Linear algebra I
Homework #1 solutions

1. Show that the diagonals of a square are orthogonal to one another.

Suppose that the side length of the square is x and that the vertices are located at

$$A(0,0), \qquad B(0,x), \qquad C(x,x), \qquad D(x,0).$$

Then the two diagonals of the square are orthogonal to one another because

$$\overrightarrow{AC} = \begin{bmatrix} x \\ x \end{bmatrix}, \qquad \overrightarrow{BD} = \begin{bmatrix} x \\ -x \end{bmatrix} \quad \Longrightarrow \quad \overrightarrow{AC} \cdot \overrightarrow{BD} = x^2 - x^2 = 0.$$

2. Find the equation of the plane that contains $A(1,3,4)$, $B(2,2,3)$ and $C(4,0,2)$.

The normal vector of the plane is given by the cross product

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \times \begin{bmatrix} 3 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}.$$

Since the plane contains the point $A(1,3,4)$, its equation is then

$$-(x-1) - (y-3) + 0(z-4) = 0 \quad \Longrightarrow \quad x + y = 4.$$

3. Find the equation of the plane that contains both the point $(1,2,1)$ and the line

$$x = 2 - t, \qquad y = 1 + 3t, \qquad z = 5 + 4t.$$

Let $P(1,2,1)$ be the given point and pick any two points on the line, say

$$Q(2,1,5) \quad \text{and} \quad R(1,4,9),$$

obtained for $t = 0$ and $t = 1$, respectively. The normal vector of the plane is then

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \times \begin{bmatrix} 0 \\ 2 \\ 8 \end{bmatrix} = \begin{bmatrix} -16 \\ -8 \\ 2 \end{bmatrix}$$

and the plane contains the point $P(1,2,1)$, so its equation is

$$-16(x-1) - 8(y-2) + 2(z-1) = 0 \quad \Longrightarrow \quad 8x + 4y - z = 15.$$

4. Consider the line through $(1,2,3)$ which is perpendicular to the plane

$$2x + 3y + 4z = 6.$$

At which point does this line intersect the plane $3x - 2y + z = 10$?

The line passes through $(1,2,3)$ with direction $\boldsymbol{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, so its equation is

$$x = 1 + 2t, \qquad y = 2 + 3t, \qquad z = 3 + 4t.$$

The point at which the line intersects the plane is the point that satisfies both the equation of the line and that of the plane. This gives

$$\begin{aligned} 3x - 2y + z = 10 \quad &\implies \quad 3(1+2t) - 2(2+3t) + (3+4t) = 10 \\ &\implies \quad 4t = 8, \end{aligned}$$

so $t = 2$ and the point of intersection is the point

$$(x, y, z) = (1 + 2t, 2 + 3t, 3 + 4t) = (5, 8, 11).$$

Linear algebra I
Homework #2 solutions

1. Find the distance between the point $A(1,2,4)$ and the plane $2x+y+2z=6$.

Consider the line through A which is perpendicular to the plane. The direction of this line is the direction of the normal vector, so the equation of the line is

$$x = 1+2t, \qquad y = 2+t, \qquad z = 4+2t.$$

Let P be the point at which the line intersects the plane. At that point, both the equation of the line and that of the plane must hold, so we must have

$$6 = 2x+y+2z = 2(1+2t)+2+t+2(4+2t) = 9t+12.$$

This gives $t = -\frac{6}{9} = -\frac{2}{3}$, while the point P has coordinates

$$(x,y,z) = (1+2t, 2+t, 4+2t).$$

Since the point A has coordinates $(1,2,4)$, the distance between them is

$$\left|\left|\overrightarrow{AP}\right|\right| = \sqrt{(2t)^2+t^2+(2t)^2} = \sqrt{9t^2} = 2.$$

2. Find a quadratic polynomial, say $f(x) = ax^2+bx+c$, such that

$$f(1) = 6, \qquad f(2) = 13, \qquad f(3) = 26.$$

The given conditions hold if and only if

$$a+b+c = 6, \qquad 4a+2b+c = 13, \qquad 9a+3b+c = 26.$$

Once we now use row reduction to solve this linear system, we find that

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 4 & 2 & 1 & 13 \\ 9 & 3 & 1 & 26 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 5 \end{bmatrix}.$$

This gives the unique solution $(a,b,c) = (3,-2,5)$ and so $f(x) = 3x^2-2x+5$.

