

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 2 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 2 \\ 5 & 4 \\ 1 & 3 \end{bmatrix}.$$

Using the definition of matrix multiplication, one easily finds that

$$AB = \begin{bmatrix} 4 & 5 & 5 \\ 3 & 3 & 7 \end{bmatrix}, \qquad AC = \begin{bmatrix} 3 & 8 \\ 6 & 6 \end{bmatrix}, \qquad BC = \begin{bmatrix} 10 & 17 \\ 15 & 22 \\ 12 & 13 \end{bmatrix}.$$

As for the last product $ACAB$, this is given by

$$AC \cdot AB = \begin{bmatrix} 3 & 8 \\ 6 & 6 \end{bmatrix} \cdot \begin{bmatrix} 4 & 5 & 5 \\ 3 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 36 & 39 & 71 \\ 42 & 48 & 72 \end{bmatrix}.$$

4. A magic square is a square matrix such that the entries in each row, each column and each of the two diagonals have the same sum. The matrix A is a typical example. So is the matrix B, but most of its entries are unknown. Can you find them?

$$A = \begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} a & 3 & b \\ c & d & 1 \\ 2 & e & 7 \end{bmatrix}.$$

To say that the matrix B is a magic square is to say that

$$a+b+3 = c+d+1 = e+9 = a+c+2 = d+e+3 = b+8 = a+d+7 = b+d+2.$$

This is a system of linear equations that one can always solve using row reduction. In our case, we have $a+b+3 = b+8$ and $b+d+2 = b+8$, so $a = 5$ and $d = 6$. This gives

$$b+8 = c+7 = e+9 = 18 \quad \Longrightarrow \quad (b,c,e) = (10,11,9)$$

and we have now determined all the entries. In particular, the magic square B is

$$B = \begin{bmatrix} 5 & 3 & 10 \\ 11 & 6 & 1 \\ 2 & 9 & 7 \end{bmatrix}.$$

Linear algebra I
Homework #4 solutions

1. For which values of a, b, c is the following matrix invertible? Explain.

$$A = \begin{bmatrix} 1 & 1 & a \\ 2 & 3 & b \\ 1 & 2 & c \end{bmatrix}.$$

To say that A is invertible is to say that its reduced row echelon form is I_3. Since

$$\begin{bmatrix} 1 & 1 & a \\ 2 & 3 & b \\ 1 & 2 & c \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & a \\ 0 & 1 & b-2a \\ 0 & 1 & c-a \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 3a-b \\ 0 & 1 & b-2a \\ 0 & 0 & a+c-b \end{bmatrix},$$

there are two possible cases. If $a + c = b$, then the last row is a row of zeros and A is not invertible. If $a + c \neq b$, then one may create a third pivot and so A is invertible.

2. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}.$$

We merge A with the identity matrix I_3 and use row reduction to get

$$\begin{aligned}
\begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{bmatrix} &\longrightarrow \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & -2 & 1 & -2 & 0 & 1 \end{bmatrix} \\
&\longrightarrow \begin{bmatrix} 1 & 0 & 1 & -3 & 2 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{bmatrix} \\
&\longrightarrow \begin{bmatrix} 1 & 0 & 0 & -5 & 4 & -1 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 1 \end{bmatrix}.
\end{aligned}$$

The inverse of A is the rightmost half of this matrix, namely

$$A^{-1} = \begin{bmatrix} -5 & 4 & -1 \\ 2 & -1 & 0 \\ 2 & -2 & 1 \end{bmatrix}.$$

3. Suppose A, B are $n \times n$ matrices and A has a column of zeros. Show that BA has a column of zeros as well and conclude that A is not invertible.

Suppose that the jth column of A is a column of zeros. Then

$$(BA)_{ij} = \sum_{k=1}^{n} B_{ik} A_{kj} = 0$$

for all indices i, so the jth column of BA consists of zeros as well. In particular, BA is not equal to I_n for any matrix B and this means that A is not invertible.

4. Consider the 2×2 matrix A_x which is defined by

$$A_x = \begin{bmatrix} 1-x & x \\ -x & 1+x \end{bmatrix}.$$

Show that $A_x A_y = A_{x+y}$ for all numbers x, y and conclude that each A_x is invertible.

