Linear algebra I
Homework #1
due Thursday, Oct. 10

1. Show that the diagonals of a square are orthogonal to one another.

Hint: Place the vertices of the square along the axes and then introduce coordinates.

2. Find the equation of the plane which contains $A(1,3,4)$, $B(2,2,3)$ and $C(4,0,2)$.

3. Find the equation of the plane which contains both the point $(1,2,1)$ and the line

$$x = 2 - t, \qquad y = 1 + 3t, \qquad z = 5 + 4t.$$

4. Consider the line through $(1,2,3)$ which is perpendicular to the plane

$$2x + 3y + 4z = 6.$$

At which point does this line intersect the plane $3x - 2y + z = 10$?

- When writing up solutions, write legibly and coherently.
- Write your name and then MATHS/TP/TSM on the first page of your homework.
- NO LATE HOMEWORK WILL BE ACCEPTED.

Linear algebra I
Homework #2
due Thursday, Oct. 17

1. Find the distance between the point $A(1,2,4)$ and the plane $2x+y+2z=6$.

2. Find a quadratic polynomial, say $f(x)=ax^2+bx+c$, such that

$$f(1)=6, \qquad f(2)=13, \qquad f(3)=26.$$

3. Solve the system of linear equations

$$\left\{\begin{array}{l} 2x-2y+2z=16 \\ 3x-4y+2z=14 \\ 2x+3y+2z=31 \end{array}\right\}.$$

4. Solve the system of linear equations

$$\left\{\begin{array}{r} x_1+2x_2+4x_3+5x_4+6x_5=2 \\ 2x_1+x_2+5x_3+7x_4+9x_5=7 \\ 2x_1+2x_2+6x_3+8x_4+9x_5=3 \\ x_1+5x_2+7x_3+2x_4+3x_5=5 \end{array}\right\}.$$

- When writing up solutions, write legibly and coherently.
- Write your name and then MATHS/TP/TSM on the first page of your homework.
- NO LATE HOMEWORK WILL BE ACCEPTED.

Linear algebra I
Homework #3
due Thursday, Oct. 24

1. Express $\boldsymbol{w}$ as a linear combination of $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ in the case that

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \qquad \boldsymbol{u}_2 = \begin{bmatrix} 4 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \qquad \boldsymbol{w} = \begin{bmatrix} 2 \\ 1 \\ 5 \\ 0 \end{bmatrix}.$$

2. Show that a system of m linear equations in $n > m$ unknowns cannot have a unique solution. Hint: count the pivots and the rows of the reduced row echelon form.

3. The trace of an $n \times n$ matrix A is the sum of its diagonal entries, namely

$$\operatorname{tr} A = A_{11} + A_{22} + \ldots + A_{nn} = \sum_{k=1}^{n} A_{kk}.$$

Show that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for all $n \times n$ matrices A, B.

4. Suppose A, B are $n \times n$ matrices and A has a row of zeros. Show that AB has a row of zeros as well and conclude that A is not invertible.

- When writing up solutions, write legibly and coherently.
- Write your name and then MATHS/TP/TSM on the first page of your homework.
- NO LATE HOMEWORK WILL BE ACCEPTED.

Linear algebra I
Homework #4
due Thursday, Oct. 31

1. Compute the determinant of the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & a & 2 \\ 2 & 1 & a \end{bmatrix}.$$

2. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix}.$$

3. Suppose A is a 3×3 matrix whose third row is the sum of the first two rows. Show that A is not invertible and find a vector $\boldsymbol{b}$ such that $A\boldsymbol{x} = \boldsymbol{b}$ has no solutions.

Hint: use row reduction for the first part; write down the equations for the second part.

4. Let A_n denote the $n \times n$ matrix whose diagonal entries are equal to 3 and all other entries are equal to 1. Show that A_n is invertible for each $n \geq 1$.

Hint: if you add the last $n - 1$ rows to the first row, then row reduction becomes somewhat easier; work out the cases $n = 2, 3$ first.

- When writing up solutions, write legibly and coherently.
- Write your name and then MATHS/TP/TSM on the first page of your homework.
- NO LATE HOMEWORK WILL BE ACCEPTED.

