4.4 Local minima and maxima

Definition 4.20 – Local minimum and maximum

We say that f has a local minimum at x_0, if there is an interval I around x_0 such that

$$f(x_0) \leq f(x) \quad \text{for all } x \in I.$$

We say that f has a local maximum at x_0, if there is an interval I around x_0 such that

$$f(x_0) \geq f(x) \quad \text{for all } x \in I.$$

Theorem 4.21 – First derivative test

Suppose that f is differentiable on some interval around the point x_0.

(a) If f' changes from being negative to being positive at the point x_0, then f changes from being decreasing to being increasing, so f has a local minimum at x_0.

(b) If f' changes from being positive to being negative at the point x_0, then f changes from being increasing to being decreasing, so f has a local maximum at x_0.

Theorem 4.22 – Second derivative test

Suppose that f is twice differentiable at the point x_0.

(a) If $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has a local minimum at x_0.

(b) If $f'(x_0) = 0$ and $f''(x_0) < 0$, then f has a local maximum at x_0.

Example 4.23 We use the second derivative test to find the local minima/maxima of

$$f(x) = x^3 + 3x^2 - 9x.$$

To find the points at which the first derivative is zero, we note that

$$f'(x) = 3x^2 + 6x - 9 = 3(x^2 + 2x - 3) = 3(x - 1)(x + 3).$$

The solutions of $f'(x) = 0$ are thus $x = 1$ and $x = -3$. When it comes to the former,

$$f''(x) = 6x + 6 \quad \Longrightarrow \quad f''(1) = 6 + 6 = 12,$$

so f attains a local minimum at the point $x = 1$. When it comes to the latter,

$$f''(x) = 6x + 6 \quad \Longrightarrow \quad f''(-3) = -18 + 6 = -12,$$

so f attains a local maximum at the point $x = -3$. $\square$

Example 4.24 We use the first derivative test to find the local minima/maxima of

$$f(x) = \frac{3x+4}{x^2+1}.$$

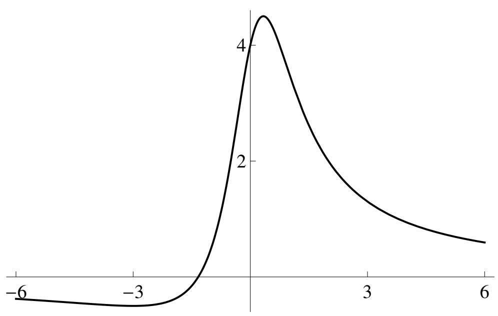
According to the quotient rule, the derivative of this function is given by

$$f'(x) = \frac{3(x^2+1) - 2x(3x+4)}{(x^2+1)^2} = \frac{-3x^2 - 8x + 3}{(x^2+1)^2}.$$

The quadratic in the numerator has roots $x_1 = 1/3$ and $x_2 = -3$, so one may also write

$$f'(x) = -\frac{3(x-x_1)(x-x_2)}{(x^2+1)^2} = \frac{(1-3x)(x+3)}{(x^2+1)^2}.$$

We now determine the sign of $f'(x)$ using the table below. When $x = -3$, the derivative changes from being negative to being positive, so f has a local minimum. When $x = 1/3$, the derivative changes from being positive to being negative, so f has a local maximum. □



	-3		$1/3$	
$1-3x$	$+$		$+$	$-$
$x+3$	$-$		$+$	$+$
$f'(x)$	$-$		$+$	$-$

Figure 4.5: The graph of $f(x) = \dfrac{3x+4}{x^2+1}$.