Using the definition of matrix multiplication, one finds that

$$\begin{aligned} A_x A_y &= \begin{bmatrix} 1-x & x \\ -x & 1+x \end{bmatrix} \begin{bmatrix} 1-y & y \\ -y & 1+y \end{bmatrix} \\ &= \begin{bmatrix} (1-x)(1-y) - xy & y - xy + x + xy \\ -x + xy - y - xy & -xy + (1+x)(1+y) \end{bmatrix} \end{aligned}$$

and this equation can be simplified to give

$$A_x A_y = \begin{bmatrix} 1-x-y & x+y \\ -x-y & 1+x+y \end{bmatrix} = A_{x+y}.$$

Noting that A_0 is merely the identity matrix I_2, we conclude that

$$A_x A_{-x} = A_0 = I_2, \qquad A_{-x} A_x = A_0 = I_2.$$

In particular, each matrix A_x is invertible and its inverse is $A_x^{-1} = A_{-x}$.

Linear algebra I
Homework #5 solutions

1. Use expansion by minors to compute the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & a & 1 \\ 1 & 3 & a \end{bmatrix}.$$

One may use expansion along the first column to find that

$$\det A = \det \begin{bmatrix} a & 1 \\ 3 & a \end{bmatrix} - \det \begin{bmatrix} 2 & 3 \\ 3 & a \end{bmatrix} + \det \begin{bmatrix} 2 & 3 \\ a & 1 \end{bmatrix}$$
$$= a^2 - 3 - (2a - 9) + 2 - 3a = a^2 - 5a + 8.$$

Alternatively, one may use expansion along the first row to get

$$\det A = \det \begin{bmatrix} a & 1 \\ 3 & a \end{bmatrix} - 2\det \begin{bmatrix} 1 & 1 \\ 1 & a \end{bmatrix} + 3\det \begin{bmatrix} 1 & a \\ 1 & 3 \end{bmatrix}$$
$$= a^2 - 3 - 2(a - 1) + 3(3 - a) = a^2 - 5a + 8.$$

2. Compute the adjoint of the matrix

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 3 & 1 \\ 1 & 2 & a \end{bmatrix}.$$

The cofactors of A are the signed determinants of the minors of A and those are

$$C_{11} = \det \begin{bmatrix} 3 & 1 \\ 2 & a \end{bmatrix} = 3a - 2, \quad C_{12} = -\det \begin{bmatrix} 2 & 1 \\ 1 & a \end{bmatrix} = 1 - 2a, \quad C_{13} = \det \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = 1,$$
$$C_{21} = -\det \begin{bmatrix} 3 & 4 \\ 2 & a \end{bmatrix} = 8 - 3a, \quad C_{22} = \det \begin{bmatrix} 1 & 4 \\ 1 & a \end{bmatrix} = a - 4, \quad C_{23} = -\det \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} = 1,$$
$$C_{31} = \det \begin{bmatrix} 3 & 4 \\ 3 & 1 \end{bmatrix} = -9, \quad C_{32} = -\det \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix} = 7, \quad C_{33} = \det \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} = -3.$$

The adjoint matrix is the transpose of the cofactor matrix C, namely

$$C = \begin{bmatrix} 3a-2 & 1-2a & 1 \\ 8-3a & a-4 & 1 \\ -9 & 7 & -3 \end{bmatrix} \implies \operatorname{adj} A = C^t = \begin{bmatrix} 3a-2 & 8-3a & -9 \\ 1-2a & a-4 & 7 \\ 1 & 1 & -3 \end{bmatrix}.$$

3. Use row reduction to compute the determinant of the following matrix. Hint: adding the last two rows to the top row should make row reduction a bit easier.

$$A = \begin{bmatrix} x & a & a \\ a & x & a \\ a & a & x \end{bmatrix}.$$

Adding the last two rows to the top row, one finds that

$$\det A = \det \begin{bmatrix} x+2a & x+2a & x+2a \\ a & x & a \\ a & a & x \end{bmatrix} = (x+2a) \cdot \det \begin{bmatrix} 1 & 1 & 1 \\ a & x & a \\ a & a & x \end{bmatrix}.$$

Once we now proceed with row reduction, we obtain an upper triangular matrix, so

$$\det A = (x+2a) \cdot \det \begin{bmatrix} 1 & 1 & 1 \\ 0 & x-a & 0 \\ 0 & 0 & x-a \end{bmatrix} = (x+2a) \cdot (x-a)^2.$$

4. Let A_n denote the $n \times n$ matrix whose diagonal entries are equal to 3 and all other entries are equal to 1. Compute the determinant of A_n.