Linear algebra I
Homework #5
due Thursday, Nov. 14

1. Compute $\det A$ using (a) expansion by minors and (b) row reduction:

$$A = \begin{bmatrix} 1 & 1 & a \\ 1 & 2 & 1 \\ 2 & a & 2 \end{bmatrix}.$$

2. Compute the adjoint and the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 1 \\ 2 & 2 & 5 \end{bmatrix}.$$

3. Suppose A is an invertible $n \times n$ matrix. Express $\det(\operatorname{adj} A)$ in terms of $\det A$.

4. Suppose A is a lower triangular matrix whose diagonal entries are all nonzero. Show that A is invertible and that its inverse is lower triangular.

Hint: Tutorial problems #2 should be useful for the first part; the second part is related to the adjoint of A.

- When writing up solutions, write legibly and coherently.
- Write your name and then MATHS/TP/TSM on the first page of your homework.
- NO LATE HOMEWORK WILL BE ACCEPTED.

Linear algebra I
Homework #6
due Thursday, Nov. 21

1. Suppose that P is an $n \times n$ permutation matrix. Show that $PP^t = I_n$.

2. The determinant of a 9×9 matrix A contains the terms

$$a_{18}a_{29}a_{37}a_{41}a_{52}a_{63}a_{76}a_{84}a_{95}, \qquad a_{13}a_{28}a_{36}a_{49}a_{52}a_{61}a_{77}a_{85}a_{94}.$$

What is the coefficient of each of these terms?

3. Determine both the null space and the column space of the matrix

$$A = \begin{bmatrix} 1 & 4 & 6 & 1 & 6 \\ 1 & 2 & 4 & 1 & 4 \\ 2 & 1 & 5 & 1 & 4 \end{bmatrix}.$$

4. Suppose that A is a square matrix whose column space is equal to its null space. Show that A^2 must be the zero matrix.

- When writing up solutions, write legibly and coherently.
- Write your name and then MATHS/TP/TSM on the first page of your homework.
- NO LATE HOMEWORK WILL BE ACCEPTED.

Linear algebra I
Homework #7
due Thursday, Nov. 28

1. Suppose that the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ form a complete set in $\mathbb{R}^n$ and that they are linearly independent. Show that $k = n$ and that the matrix whose columns are these vectors is invertible.

2. Is the matrix A a linear combination of the other three matrices? Explain.

$$A = \begin{bmatrix} 4 & 9 \\ 8 & 5 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}.$$

3. Show that the following matrices are linearly independent in M_{22}.

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

4. Suppose $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ are linearly independent vectors of a vector space V. Show that the vectors $\boldsymbol{u}$, $\boldsymbol{u}+\boldsymbol{v}$, $\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}$ are linearly independent as well.

- When writing up solutions, write legibly and coherently.
- Write your name and then MATHS/TP/TSM on the first page of your homework.
- NO LATE HOMEWORK WILL BE ACCEPTED.

Linear algebra I
Homework #8
due Thursday, Dec. 5

1. Let U be the set of all polynomials $f \in P_3$ such that $f(0) = f(1)$. Show that U is a subspace of P_3 and find a basis for it.

2. Show that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ form a basis of $\mathbb{R}^3$ and then find the coordinate vector of $\boldsymbol{v}$ with respect to this basis when

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \qquad \boldsymbol{v} = \begin{bmatrix} 7 \\ 5 \\ 5 \end{bmatrix}.$$

3. Show that $\boldsymbol{w}_1, \boldsymbol{w}_2$ form a basis of $\mathbb{R}^2$ when

$$\boldsymbol{w}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{w}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Compute the coordinate vectors of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ with respect to this basis.

4. Let $\boldsymbol{w}_1, \boldsymbol{w}_2$ be as above. Find a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$T(\boldsymbol{w}_1) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \qquad T(\boldsymbol{w}_2) = \begin{bmatrix} 3 \\ 9 \end{bmatrix}.$$

Hint: express each of $\boldsymbol{e}_1, \boldsymbol{e}_2$ as a linear combination of $\boldsymbol{w}_1, \boldsymbol{w}_2$ and then use linearity to determine each of $T(\boldsymbol{e}_1), T(\boldsymbol{e}_2)$.

- When writing up solutions, write legibly and coherently.
- Write your name and then MATHS/TP/TSM on the first page of your homework.
- NO LATE HOMEWORK WILL BE ACCEPTED.