3. Solve the system of linear equations

$$\left\{\begin{aligned} 2x - 2y + 2z &= 16 \\ 3x - 4y + 2z &= 14 \\ 2x + 3y + 2z &= 31 \end{aligned}\right\}.$$

Using row reduction of the associated matrix, we get

$$\begin{bmatrix} 2 & -2 & 2 & 16 \\ 3 & -4 & 2 & 14 \\ 2 & 3 & 2 & 31 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 1 & 8 \\ 3 & -4 & 2 & 14 \\ 2 & 3 & 2 & 31 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 1 & 8 \\ 0 & -1 & -1 & -10 \\ 0 & 5 & 0 & 15 \end{bmatrix}$$

and this can be further reduced to give

$$\begin{bmatrix} 1 & 0 & 2 & 18 \\ 0 & 1 & 1 & 10 \\ 0 & 0 & -5 & -35 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 18 \\ 0 & 1 & 1 & 10 \\ 0 & 0 & 1 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 7 \end{bmatrix}.$$

In particular, the system has a unique solution which is given by $(x, y, z) = (4, 3, 7)$.

4. Solve the system of linear equations

$$\left\{\begin{aligned} x_1 + 2x_2 + 4x_3 + 5x_4 + 6x_5 &= 2 \\ 2x_1 + x_2 + 5x_3 + 7x_4 + 9x_5 &= 7 \\ 2x_1 + 2x_2 + 6x_3 + 8x_4 + 9x_5 &= 3 \\ x_1 + 5x_2 + 7x_3 + 2x_4 + 3x_5 &= 5 \end{aligned}\right\}.$$

In this case, row reduction leads to the reduced row echelon form

$$\begin{bmatrix} 1 & 2 & 4 & 5 & 6 & 2 \\ 2 & 1 & 5 & 7 & 9 & 7 \\ 2 & 2 & 6 & 8 & 9 & 3 \\ 1 & 5 & 7 & 2 & 3 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 4 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}.$$

Note that the corresponding system of equations reads

$$x_1 + 2x_3 = 4, \qquad x_2 + x_3 = 0, \qquad x_4 = -4, \qquad x_5 = 3.$$

Thus, x_3 is the only free variable and the solution of the system is

$$x_1 = 4 - 2x_3, \qquad x_2 = -x_3, \qquad x_4 = -4, \qquad x_5 = 3.$$

Linear algebra I
Homework #3 solutions

1. Express $\boldsymbol{w}$ as a linear combination of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ in the case that

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \qquad \boldsymbol{u}_2 = \begin{bmatrix} 4 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \qquad \boldsymbol{w} = \begin{bmatrix} 2 \\ 1 \\ 5 \\ 0 \end{bmatrix}.$$

Let A denote the matrix whose columns are the vectors $\boldsymbol{u}_i$. In order to express $\boldsymbol{w}$ as a linear combination of these vectors, we have to solve the system $A\boldsymbol{x} = \boldsymbol{w}$. Since

$$\begin{bmatrix} 1 & 4 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 1 & 5 \\ 2 & 1 & 3 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

the unique solution is $(x_1, x_2, x_3) = (-5, 1, 3)$ and this means that $\boldsymbol{w} = -5\boldsymbol{u}_1 + \boldsymbol{u}_2 + 3\boldsymbol{u}_3$.

2. Show that a system of m linear equations in $n > m$ unknowns cannot have a unique solution. Hint: count the pivots and the rows of the reduced row echelon form.

If the system has a unique solution, then every variable is associated with a pivot, so the reduced row echelon form contains n pivots. These appear in m rows and there is at most one pivot per row, so the total number of pivots is at most $m < n$, a contradiction.

3. The trace of an $n \times n$ matrix A is the sum of its diagonal entries, namely

$$\operatorname{tr} A = A_{11} + A_{22} + \ldots + A_{nn} = \sum_{k=1}^{n} A_{kk}.$$

Show that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for all $n \times n$ matrices A, B.

Using the definition of matrix multiplication, we get

$$\operatorname{tr}(AB) = \sum_{k=1}^{n} (AB)_{kk} = \sum_{k=1}^{n} \sum_{m=1}^{n} A_{km} B_{mk}$$

and then a similar computation gives

$$\operatorname{tr}(BA) = \sum_{m=1}^{n} (BA)_{mm} = \sum_{m=1}^{n} \sum_{k=1}^{n} B_{mk} A_{km} = \operatorname{tr}(AB).$$

4. Suppose A, B are $n \times n$ matrices and A has a row of zeros. Show that AB has a row of zeros as well and conclude that A is not invertible.