Adding the last $n-1$ rows to the top row, one finds that

$$\det A_n = \det \begin{bmatrix} n+2 & n+2 & n+2 & \cdots & n+2 \\ 1 & 3 & 1 & \cdots & 1 \\ 1 & 1 & 3 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 3 \end{bmatrix} = (n+2) \cdot \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 3 & 1 & \cdots & 1 \\ 1 & 1 & 3 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 3 \end{bmatrix}.$$

Next, we proceed with row reduction. This leads to an upper triangular matrix, so

$$\det A_n = (n+2) \cdot \det \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{bmatrix} = (n+2) \cdot 2^{n-1}.$$

Linear algebra I
Homework #6 solutions

1. The determinant of a 6×6 matrix A contains the terms

$$a_{14}a_{25}a_{32}a_{46}a_{53}a_{61}, \qquad a_{16}a_{25}a_{34}a_{42}a_{53}a_{61}, \qquad a_{15}a_{26}a_{34}a_{43}a_{51}a_{62}.$$

What is the coefficient of each of these terms?

The coefficient of the first term is given by

$$\operatorname{sign}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 2 & 6 & 3 & 1 \end{pmatrix} = \operatorname{sign}(146)(253) = (-1)^{2+2} = 1.$$

The coefficient of the second term is given by

$$\operatorname{sign}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 2 & 3 & 1 \end{pmatrix} = \operatorname{sign}(16)(2534) = (-1)^{1+3} = 1.$$

The coefficient of the third term is given by

$$\operatorname{sign}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 4 & 3 & 1 & 2 \end{pmatrix} = \operatorname{sign}(15)(26)(34) = (-1)^{1+1+1} = -1.$$

2. For which values of x is A invertible? Determine the inverse for all such values.

$$A = \begin{bmatrix} 1 & 2 & x \\ 2 & 5 & 1 \\ 1 & x & 2 \end{bmatrix}.$$

To say that A is invertible is to say that $\det A \neq 0$. In this case, we have

$$\det A = 2x^2 - 6x + 4 = 2(x^2 - 3x + 2) = 2(x-1)(x-2),$$

so A is invertible if and only if $x \neq 1, 2$. To find the inverse of A, we use the formula that relates the inverse with the adjoint. As one can easily check, the cofactor matrix is

$$C = \begin{bmatrix} 10 - x & -3 & 2x - 5 \\ x^2 - 4 & 2 - x & 2 - x \\ 2 - 5x & 2x - 1 & 1 \end{bmatrix}.$$

The adjoint of A is the transpose of the matrix C, while the inverse of A is

$$A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj} A = \frac{1}{2x^2 - 6x + 4} \cdot \begin{bmatrix} 10 - x & x^2 - 4 & 2 - 5x \\ -3 & 2 - x & 2x - 1 \\ 2x - 5 & 2 - x & 1 \end{bmatrix}.$$

3. Determine the inverse of each of the following matrices.

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \qquad B = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}, \qquad C = B^{-1}AB.$$

Since $\det A = \cos^2\theta + \sin^2\theta = 1$ and $\det B = -\cos^2\theta - \sin^2\theta = -1$, one has

$$A^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \qquad B^{-1} = -\begin{bmatrix} -\cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

To find the inverse of C, we recall that $(A_1A_2A_3)^{-1} = A_3^{-1}A_2^{-1}A_1^{-1}$. This gives

$$\begin{aligned} C^{-1} = B^{-1}A^{-1}B &= \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \end{aligned}$$

4. Suppose A is an invertible matrix with $\det A = 1$. Show that $\operatorname{adj}(\operatorname{adj} A) = A$.

First of all, we use the identity $A \cdot \operatorname{adj} A = (\det A)I_n$ to find that

$$A \cdot \operatorname{adj} A = I_n \quad\Longrightarrow\quad \operatorname{adj} A = A^{-1} \quad\Longrightarrow\quad \det\operatorname{adj} A = \det A^{-1} = 1.$$

Next, we use the same identity with the matrix A replaced by its adjoint. This gives

$$\operatorname{adj} A \cdot \operatorname{adj}(\operatorname{adj} A) = (\det\operatorname{adj} A)I_n = I_n \quad\Longrightarrow\quad \operatorname{adj}(\operatorname{adj} A) = (\operatorname{adj} A)^{-1} = A.$$

Linear algebra I
Homework #7 solutions

1. Express $\boldsymbol{w}$ as a linear combination of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ in the case that

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \qquad \boldsymbol{w} = \begin{bmatrix} 5 \\ 4 \\ 5 \\ 7 \end{bmatrix}.$$