Example 4.25 We use the second derivative test to find the local minima/maxima of

$$f(x) = x^4 - 4x^2 + 3.$$

To find the points at which the first derivative is zero, we note that

$$f'(x) = 4x^3 - 8x = 4x(x^2-2) = 4x(x-\sqrt{2})(x+\sqrt{2}).$$

The solutions of $f'(x) = 0$ are thus $x = 0$ and $x = \pm\sqrt{2}$. When it comes to the first point,

$$f''(x) = 12x^2 - 8 \quad\Longrightarrow\quad f''(0) = -8,$$

so f attains a local maximum at the point $x = 0$. When it comes to the other two points,

$$f''(x) = 12x^2 - 8 \quad\Longrightarrow\quad f''(\pm\sqrt{2}) = 12 \cdot 2 - 8 = 16,$$

so f attains a local minimum at each of the points $x = \pm\sqrt{2}$. □

4.5 Global minima and maxima

Definition 4.26 – Global minimum and maximum

Consider a function f with domain A and let $x_0 \in A$ be a given point.

(a) We say that f has a global minimum at x_0, if $f(x_0) \leq f(x)$ for all $x \in A$.

(b) We say that f has a global maximum at x_0, if $f(x_0) \geq f(x)$ for all $x \in A$.

Theorem 4.27 – Unique change of sign

Suppose that f is differentiable and f' changes sign exactly once. If f' changes from being negative to being positive at x_0, then f has a global minimum at x_0. If f' changes from being positive to being negative at x_0, then f has a global maximum at x_0.

Theorem 4.28 – Continuous functions on finite intervals

Suppose that f is continuous on the finite interval $[a, b]$. Then f attains both a global minimum and a global maximum. In fact, these may only occur at the endpoints a, b, the points at which $f'(x)$ is zero and the points at which $f'(x)$ does not exist.

Example 4.29 We find the global minimum/maximum values that are attained by

$$f(x) = x^3 - 3x, \qquad 0 \leq x \leq 2.$$

This function is differentiable at all points and its derivative is given by

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x + 1)(x - 1).$$

Thus, the global minimum/maximum values may only occur at the points

$$x = -1, \qquad x = 1, \qquad x = 0, \qquad x = 2.$$

We exclude the leftmost point, as it does not lie in the given interval, and we compute

$$f(1) = 1^3 - 3 = -2, \qquad f(0) = 0, \qquad f(2) = 2^3 - 3 \cdot 2 = 2.$$

This means that the minimum value is $f(1) = -2$ and the maximum value is $f(2) = 2$. $\square$

Example 4.30 We use Theorem 4.27 in order to establish the inequality

$$xe^{-x} \leq e^{-1} \quad \text{for all } x.$$

Consider the function f that is defined by $f(x) = xe^{-x}$. As one can easily check,

$$f'(x) = e^{-x} + x \cdot (e^{-x}) \cdot (-x)' = e^{-x} - xe^{-x} = (1 - x)e^{-x}.$$

This implies that $f'(x)$ is positive when $x < 1$ and $f'(x)$ is negative when $x > 1$. Thus, f attains a global maximum at the point $x = 1$ and one has $f(x) \leq f(1) = e^{-1}$ for all x. $\square$

Example 4.31 We find the global minimum and maximum values that are attained by

$$f(x) = \sin x + \cos x, \qquad 0 \leq x \leq 2\pi.$$

This function is differentiable at all points and its derivative is given by

$$f'(x) = \cos x - \sin x.$$

To say that $f'(x) = 0$ is to say that $\sin x = \cos x$ and this is true precisely when $\tan x = 1$. Thus, the only points at which the minimum/maximum values may occur are the points

$$x = 0, \qquad x = 2\pi, \qquad x = \pi/4, \qquad x = 5\pi/4.$$

As one can easily check, the corresponding values of $f(x)$ are

$$f(0) = f(2\pi) = 1, \qquad f(\pi/4) = \sqrt{2}, \qquad f(5\pi/4) = -\sqrt{2}.$$

Thus, the minimum value is $f(5\pi/4) = -\sqrt{2}$ and the maximum value is $f(\pi/4) = \sqrt{2}$. □

Example 4.32 We use Theorem 4.27 in order to establish the inequality

$$e^x \geq x + 1 \quad \text{for all } x.$$

Consider the function f that is defined by $f(x) = e^x - x - 1$. Its derivative is

$$f'(x) = e^x - 1 = e^x - e^0,$$

so it is negative when $x < 0$ and it is positive when $x > 0$. This implies that f attains a global minimum at the point $x = 0$. In particular, one has $f(x) \geq f(0) = 0$ for all x. □