Suppose that the ith row of A is a row of zeros. Then we have

$$(AB)_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \ldots + A_{in}B_{nj} = 0$$

for each j, so the ith row of AB is zero as well. Since the identity matrix does not have a row of zeros, we conclude that $AB \neq I_n$ and that A is not invertible.

Linear algebra I
Homework #4 solutions

1. Compute the determinant of the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & a & 2 \\ 2 & 1 & a \end{bmatrix}.$$

Using row reduction to compute the determinant, one finds that

$$\begin{aligned} \det A &= \det \begin{bmatrix} 1 & 1 & 2 \\ 0 & a-1 & 0 \\ 0 & -1 & a-4 \end{bmatrix} = (a-1)\cdot\det \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & -1 & a-4 \end{bmatrix} \\ &= (a-1)\cdot\det \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & a-4 \end{bmatrix} = (a-1)(a-4)\cdot\det \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= (a-1)(a-4). \end{aligned}$$

2. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix}.$$

To find the inverse of A, one can use row reduction to get

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 2 & 2 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -1 & 5 & -7 \\ 0 & 0 & 1 & 1 & -3 & 4 \end{array}\right].$$

In view of this computation, we may conclude that

$$A^{-1} = \begin{bmatrix} 0 & -1 & 2 \\ -1 & 5 & -7 \\ 1 & -3 & 4 \end{bmatrix}.$$

3. Suppose A is a 3×3 matrix whose third row is the sum of the first two rows. Show that A is not invertible and find a vector $\boldsymbol{b}$ such that $A\boldsymbol{x} = \boldsymbol{b}$ has no solutions.

Subtracting the first two rows from the last row, one finds that

$$A = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \alpha_1+\beta_1 & \alpha_2+\beta_2 & \alpha_3+\beta_3 \end{bmatrix} \longrightarrow \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ 0 & 0 & 0 \end{bmatrix}.$$

In particular, the reduced row echelon form of A has a row of zeros, so A is not invertible. This proves the first part; to prove the second part, we consider the equations

$$\left\{ \begin{array}{c} \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = b_1 \\ \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = b_2 \\ \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3 = b_3 \end{array} \right\},$$

where $\gamma_i = \alpha_i + \beta_i$ for each i. Adding the first two equations gives

$$b_1 + b_2 = (\alpha_1+\beta_1)x_1 + (\alpha_2+\beta_2)x_2 + (\alpha_3+\beta_3)x_3 = \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3 x_3$$

and this contradicts the third equation, unless $b_1 + b_2 = b_3$. Thus, the system $A\boldsymbol{x} = \boldsymbol{b}$ has no solutions whenever the entries of the vector $\boldsymbol{b}$ are such that $b_1 + b_2 \neq b_3$.

4. Let A_n denote the $n \times n$ matrix whose diagonal entries are equal to 3 and all other entries are equal to 1. Show that A_n is invertible for each $n \geq 1$.

First, we add the last $n-1$ rows to the first row and divide by $n+2$ to get

$$A_n = \begin{bmatrix} 3 & 1 & \cdots & 1 \\ 1 & 3 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} n+2 & n+2 & \cdots & n+2 \\ 1 & 3 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 3 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 3 \end{bmatrix}.$$

Next, we subtract the first row from every other row to arrive at

$$A_n \longrightarrow \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 3 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & \cdots & 1 \\ & 2 & & \\ & & \ddots & \\ & & & 2 \end{bmatrix}.$$

Dividing the last $n-1$ rows by 2, we can then easily clear the entries above the diagonal. Thus, the reduced row echelon form of A_n is the identity matrix, so A_n is invertible.

Linear algebra I
Homework #5 solutions

1. Compute $\det A$ using (a) expansion by minors and (b) row reduction:

$$A = \begin{bmatrix} 1 & 1 & a \\ 1 & 2 & 1 \\ 2 & a & 2 \end{bmatrix}.$$

(a) Using expansion by minors on the first column, we get

$$\begin{aligned} \det A &= \det \begin{bmatrix} 2 & 1 \\ a & 2 \end{bmatrix} - \det \begin{bmatrix} 1 & a \\ a & 2 \end{bmatrix} + 2 \det \begin{bmatrix} 1 & a \\ 2 & 1 \end{bmatrix} \\ &= (4-a) - (2-a^2) + 2(1-2a) = a^2 - 5a + 4. \end{aligned}$$