Let B denote the matrix whose columns are the vectors $\boldsymbol{v}_i$. In order to express $\boldsymbol{w}$ as a linear combination of these vectors, one has to solve the system $B\boldsymbol{x} = \boldsymbol{w}$. Since

$$\begin{bmatrix} 1 & 1 & 1 & 5 \\ 1 & 2 & 2 & 4 \\ 1 & 1 & 1 & 5 \\ 1 & 2 & 3 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

the unique solution is $(x_1, x_2, x_3) = (6, -4, 3)$ and this means that $\boldsymbol{w} = 6\boldsymbol{v}_1 - 4\boldsymbol{v}_2 + 3\boldsymbol{v}_3$.

2. Do the following vectors form a complete set for $\mathbb{R}^4$? Explain.

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \qquad \boldsymbol{v}_4 = \begin{bmatrix} 2 \\ 8 \\ 8 \\ 0 \end{bmatrix}.$$

If B is the matrix whose columns are the vectors $\boldsymbol{v}_i$, then row reduction gives

$$B = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 1 & 1 & 8 \\ 3 & 2 & 3 & 8 \\ 1 & 1 & 2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since there is no pivot in the last row, the vectors $\boldsymbol{v}_i$ do not form a complete set for $\mathbb{R}^4$.

3. Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly independent vectors in $\mathbb{R}^3$ and let

$$\boldsymbol{w}_1 = \boldsymbol{v}_2 + \boldsymbol{v}_3 - \boldsymbol{v}_1, \qquad \boldsymbol{w}_2 = \boldsymbol{v}_1 + \boldsymbol{v}_3 - \boldsymbol{v}_2, \qquad \boldsymbol{w}_3 = \boldsymbol{v}_1 + \boldsymbol{v}_2 - \boldsymbol{v}_3.$$

Show that the vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ are linearly independent as well.

Suppose that $x_1\boldsymbol{w}_1 + x_2\boldsymbol{w}_2 + x_3\boldsymbol{w}_3 = 0$ for some scalars x_i, namely

$$x_1(\boldsymbol{v}_2 + \boldsymbol{v}_3 - \boldsymbol{v}_1) + x_2(\boldsymbol{v}_1 + \boldsymbol{v}_3 - \boldsymbol{v}_2) + x_3(\boldsymbol{v}_1 + \boldsymbol{v}_2 - \boldsymbol{v}_3) = 0.$$

Rearranging terms, one may then express the last equation in the form

$$(x_2 + x_3 - x_1)\boldsymbol{v}_1 + (x_1 + x_3 - x_2)\boldsymbol{v}_2 + (x_1 + x_2 - x_3)\boldsymbol{v}_3 = 0.$$

Since the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly independent, this implies that

$$x_2 + x_3 - x_1 = x_1 + x_3 - x_2 = x_1 + x_2 - x_3 = 0.$$

It easily follows that each x_i is zero, as row reduction of the associated matrix gives

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

4. Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ form a complete set for $\mathbb{R}^3$ and let

$$\boldsymbol{w}_1 = \boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3, \qquad \boldsymbol{w}_2 = \boldsymbol{v}_1 + \boldsymbol{v}_3, \qquad \boldsymbol{w}_3 = \boldsymbol{v}_1 + \boldsymbol{v}_2.$$

Show that the vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ form a complete set for $\mathbb{R}^3$ as well.

Let $\boldsymbol{v} \in \mathbb{R}^3$ be arbitrary. Since the vectors $\boldsymbol{v}_i$ form a complete set, one certainly has

$$\boldsymbol{v} = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + a_3\boldsymbol{v}_3$$

for some scalars a_i. We wish to express $\boldsymbol{v}$ as a linear combination of the vectors $\boldsymbol{w}_i$, say

$$\begin{aligned} \boldsymbol{v} = x_1\boldsymbol{w}_1 + x_2\boldsymbol{w}_2 + x_3\boldsymbol{w}_3 &= x_1(\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3) + x_2(\boldsymbol{v}_1 + \boldsymbol{v}_3) + x_3(\boldsymbol{v}_1 + \boldsymbol{v}_2) \\ &= (x_1 + x_2 + x_3)\boldsymbol{v}_1 + (x_1 + x_3)\boldsymbol{v}_2 + (x_1 + x_2)\boldsymbol{v}_3. \end{aligned}$$

In view of the last two equations, it suffices to find scalars x_1, x_2, x_3 such that

$$x_1 + x_2 + x_3 = a_1, \qquad x_1 + x_3 = a_2, \qquad x_1 + x_2 = a_3.$$

This is a linear system that can be easily solved to obtain the unique solution

$$x_1 = a_2 + a_3 - a_1, \qquad x_2 = a_1 - a_2, \qquad x_3 = a_1 - a_3.$$

Thus, every vector $\boldsymbol{v} \in \mathbb{R}^3$ may be expressed as a linear combination of the vectors $\boldsymbol{w}_i$.