Example 4.33 We find the global minimum and maximum values that are attained by

$$f(x) = x\sqrt{4 - x^2}, \qquad -2 \leq x \leq 2.$$

Using both the product rule and the chain rule, one may easily check that

$$f'(x) = \sqrt{4 - x^2} + x \cdot \frac{-2x}{2\sqrt{4 - x^2}} = \sqrt{4 - x^2} - \frac{x^2}{\sqrt{4 - x^2}} = \frac{4 - 2x^2}{\sqrt{4 - x^2}}.$$

Thus, the only points at which the minimum/maximum values may occur are the points

$$x = -2, \qquad x = 2, \qquad x = -\sqrt{2}, \qquad x = \sqrt{2}.$$

The corresponding values that are attained by $f(x)$ are given by

$$f(-2) = f(2) = 0, \qquad f(-\sqrt{2}) = -2, \qquad f(\sqrt{2}) = 2.$$

This makes $f(-\sqrt{2}) = -2$ the minimum value and $f(\sqrt{2}) = 2$ the maximum value. □

4.6 Optimisation

- There are several standard problems that ask for the minimum/maximum value of a function which represents a physical quantity such as length, area and volume.
- To solve these problems, one introduces the variables of interest and expresses them in terms of a single variable x. This variable is not usually arbitrary, as length, area and volume must be non-negative. One must thus determine the exact restrictions on x.

Example 4.34 Out of all rectangles of perimeter 40, which one has the largest area? To answer this question, we denote by x, y the two sides of the rectangle and we note that

$$2x + 2y = 40 \quad \Longrightarrow \quad x + y = 20 \quad \Longrightarrow \quad y = 20 - x.$$

To maximise the area A of the rectangle, we first express it in terms of x, namely

$$A = xy = x(20 - x) = 20x - x^2.$$

Next, we determine the restrictions on x. Since the lengths x, y must be non-negative, we need to have $x \geq 0$ and $y = 20 - x \geq 0$. Thus, we are looking for the maximum value of

$$A(x) = 20x - x^2, \qquad 0 \leq x \leq 20.$$

Since $A'(x) = 20 - 2x$, the only points at which the maximum value may occur are the points

$$x = 0, \qquad x = 20, \qquad x = 10.$$

The corresponding values that are attained by $A(x)$ are given by

$$A(0) = A(20) = 0, \qquad A(10) = 200 - 100 = 100.$$

Thus, the largest area arises when $x = y = 10$, in which case the rectangle is a square. $\square$

Example 4.35 If a right triangle has a hypotenuse of length 6, how large can its area be? In this case, we let x, y be the other two sides and we use Pythagoras' theorem to get

$$x^2 + y^2 = 6^2 \quad \Longrightarrow \quad y^2 = 36 - x^2 \quad \Longrightarrow \quad y = \sqrt{36 - x^2}.$$

Eliminating y, one may express the area of the right triangle in the form

$$A = \frac{xy}{2} = \frac{x\sqrt{36 - x^2}}{2}.$$

Since A becomes maximum if and only if A^2 becomes maximum, it suffices to maximise

$$f(x) = \frac{x^2(36 - x^2)}{4} = 9x^2 - \frac{x^4}{4}, \qquad 0 \leq x \leq 6.$$

The derivative of this function is easy to compute and one has

$$f'(x) = 18x - x^3 = x(18 - x^2).$$

Thus, the only points at which the maximum value may occur are the points

$$x = 0, \qquad x = 6, \qquad x = \sqrt{18}.$$

The corresponding values that are attained by $f(x)$ are given by

$$f(0) = f(6) = 0, \qquad f(\sqrt{18}) = \frac{18 \cdot 18}{4} = 9^2.$$

In particular, the maximum value is $f(\sqrt{18}) = 9^2$ and the largest possible area is 9. $\square$

