Linear algebra I
Homework #1
due Thursday, Oct. 5

1. Show that $A(5,3,4)$, $B(1,0,2)$ and $C(3,-4,4)$ are the vertices of a right triangle.

2. Find the equation of the plane that passes through the points

$$A(2,4,3), \qquad B(2,3,5), \qquad C(3,2,1).$$

3. Find the equation of the plane which contains both the point $(1,2,3)$ and the line

$$x = 3+t, \qquad y = 1+2t, \qquad z = 2t.$$

4. Which point of the plane $x - 2y + 3z = 1$ is closest to the point $(1,2,6)$?

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- Write your name and course (Maths, TP, TSM) on the first page of your homework.
- NO LATE HOMEWORK WILL BE ACCEPTED.

Linear algebra I
Homework #2
due Thursday, Oct. 12

1. Solve the system of linear equations

$$\left\{ \begin{aligned} x + 4y - 2z &= 15 \\ 2x - 3y + 5z &= 11 \\ 3x + 6y - 4z &= 23 \end{aligned} \right\}.$$

2. Find a quadratic polynomial, say $f(x) = ax^2 + bx + c$, such that

$$f(1) = 1, \qquad f(2) = 9, \qquad f(3) = 27.$$

3. The intersection of the following planes is a line. Find the equation of this line.

$$x + 2y + 3z = 1, \qquad 2x + 3y + 5z = 2, \qquad 2x + y + 3z = 2.$$

4. Solve the system of linear equations

$$\left\{ \begin{aligned} x_1 + 2x_2 + 4x_3 + 6x_4 - 2x_5 &= 9 \\ 2x_1 + 2x_2 + 3x_3 + 8x_4 - x_5 &= 8 \\ 3x_1 + 3x_2 + 2x_3 + 7x_4 - 4x_5 &= 7 \end{aligned} \right\}.$$

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Linear algebra I
Homework #3
due Thursday, Oct. 19

1. Let a, b be some fixed parameters. Solve the system of linear equations

$$\left\{ \begin{aligned} x + ay &= 2 \\ bx + 2y &= 3 \end{aligned} \right\}.$$

2. For which value of a does the following system fail to have solutions?

$$\left\{ \begin{aligned} x + 2y + 3z &= 4 \\ 2x + 3y + 4z &= 3 \\ 2x + 4y + az &= 2 \end{aligned} \right\}.$$

3. Compute the products AB, AC, BC and $ACAB$ in the case that

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 2 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 2 \\ 5 & 4 \\ 1 & 3 \end{bmatrix}.$$

4. A magic square is a square matrix such that the entries in each row, each column and each of the two diagonals have the same sum. The matrix A is a typical example. So is the matrix B, but most of its entries are unknown. Can you find them?

$$A = \begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} a & 3 & b \\ c & d & 1 \\ 2 & e & 7 \end{bmatrix}.$$

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Linear algebra I
Homework #4
due Thursday, Oct. 26

1. For which values of a, b, c is the following matrix invertible? Explain.

$$A = \begin{bmatrix} 1 & 1 & a \\ 2 & 3 & b \\ 1 & 2 & c \end{bmatrix}.$$

2. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}.$$

3. Suppose A, B are $n \times n$ matrices and A has a column of zeros. Show that BA has a column of zeros as well and conclude that A is not invertible.

4. Consider the 2×2 matrix A_x which is defined by

$$A_x = \begin{bmatrix} 1-x & x \\ -x & 1+x \end{bmatrix}.$$

Show that $A_x A_y = A_{x+y}$ for all numbers x, y and conclude that each A_x is invertible.

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Linear algebra I
Homework #5
due Thursday, Nov. 2

1. Use expansion by minors to compute the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & a & 1 \\ 1 & 3 & a \end{bmatrix}.$$

2. Compute the adjoint of the matrix

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 3 & 1 \\ 1 & 2 & a \end{bmatrix}.$$

3. Use row reduction to compute the determinant of the following matrix. Hint: adding the last two rows to the top row should make row reduction a bit easier.

$$A = \begin{bmatrix} x & a & a \\ a & x & a \\ a & a & x \end{bmatrix}.$$

4. Let A_n denote the $n \times n$ matrix whose diagonal entries are equal to 3 and all other entries are equal to 1. Compute the determinant of A_n.