(b) Using row reduction, one similarly finds that

$$\begin{aligned} \det A = \det \begin{bmatrix} 1 & 1 & a \\ 0 & 1 & 1-a \\ 0 & a-2 & 2(1-a) \end{bmatrix} &= \det \begin{bmatrix} 1 & 0 & 2a-1 \\ 0 & 1 & 1-a \\ 0 & 0 & (4-a)(1-a) \end{bmatrix} \\ = (4-a)(1-a) \cdot \det \begin{bmatrix} 1 & 0 & 2a-1 \\ 0 & 1 & 1-a \\ 0 & 0 & 1 \end{bmatrix} &= a^2 - 5a + 4. \end{aligned}$$

2. Compute the adjoint and the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 1 \\ 2 & 2 & 5 \end{bmatrix}.$$

The cofactors of the given matrix are

$$C_{11} = \det \begin{bmatrix} 2 & 1 \\ 2 & 5 \end{bmatrix} = 8, \qquad C_{12} = -\det \begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix} = -3, \qquad C_{13} = \det \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} = -2,$$

$$C_{21} = -\det \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = 1, \qquad C_{22} = \det \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = -1, \qquad C_{23} = -\det \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = 0,$$

$$C_{31} = \det \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} = -5, \qquad C_{32} = -\det \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} = 2, \qquad C_{33} = \det \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = 1.$$

This means that the adjoint of A is

$$\operatorname{adj} A = \begin{bmatrix} 8 & -3 & -2 \\ 1 & -1 & 0 \\ -5 & 2 & 1 \end{bmatrix}^t = \begin{bmatrix} 8 & 1 & -5 \\ -3 & -1 & 2 \\ -2 & 0 & 1 \end{bmatrix}.$$

On the other hand, one can easily compute the determinant

$$\det A = \det \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 1 \\ 2 & 2 & 5 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{bmatrix} = -1,$$

so the inverse of the given matrix is

$$A^{-1} = -\operatorname{adj} A = \begin{bmatrix} -8 & -1 & 5 \\ 3 & 1 & -2 \\ 2 & 0 & -1 \end{bmatrix}.$$

3. Suppose A is an invertible $n \times n$ matrix. Express $\det(\operatorname{adj} A)$ in terms of $\det A$.

Using the identity $A \cdot \operatorname{adj} A = (\det A) I_n$, we get

$$\det(A \cdot \operatorname{adj} A) = (\det A)^n \quad \Longrightarrow \quad (\det A) \cdot \det(\operatorname{adj} A) = (\det A)^n.$$

Since A is invertible, however, $\det A \neq 0$ and this implies $\det(\operatorname{adj} A) = (\det A)^{n-1}$.

4. Suppose A is a lower triangular matrix whose diagonal entries are all nonzero. Show that A is invertible and that its inverse is lower triangular.

Since A is lower triangular, its determinant is the product of its diagonal entries and those are all nonzero. Thus, the determinant of A is nonzero and A is invertible.

There are essentially two ways to show that the inverse is lower triangular. First of all, one can try to compute the inverse using row reduction on the augmented matrix

$$[A|I_n] = \left[\begin{array}{cccc|cccc} a_{11} & & & & 1 & & & \\ a_{21} & a_{22} & & & & 1 & & \\ \vdots & \vdots & \ddots & & & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} & & & & 1 \end{array}\right].$$

In this case, we have to divide the first row by a_{11}, clear all entries below the pivot and then proceed in the same way. These operations only affect the entries below the diagonal, so the lower triangular matrix on the right remains lower triangular at each step. As this matrix will eventually become the inverse of A, the inverse of A is lower triangular as well.

Linear algebra I
Homework #6 solutions

1. Suppose that P is an $n\times n$ permutation matrix. Show that $PP^t = I_n$.

We compute the (i,j)th entry of PP^t, which is given by

$$(PP^t)_{ij} = \sum_{k=1}^{n} P_{ik}(P^t)_{kj} = \sum_{k=1}^{n} P_{ik}P_{jk}.$$

When $i\neq j$, each summand is zero because both the ith row and the jth row contain a single nonzero entry, but those occur in different columns. When $i=j$, on the other hand, the sum is equal to 1 for similar reasons. This implies that $PP^t = I_n$.