Example 4.36 A cylinder is formed when a rectangle is rotated around one of its sides. If the rectangle has perimeter 6, then how large can the volume of the cylinder be? Let x be the side of the rectangle which lies along the line of rotation and let y be the other side. Then x becomes the height of the cylinder and y becomes the radius, so the volume of the cylinder is given by $V = \pi y^2 x$. Since $2x + 2y = 6$ by assumption, one may write

$$V(y) = \pi y^2 x = \pi y^2(3 - y) = 3\pi y^2 - \pi y^3.$$

To ensure that the lengths x, y are non-negative, we need to assume that $0 \leq y \leq 3$. Since

$$V'(y) = 6\pi y - 3\pi y^2 = 3\pi y(2 - y),$$

the only points at which the maximum value may occur are the points

$$y = 0, \qquad y = 3, \qquad y = 2.$$

Since $V(0) = V(3) = 0$ and $V(2) = 4\pi$, the largest possible volume is $V(2) = 4\pi$. $\square$

Example 4.37 We find the point on the line $y = 2x + 1$ which is closest to $A(3, 6)$. In this case, we need to minimise the distance d between (x, y) and $(3, 6)$, namely

$$d = \sqrt{(x-3)^2 + (y-6)^2} = \sqrt{(x-3)^2 + (2x-5)^2}.$$

Since d becomes minimum if and only if d^2 becomes minimum, it suffices to minimise

$$f(x) = (x-3)^2 + (2x-5)^2.$$

The derivative of this function is easily found to be

$$f'(x) = 2(x-3) + 2(2x-5) \cdot 2 = 2(x - 3 + 4x - 10) = 2(5x - 13).$$

Thus, $f'(x)$ is negative when $x < 13/5$ and positive when $x > 13/5$. This implies that f has a global minimum at $x = 13/5$, so the closest point on the line is $(13/5, 27/5)$. $\square$

Example 4.38 Out of all rectangles of area $a > 0$, which one has the smallest perimeter? To answer this question, we let x, y be the two sides of the rectangle and we note that $xy = a$. The perimeter of the rectangle is $2x + 2y$ and this can be expressed in the form

$$P(x) = 2x + 2y = 2x + \frac{2a}{x}.$$

The only restriction on x is that $x > 0$, while the derivative of $P(x)$ is

$$P'(x) = 2 - \frac{2a}{x^2} = \frac{2(x^2 - a)}{x^2}.$$

This gives $P'(x) < 0$ when $0 < x < \sqrt{a}$ and $P'(x) > 0$ when $x > \sqrt{a}$, so $P(x)$ attains its minimum when $x = y = \sqrt{a}$. The rectangle of smallest perimeter is thus a square. $\square$

4.7 Related rates

- Suppose that two or more variables are related by some equation. Then one may use implicit differentiation to see that their derivatives are related as well.
- This situation arises frequently when quantities such as length, area and volume are varying with time. We shall thus mainly focus on functions of time t.

Example 4.39 If the radius of a circle is increasing at the rate of 1 cm/sec, how fast is the area of the circle changing when the radius is 3 cm? In this case, the variables of interest are the radius $r(t)$ and the area $A(t)$. They are related by the usual formula

$$A(t) = \pi r(t)^2$$

and one may differentiate both sides of this equation to find that

$$A'(t) = \pi \cdot 2r(t) \cdot r'(t).$$

At the given moment, $r'(t) = 1$ and $r(t) = 3$, so it easily follows that $A'(t) = 6\pi$. $\square$

Example 4.40 A ladder that is 10 ft long is resting against a wall. If its base starts sliding along the floor at the rate of 1 ft/sec, how fast is the top of the ladder sliding down the wall when the base is 6 ft away from the wall? To solve this problem, let x be the horizontal distance between the base and the wall, and let y be the vertical distance between the top of the ladder and the floor. According to Pythagoras' theorem, one must have

$$x(t)^2 + y(t)^2 = 10^2 \quad \Longrightarrow \quad 2x(t)x'(t) + 2y(t)y'(t) = 0.$$