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Linear algebra I
Homework #6
due Thursday, Nov. 16

1. The determinant of a 6×6 matrix A contains the terms

$$a_{14}a_{25}a_{32}a_{46}a_{53}a_{61}, \qquad a_{16}a_{25}a_{34}a_{42}a_{53}a_{61}, \qquad a_{15}a_{26}a_{34}a_{43}a_{51}a_{62}.$$

What is the coefficient of each of these terms?

2. For which values of x is A invertible? Determine the inverse for all such values.

$$A = \begin{bmatrix} 1 & 2 & x \\ 2 & 5 & 1 \\ 1 & x & 2 \end{bmatrix}.$$

3. Determine the inverse of each of the following matrices.

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \qquad B = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}, \qquad C = B^{-1}AB.$$

4. Suppose A is an invertible matrix with $\det A = 1$. Show that $\operatorname{adj}(\operatorname{adj} A) = A$.

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Linear algebra I
Homework #7
due Thursday, Nov. 23

1. Express $\boldsymbol{w}$ as a linear combination of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ in the case that

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \qquad \boldsymbol{w} = \begin{bmatrix} 5 \\ 4 \\ 5 \\ 7 \end{bmatrix}.$$

2. Do the following vectors form a complete set for $\mathbb{R}^4$? Explain.

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \qquad \boldsymbol{v}_4 = \begin{bmatrix} 2 \\ 8 \\ 8 \\ 0 \end{bmatrix}.$$

3. Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly independent vectors in $\mathbb{R}^3$ and let

$$\boldsymbol{w}_1 = \boldsymbol{v}_2 + \boldsymbol{v}_3 - \boldsymbol{v}_1, \qquad \boldsymbol{w}_2 = \boldsymbol{v}_1 + \boldsymbol{v}_3 - \boldsymbol{v}_2, \qquad \boldsymbol{w}_3 = \boldsymbol{v}_1 + \boldsymbol{v}_2 - \boldsymbol{v}_3.$$

Show that the vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ are linearly independent as well.

4. Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ form a complete set for $\mathbb{R}^3$ and let

$$\boldsymbol{w}_1 = \boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3, \qquad \boldsymbol{w}_2 = \boldsymbol{v}_1 + \boldsymbol{v}_3, \qquad \boldsymbol{w}_3 = \boldsymbol{v}_1 + \boldsymbol{v}_2.$$

Show that the vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ form a complete set for $\mathbb{R}^3$ as well.

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Linear algebra I
Homework #8
due Thursday, Nov. 30

1. Find a basis for both the null space and the column space of the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 3 & 6 & 9 \\ 2 & 1 & 6 & 1 \end{bmatrix}.$$

2. Does the vector $\boldsymbol{w}$ belong to the column space of A? Explain.

$$\boldsymbol{w} = \begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 3 & 9 \\ 2 & 1 & 7 \end{bmatrix}.$$

3. Express the polynomial $f(x) = x^2 + 4x - 6$ as a linear combination of

$$f_1(x) = x^2 + x, \qquad f_2(x) = x^2 + 1, \qquad f_3(x) = x + 2.$$

4. Show that the following matrices are linearly independent in M_{22}.

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

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Linear algebra I
Homework #9
due Thursday, Dec. 7

1. Let P_3 be the set of all polynomials of degree at most 3 and let

$$U = \{f \in P_3 : f(2) = f(1) = 2f(0)\}.$$

Show that U is a subspace of P_3 and find a basis for it.

2. Show that the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ form a basis of $\mathbb{R}^3$ and compute the coordinate vector of $\boldsymbol{w}$ with respect to this basis when

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}, \qquad \boldsymbol{w} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}.$$

3. Suppose $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are linearly independent vectors in a vector space V and let

$$\boldsymbol{w}_1 = \boldsymbol{v}_2 + a\boldsymbol{v}_3, \qquad \boldsymbol{w}_2 = \boldsymbol{v}_1 + a\boldsymbol{v}_3, \qquad \boldsymbol{w}_3 = \boldsymbol{v}_1 + a\boldsymbol{v}_2.$$

For which values of a are the vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ linearly independent?

4. Find a linear transformation $T\colon \mathbb{R}^2 \to M_{23}$ such that

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 1 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 0 \end{bmatrix}.$$

- This assignment is due by Thursday noon, either in class or else in my office.
- Write your name and course (Maths, TP, TSM) on the first page of your homework.
- NO LATE HOMEWORK WILL BE ACCEPTED.