2. The determinant of a 9×9 matrix A contains the terms

$$a_{18}a_{29}a_{37}a_{41}a_{52}a_{63}a_{76}a_{84}a_{95}, \qquad a_{13}a_{28}a_{36}a_{49}a_{52}a_{61}a_{77}a_{85}a_{94}.$$

What is the coefficient of each of these terms?

The coefficient of the first term is

$$\operatorname{sign}\begin{pmatrix}1&2&3&4&5&6&7&8&9\\8&9&7&1&2&3&6&4&5\end{pmatrix} = \operatorname{sign}(184)(295)(376) = (-1)^{2+2+2} = 1.$$

The coefficient of the second term is

$$\operatorname{sign}\begin{pmatrix}1&2&3&4&5&6&7&8&9\\3&8&6&9&2&1&7&5&4\end{pmatrix} = \operatorname{sign}(136)(285)(49) = (-1)^{2+2+1} = -1.$$

3. Determine both the null space and the column space of the matrix

$$A = \begin{bmatrix}1&4&6&1&6\\1&2&4&1&4\\2&1&5&1&4\end{bmatrix}.$$

The reduced row echelon form of A is

$$\begin{bmatrix}1&0&2&0&1\\0&1&1&0&1\\0&0&0&1&1\end{bmatrix}.$$

Thus, the null space and the column space of A are

$$\mathcal{N}(A) = \operatorname{Span}\left\{\begin{bmatrix}-2\\-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\-1\\0\\-1\\1\end{bmatrix}\right\}, \qquad \mathcal{C}(A) = \operatorname{Span}\left\{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}4\\2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}.$$

4. Suppose that A is a square matrix whose column space is equal to its null space. Show that A^2 must be the zero matrix.

Since $A\boldsymbol{e}_j$ is the jth column of A, it belongs to the column space of A, hence also to the null space. This means that $A(A\boldsymbol{e}_j) = 0$ for all j and thus $A^2\boldsymbol{e}_j = 0$ for all j. In particular, every column of A^2 must be zero and so A^2 is the zero matrix.

Linear algebra I
Homework #7 solutions

1. Suppose that the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ form a complete set in $\mathbb{R}^n$ and that they are linearly independent. Show that $k = n$ and that the matrix whose columns are these vectors is invertible.

According to the theorems we proved in class, one needs to have $k \geq n$ for the vectors to form a complete set and $k \leq n$ for the vectors to be linearly independent. This proves the first part. To prove the second part, let A be the matrix whose columns are the given vectors. The reduced row echelon form of A must have a pivot in each row and column, so the reduced row echelon form is the identity matrix and A is invertible.

2. Is the matrix A a linear combination of the other three matrices? Explain.

$$A = \begin{bmatrix} 4 & 9 \\ 8 & 5 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}.$$

We try to find coefficients x_1, x_2, x_3 such that $x_1 B_1 + x_2 B_2 + x_3 B_3 = A$, namely

$$\begin{bmatrix} x_1 + x_2 + x_3 & 2x_1 + 3x_2 + x_3 \\ x_1 + 2x_2 + 3x_3 & x_1 + x_2 + 2x_3 \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ 8 & 5 \end{bmatrix}.$$

This leads to the system of equations

$$\left\{ \begin{aligned} x_1 + x_2 \quad + x_3 \quad &= 4 \\ 2x_1 + 3x_2 + x_3 \quad &= 9 \\ x_1 + 2x_2 + 3x_3 &= 8 \\ x_1 + x_2 \quad + 2x_3 &= 5 \end{aligned} \right\}$$

and we can use row reduction to get

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 3 & 1 & 9 \\ 1 & 2 & 3 & 8 \\ 1 & 1 & 2 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, the system has the unique solution $(x_1, x_2, x_3) = (1, 2, 1)$ and so $A = B_1 + 2B_2 + B_3$.

3. Show that the following matrices are linearly independent in M_{22}.

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Suppose that some linear combination of these matrices gives the zero matrix, say

$$x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4 = 0$$

for some coefficients x_1, x_2, x_3, x_4. Expanding the left hand side, we then get

$$\begin{bmatrix} x_1 + x_2 + x_3 & x_3 + x_4 \\ x_2 + x_3 + x_4 & x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and this leads to the system of equations

$$x_1 + x_2 + x_3 = 0 = x_1 + x_2, \qquad x_3 + x_4 = 0 = x_2 + x_3 + x_4.$$

Based on the leftmost equations, we must have $x_3 = 0$. Based on the rightmost equations, we must also have $x_2 = 0$. It now easily follows that x_1 and x_4 are zero as well.