At the given moment, $x'(t) = 1$ and $x(t) = 6$, so the last equation gives

$$y(t)y'(t) = -x(t)x'(t) = -6 \quad \Longrightarrow \quad y'(t) = -\frac{6}{y(t)}.$$

Using Pythagoras' theorem to determine the remaining variable $y(t)$, we conclude that

$$y(t) = \sqrt{10^2 - x(t)^2} = \sqrt{10^2 - 6^2} = 8 \quad \Longrightarrow \quad y'(t) = -\frac{6}{8} = -\frac{3}{4}. \qquad \square$$

Example 4.41 A girl flies a kite at a constant height of 30 metres above her hand and the wind is carrying the kite horizontally at a rate of 2 m/sec. How fast must she let out the string when the kite is 50 metres away from her? Let $x(t)$ be the horizontal distance between the girl and the kite, and let $z(t)$ be the length of the string. We must then have

$$x(t)^2 + 30^2 = z(t)^2 \quad \Longrightarrow \quad 2x(t)x'(t) = 2z(t)z'(t).$$

Since $x'(t) = 2$ and $z(t) = 50$ by assumption, it easily follows that

$$z'(t) = \frac{x(t)x'(t)}{z(t)} = \frac{2\sqrt{50^2 - 30^2}}{50} = \frac{2 \cdot 40}{50} = \frac{8}{5}. \qquad \square$$

4.8 Linear approximation

Definition 4.42 – Linear approximation

Given a function f which is differentiable at the point x_0, we say that

$$L(x) = f'(x_0) \cdot (x - x_0) + f(x_0) \tag{4.8}$$

is the tangent line approximation or linear approximation of f at the point x_0.

- The linear approximation is merely the linear function that best approximates $f(x)$ near the point x_0. In fact, the points x which are sufficiently close to x_0 satisfy

$$\frac{f(x) - f(x_0)}{x - x_0} \approx f'(x_0) \quad \Longrightarrow \quad f(x) \approx f'(x_0) \cdot (x - x_0) + f(x_0).$$

Example 4.43 The linear approximation to $f(x) = \sin x$ at the point $x_0 = 0$ is given by

$$L(x) = f'(0) \cdot (x - 0) + f(0).$$

Since $f(0) = \sin 0 = 0$ and $f'(0) = \cos 0 = 1$, the linear approximation is thus $L(x) = x$. One may use this approximation to argue that $\sin x \approx x$ for all small enough x. □

Example 4.44 We find the linear approximation to $f(x)$ at the point x_0 in the case that

$$f(x) = \frac{4x^2 - 5x - 1}{x + 1}, \qquad x_0 = 1.$$

To find the derivative of $f(x)$ at the given point, we use the quotient rule to get

$$f'(x) = \frac{(8x - 5) \cdot (x + 1) - (4x^2 - 5x - 1)}{(x + 1)^2} \quad \Longrightarrow \quad f'(1) = \frac{3 \cdot 2 + 2}{4} = 2.$$

Since $f(1) = -2/2 = -1$, the linear approximation is $L(x) = 2(x - 1) - 1 = 2x - 3$. □

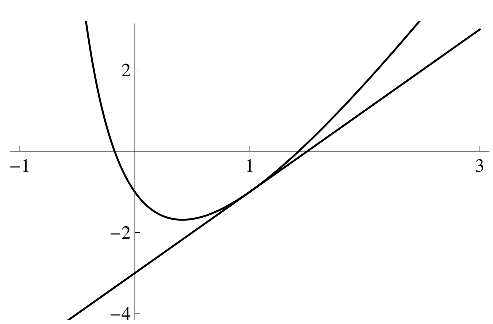


Figure 4.6: The graph of $f(x) = \dfrac{4x^2 - 5x - 1}{x + 1}$ and its tangent line at $x = 1$.