Linear algebra I
Homework #8 solutions

1. Find a basis for both the null space and the column space of the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 3 & 6 & 9 \\ 2 & 1 & 6 & 1 \end{bmatrix}.$$

The reduced row echelon form of A is easily found to be

$$R = \begin{bmatrix} 1 & 0 & 4 & -2 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the pivots appear in the first and the second columns, this implies that

$$\mathcal{C}(A) = \operatorname{Span}\{A\boldsymbol{e}_1, A\boldsymbol{e}_2\} = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

The null space of A is the same as the null space of R. It can be expressed in the form

$$\mathcal{N}(A) = \left\{ \begin{bmatrix} -4x_3 + 2x_4 \\ 2x_3 - 5x_4 \\ x_3 \\ x_4 \end{bmatrix} : x_3, x_4 \in \mathbb{R} \right\} = \operatorname{Span}\left\{ \begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

2. Does the vector $\boldsymbol{w}$ belong to the column space of A? Explain.

$$\boldsymbol{w} = \begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 3 & 9 \\ 2 & 1 & 7 \end{bmatrix}.$$

A vector $\boldsymbol{w}$ belongs to the column space of A if and only if $\boldsymbol{w}$ is a linear combination of the columns of A. Let us then consider the system $A\boldsymbol{x} = \boldsymbol{w}$. Using row reduction, we get

$$\begin{bmatrix} 1 & 1 & 4 & 5 \\ 2 & 3 & 9 & 7 \\ 2 & 1 & 7 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the last equation reads $0 = 1$. This means that the system $A\boldsymbol{x} = \boldsymbol{w}$ has no solutions. In other words, $\boldsymbol{w}$ is not a linear combination of the columns of A.

3. Express the polynomial $f(x) = x^2 + 4x - 6$ as a linear combination of

$$f_1(x) = x^2 + x, \qquad f_2(x) = x^2 + 1, \qquad f_3(x) = x + 2.$$

We need to find scalars a, b, c such that $af_1 + bf_2 + cf_3 = f$ and this gives

$$a(x^2 + x) + b(x^2 + 1) + c(x + 2) = x^2 + 4x - 6.$$

Comparing the coefficients of these polynomials, one obtains the system of equations

$$a + b = 1, \qquad a + c = 4, \qquad b + 2c = -6.$$

This can be easily solved using row reduction which gives

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 4 \\ 0 & 1 & 2 & -6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

In particular, the unique solution is $(a, b, c) = (5, -4, -1)$ and so $f = 5f_1 - 4f_2 - f_3$.

4. Show that the following matrices are linearly independent in M_{22}.

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Suppose that some linear combination of these matrices gives the zero matrix, say

$$x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4 = 0$$

for some scalars x_1, x_2, x_3, x_4. One may then simplify this equation to obtain the system

$$\begin{bmatrix} x_1 + x_3 + x_4 & x_1 + x_2 + x_4 \\ x_3 + x_4 & x_2 + x_3 + x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \implies \left\{ \begin{aligned} x_1 + x_3 + x_4 &= 0 \\ x_1 + x_2 + x_4 &= 0 \\ x_3 + x_4 &= 0 \\ x_2 + x_3 + x_4 &= 0 \end{aligned} \right\}.$$

Note that $x_2 = 0$ by the last two equations and thus $x_3 = x_2 = 0$ by the first two. It easily follows that $x_1 = x_4 = 0$ as well, so the given matrices are linearly independent.

Linear algebra I
Homework #9 solutions

1. Let P_3 be the set of all polynomials of degree at most 3 and let

$$U = \{f \in P_3 : f(2) = f(1) = 2f(0)\}.$$

Show that U is a subspace of P_3 and find a basis for it.