4. Suppose $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ are linearly independent vectors of a vector space V. Show that the vectors $\boldsymbol{u}$, $\boldsymbol{u}+\boldsymbol{v}$, $\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}$ are linearly independent as well.

Suppose that some linear combination of these vectors is zero, say

$$x\boldsymbol{u} + y(\boldsymbol{u}+\boldsymbol{v}) + z(\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}) = 0 \quad \Longrightarrow \quad (x+y+z)\boldsymbol{u} + (y+z)\boldsymbol{v} + z\boldsymbol{w} = 0.$$

Since $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ are linearly independent by assumption, this actually implies

$$x + y + z = y + z = z = 0 \quad \Longrightarrow \quad x = y = z = 0.$$

Linear algebra I
Homework #8 solutions

1. Let U be the set of all polynomials $f \in P_3$ such that $f(0) = f(1)$. Show that U is a subspace of P_3 and find a basis for it.

By assumption, U consists of all polynomials $f(x) = ax^3 + bx^2 + cx + d$ such that

$$f(0) = f(1) \quad \Longleftrightarrow \quad d = a + b + c + d \quad \Longleftrightarrow \quad c = -a - b.$$

In other words, U consists of all polynomials of the form

$$f(x) = ax^3 + bx^2 - (a+b)x + d = a(x^3 - x) + b(x^2 - x) + d$$

so $U = \mathrm{Span}\{x^3 - x, x^2 - x, 1\}$. Since U is the span of three polynomials, it is certainly a subspace. To show that these polynomials form a basis, suppose that

$$a(x^3 - x) + b(x^2 - x) + d = 0$$

for some scalars a, b, d. Comparing the coefficients of $x^3, x^2, 1$ on both sides, it is then easy to see that $a = b = d = 0$. Thus, the three polynomials are also linearly independent.

2. Show that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ form a basis of $\mathbb{R}^3$ and then find the coordinate vector of $\boldsymbol{v}$ with respect to this basis when

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \qquad \boldsymbol{v} = \begin{bmatrix} 7 \\ 5 \\ 5 \end{bmatrix}.$$

The coordinate vector of $\boldsymbol{v}$ is the rightmost column in the row reduction

$$\begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 7 \\ 2 & 1 & 1 & 5 \\ 1 & 3 & 3 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix}.$$

To show that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ form a basis of $\mathbb{R}^3$, we have to show that the matrix whose columns are these vectors is invertible. In view of the row reduction above, the reduced row echelon form of that matrix is the identity, so the matrix is certainly invertible.

3. Show that $\boldsymbol{w}_1, \boldsymbol{w}_2$ form a basis of $\mathbb{R}^2$ when

$$\boldsymbol{w}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{w}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Compute the coordinate vectors of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ with respect to this basis.

The vectors $\boldsymbol{w}_1, \boldsymbol{w}_2$ form a basis of $\mathbb{R}^2$ because

$$\det \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} = 2 - 3 \neq 0.$$

The coordinate vectors of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are the two rightmost columns in the row reduction

$$\begin{bmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \boldsymbol{e}_1 & \boldsymbol{e}_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 1 & -2 \end{bmatrix}.$$

4. Let $\boldsymbol{w}_1, \boldsymbol{w}_2$ be as above. Find a linear transformation $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$T(\boldsymbol{w}_1) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \qquad T(\boldsymbol{w}_2) = \begin{bmatrix} 3 \\ 9 \end{bmatrix}.$$

We know that T is left multiplication by a matrix A whose ith column is $T(\boldsymbol{e}_i)$. Let us then express each $\boldsymbol{e}_i$ in terms of the given vectors. By the previous problem, we have

$$\boldsymbol{e}_1 = -\boldsymbol{w}_1 + \boldsymbol{w}_2 \quad \text{and} \quad \boldsymbol{e}_2 = 3\boldsymbol{w}_1 - 2\boldsymbol{w}_2.$$

Using the fact that T is linear, we conclude that

$$T(\boldsymbol{e}_1) = -T(\boldsymbol{w}_1) + T(\boldsymbol{w}_2) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \qquad T(\boldsymbol{e}_2) = 3T(\boldsymbol{w}_1) - 2T(\boldsymbol{w}_2) = \begin{bmatrix} -3 \\ -3 \end{bmatrix}.$$

These are the two columns of the matrix A, hence

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & -3 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x - 3y \\ 4x - 3y \end{bmatrix}.$$