4.9 Newton's method

- Newton's method is a standard approach for approximating the roots of an equation of the form $f(x) = 0$. The method does not always work, but it works quite often.
- The idea is to start with an initial guess x_1, find the tangent line to f at that point and determine the point x_2 at which the line meets the x-axis. One may then use x_2 as a second guess and proceed in this manner to obtain the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{for each } n \geq 1. \tag{4.9}$$

Example 4.45 We use Newton's method to approximate $\sqrt{2}$. In this case, we need to approximate the positive root of $f(x) = x^2 - 2$ and equation (4.9) has the form

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = x_n - \frac{x_n}{2} + \frac{1}{x_n} = \frac{x_n}{2} + \frac{1}{x_n}.$$

Starting with $x_1 = 1$ as our initial guess, we apply this equation repeatedly to get

$$\begin{array}{lll} x_1 = 1, & x_2 = 1.5, & x_3 = 1.41666667, \\ x_4 = 1.41421569, & x_5 = 1.41421356, & x_6 = 1.41421356. \end{array}$$

Based on these calculations, we find that $\sqrt{2}$ is approximately 1.41421 within five decimal places. In fact, the same conclusion may be reached by taking $x_1 = 2$, for instance. If one starts with $x_1 = -1$, then Newton's method leads to $-\sqrt{2}$, the other root of f. □

Example 4.46 Consider the polynomial $f(x) = x^3 - 3x + 1$ which is continuous with

$$f(0) = 1, \qquad f(1) = 1 - 3 + 1 = -1.$$

Since $f(0)$ and $f(1)$ have opposite signs, f has a root that lies in $(0, 1)$. In fact, this root is unique by Rolle's theorem because $f'(x) = 3(x^2 - 1)$ has no roots in the given interval. Let us now use Newton's method to approximate the unique root. Equation (4.9) gives

$$x_{n+1} = x_n - \frac{x_n^3 - 3x_n + 1}{3x_n^2 - 3}$$

and the fraction is not defined when $x_n^2 = 1$. Starting with the initial guess $x_1 = 0$, we get

$$\begin{array}{lll} x_1 = 0, & x_2 = 0.33333333, & x_3 = 0.34722222, \\ x_4 = 0.34729635, & x_5 = 0.34729635, & x_6 = 0.34729635. \end{array}$$

This gives an approximation of the root which is accurate to eight decimal places. □

Example 4.47 We show that Newton's method fails in the case that $f(x) = x^{1/3}$. If one uses equation (4.9) to approximate the root of $f(x) = 0$, one finds that

$$x_{n+1} = x_n - \frac{x_n^{1/3}}{x_n^{-2/3} \cdot 1/3} = x_n - 3x_n = -2x_n.$$

For instance, the initial guess $x_1 = 1$ gives $x_2 = -2$, $x_3 = 4$ and so on. The points x_n are thus getting larger in absolute value and they fail to approach a limiting value. □

Chapter 5

Integration

5.1 Definite integral

Definition 5.1 – Integrability

Suppose that f is defined on $[a, b]$ and let $x_0, x_1, \ldots, x_n$ be the points that divide $[a, b]$ into n intervals of length $\Delta x = \frac{b-a}{n}$. We say that f is integrable on $[a, b]$, if the limit

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^n f(x_k^*)\Delta x \tag{5.1}$$

exists and its value does not depend on which point x_k^* is chosen from each $[x_{k-1}, x_k]$.

- The sum that appears in definition (5.1) is also known as a Riemann sum. When f is positive, it gives the total area of the rectangles with height $f(x_k^*)$ and base Δx. The limit of their sum should be the area of the region that lies below the graph of f.
- There are very few functions for which the Riemann sums can be computed explicitly.