To say that $f(x) = ax^3 + bx^2 + cx + d$ satisfies the first condition is to say that

$$f(2) = f(1) \quad\Longrightarrow\quad 8a + 4b + 2c + d = a + b + c + d \quad\Longrightarrow\quad c = -7a - 3b.$$

To say that $f(x) = ax^3 + bx^2 + cx + d$ satisfies the second condition is to say that

$$f(1) = 2f(0) \quad\Longrightarrow\quad a + b + c + d = 2d \quad\Longrightarrow\quad d = a + b + c.$$

It easily follows that U is a subspace because U can be expressed in the form

$$\begin{aligned} U &= \{ax^3 + bx^2 - 7ax - 3bx - 6a - 2b : a, b \in \mathbb{R}\} \\ &= \operatorname{Span}\{x^3 - 7x - 6, x^2 - 3x - 2\}. \end{aligned}$$

To show that the last two polynomials are linearly independent, we note that

$$a(x^3 - 7x - 6) + b(x^2 - 3x - 2) = ax^3 + bx^2 - (7a + 3b)x - (6a + 2b).$$

If this linear combination happens to be zero, then the coefficients of x^3, x^2 must both be zero and so $a = b = 0$. We conclude that $x^3 - 7x - 6$ and $x^2 - 3x - 2$ form a basis of U.

2. Show that the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ form a basis of $\mathbb{R}^3$ and compute the coordinate vector of $\boldsymbol{w}$ with respect to this basis when

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}, \qquad \boldsymbol{w} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}.$$

The first three vectors form a basis of $\mathbb{R}^3$ because

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \\ 2 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & -4 & -5 \end{bmatrix} = \det \begin{bmatrix} -5 & -5 \\ -4 & -5 \end{bmatrix} = 25 - 20 = 5.$$

The coordinate vector of $\boldsymbol{w}$ is merely the last column in the row reduction

$$\begin{bmatrix} | & | & | & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{w} \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \\ 2 & 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

3. Suppose $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly independent vectors in a vector space V and let

$$\boldsymbol{w}_1 = \boldsymbol{v}_2 + a\boldsymbol{v}_3, \qquad \boldsymbol{w}_2 = \boldsymbol{v}_1 + a\boldsymbol{v}_3, \qquad \boldsymbol{w}_3 = \boldsymbol{v}_1 + a\boldsymbol{v}_2.$$

For which values of a are the vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ linearly independent?

Consider the vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ and their linear combination

$$\begin{aligned} x_1\boldsymbol{w}_1 + x_2\boldsymbol{w}_2 + x_3\boldsymbol{w}_3 &= x_1(\boldsymbol{v}_2 + a\boldsymbol{v}_3) + x_2(\boldsymbol{v}_1 + a\boldsymbol{v}_3) + x_3(\boldsymbol{v}_1 + a\boldsymbol{v}_2) \\ &= (x_2 + x_3)\boldsymbol{v}_1 + (x_1 + ax_3)\boldsymbol{v}_2 + (ax_1 + ax_2)\boldsymbol{v}_3. \end{aligned}$$

We need to ensure that this expression is zero if and only if $x_1 = x_2 = x_3 = 0$. Since the vectors $\boldsymbol{v}_i$ are linearly independent, the above expression is zero if and only if

$$x_2 + x_3 = 0, \qquad x_1 + ax_3 = 0, \qquad ax_1 + ax_2 = 0.$$

This is a linear system that can also be written in the form $A\boldsymbol{x} = 0$, namely

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & a \\ a & a & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

To say that $A\boldsymbol{x} = 0$ has only the trivial solution $\boldsymbol{x} = 0$ is to say that A is invertible. Since

$$\det A = \det \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & a \\ a & a & 0 \end{bmatrix} = a^2 + a,$$

the vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ are linearly independent if and only if $a \neq -1, 0$.

4. Find a linear transformation $T\colon \mathbb{R}^2 \to M_{23}$ such that

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 1 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \end{bmatrix}.$$

We are given $T(\boldsymbol{v}_1), T(\boldsymbol{v}_2)$ and we need to determine $T(\boldsymbol{x})$ for any vector $\boldsymbol{x}$, so we need to express $\boldsymbol{x}$ as a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. Using row reduction, we now get

$$\begin{bmatrix} | & | & | \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{x} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 & x \\ 1 & 1 & y \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2y - x \\ 0 & 1 & x - y \end{bmatrix}.$$

This implies that $\boldsymbol{x} = (2y - x)\boldsymbol{v}_1 + (x - y)\boldsymbol{v}_2$, so we also have

$$T(\boldsymbol{x}) = (2y - x)\begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 1 \end{bmatrix} + (x - y)\begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} y & x + y & 3y - x \\ x & x - y & 2y - x \end{bmatrix}.$$