Example 5.2 We use the definition of integrability to show that every constant function is integrable. Indeed, suppose that $f(x) = c$ is constant for all x. Then equation (5.1) gives

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^n c\,\Delta x = \lim_{n\to\infty} \sum_{k=1}^n \frac{c(b-a)}{n} = \lim_{n\to\infty} n \cdot \frac{c(b-a)}{n} = c(b-a). \qquad \square$$

Example 5.3 We use the definition of integrability to show that

$$\int_0^b x\,dx = \frac{b^2}{2} \quad \text{for all } b > 0.$$

Consider the points $x_0, x_1, \ldots, x_n$ that divide $[0, b]$ into n intervals of equal length. These are given by $x_k = k\Delta x$ for each $0 \le k \le n$ and $\Delta x = b/n$, so it easily follows that

$$x_{k-1}\,\Delta x \le x_k^*\,\Delta x \le x_k\,\Delta x \quad \Longrightarrow \quad (k-1)\frac{b^2}{n^2} \le x_k^*\,\Delta x \le k\frac{b^2}{n^2}.$$

Adding up these inequalities over all possible values of $1 \leq k \leq n$, we conclude that

$$\frac{b^2}{n^2} \sum_{k=1}^{n} (k-1) \leq \sum_{k=1}^{n} x_k^* \,\Delta x \leq \frac{b^2}{n^2} \sum_{k=1}^{n} k. \tag{5.2}$$

The sum that appears in the right hand side of (5.2) is a standard sum whose value is

$$\sum_{k=1}^{n} k = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$

The sum that appears in the left hand side of (5.2) is practically the same because

$$\sum_{k=1}^{n} (k-1) = 0 + 1 + \ldots + (n-1) = \frac{(n-1)n}{2}.$$

In particular, one may simplify equation (5.2) to establish the estimates

$$\frac{b^2}{n^2} \cdot \frac{(n-1)n}{2} \leq \sum_{k=1}^{n} x_k^* \,\Delta x \leq \frac{b^2}{n^2} \cdot \frac{n(n+1)}{2}. \tag{5.3}$$

To compute the integral of $f(x) = x$, it remains to take the limit as $n \to \infty$. In this case,

$$\lim_{n\to\infty} \frac{b^2}{n^2} \cdot \frac{(n-1)n}{2} = \frac{b^2}{2} = \lim_{n\to\infty} \frac{b^2}{n^2} \cdot \frac{n(n+1)}{2},$$

so one may apply the squeeze theorem to find that $\int_0^b x\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} x_k^* \,\Delta x = \frac{b^2}{2}$. □

Example 5.4 We show that f is not integrable on any interval $[a, b]$ when

$$f(x) = \left\{ \begin{array}{ll} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{array} \right\}.$$

Were f integrable on $[a, b]$, one would be able to express its integral in the form

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*) \,\Delta x$$

for an arbitrary choice of points $x_k^* \in [x_{k-1}, x_k]$. On the other hand, this interval contains both rational and irrational numbers. If we choose the points x_k^* to be irrational, then

$$f(x_k^*) = 0 \quad \Longrightarrow \quad \sum_{k=1}^{n} f(x_k^*) \,\Delta x = 0 \quad \Longrightarrow \quad \int_a^b f(x)\,dx = 0.$$

If we choose the points x_k^* to be rational, then we similarly get

$$f(x_k^*) = 1 \quad \Longrightarrow \quad \sum_{k=1}^{n} f(x_k^*) \,\Delta x = n\,\Delta x = b - a \quad \Longrightarrow \quad \int_a^b f(x)\,dx = b - a.$$

This gives two different values for the same integral, so f is not integrable on $[a, b]$. □

5.2 Rules of integration

Theorem 5.5 – Linearity

If the functions f, g are integrable on $[a, b]$, then so is their sum, and one has

$$\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$

If a function f is integrable on $[a, b]$, then cf is integrable for any constant c and

$$\int_a^b cf(x)\, dx = c \int_a^b f(x)\, dx.$$

Theorem 5.6 – Integrals and inequalities

If the functions f, g are integrable on $[a, b]$ and $f(x) \leq g(x)$ for all $a \leq x \leq b$, then

$$\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx.$$

Theorem 5.7 – Continuous implies integrable

If a function f is continuous on $[a, b]$, then f is also integrable on $[a, b]$.

Definition 5.8 – Arbitrary limits of integration

The definition of integrability on $[a, b]$ implicitly assumes that $a < b$. However, one may extend this definition to any limits of integration using the formulas

$$\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx, \qquad \int_a^a f(x)\, dx = 0.$$

- The first two theorems are easy to prove, but the proof of Theorem 5.7 is difficult.
- Theorem 5.6 implies the triangle inequality for integrals which states that

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx$$

for any continuous function f. In fact, one has $-|f(x)| \leq f(x) \leq |f(x)|$ and so

$$-\int_a^b |f(x)|\, dx \leq \int_a^b f(x)\, dx \leq \int_a^b |f(x)|\, dx.$$

Writing the last equation as $-A \leq B \leq A$, we conclude that $|B| \leq A$, as needed.

5.3 Fundamental theorem of calculus

Theorem 5.9 – Fundamental theorem of calculus, part 1

If f is continuous on $[a, b]$ and F is a function whose derivative is f, then

$$\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = F(b) - F(a).$$

Theorem 5.10 – Mean value theorem for integrals

If f is continuous on $[a, b]$, then there exists a point $a \leq c \leq b$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

Theorem 5.11 – Fundamental theorem of calculus, part 2

Suppose that f is continuous on $[a, b]$ and let $F(x)$ denote its definite integral

$$F(x) = \int_a^x f(t)\,dt, \qquad a \leq x \leq b.$$

Then F is a function whose derivative is f. In other words, one has $F'(x) = f(x)$.

- An antiderivative of f is a function F whose derivative is f. It is denoted by

  $$F(x) = \int f(x)\,dx,$$

  an integral without limits, and it is also known as the indefinite integral of f.
- The difference $F(b) - F(a)$ is frequently denoted by $[F(x)]_a^b$. One may thus write

  $$\int_a^b F'(x)\,dx = F(b) - F(a) = [F(x)]_a^b\,.$$

Example 5.12 We use the fundamental theorem of calculus to compute the integral

$$I = \int_1^2 (3x^2 - 2)\,dx.$$

Since $F(x) = x^3 - 2x$ is such that $F'(x) = 3x^2 - 2$, one may apply Theorem 5.9 to get

$$I = \int_1^2 F'(x)\,dx = [F(x)]_1^2 = F(2) - F(1) = 4 - (-1) = 5. \qquad \square$$

5.4 Integrals of standard functions

- The following list includes the antiderivatives of some standard functions.
- In each case, the antiderivative is expressed in terms of an arbitrary constant C.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1 \qquad \int x^{-1}\,dx = \ln|x| + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \cot x\,dx = \ln|\sin x| + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x\,\tan x\,dx = \sec x + C \qquad \int \csc x\,\cot x\,dx = -\csc x + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C \qquad \int \csc x\,dx = -\ln|\csc x + \cot x| + C$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C \qquad \int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

$$\int e^{ax}\,dx = \frac{e^{ax}}{a} + C, \quad a \neq 0 \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C, \quad 0 < a \neq 1$$

- The formulas above can be verified using differentiation. For instance, one has

$$\Big[\ln|\sec x|\Big]' = \frac{(\sec x)'}{\sec x} = \frac{\sec x \tan x}{\sec x} = \tan x.$$

  This makes $\ln|\sec x|$ an antiderivative of $\tan x$, so $\int \tan x\,dx = \ln|\sec x| + C$.

- The formula for the integral of $\sec x$ can be similarly verified by checking that

$$\Big[\ln|\sec x + \tan x|\Big]' = \frac{(\sec x + \tan x)'}{\sec x + \tan x} = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x.$$

Example 5.13 Using the formula for the antiderivative of x^n, one finds that

$$\int_0^2 x^2(\sqrt{x}+4)\,dx = \int_0^2 (x^{5/2} + 4x^2)\,dx = \left[\frac{2}{7}x^{7/2} + \frac{4x^3}{3}\right]_0^2 = \frac{16\sqrt{2}}{7} + \frac{32}{3}.$$

Using the formula for the antiderivative of tangent, one similarly finds that

$$\int_0^{\pi/4} \tan x\,dx = \Big[\ln|\sec x|\Big]_0^{\pi/4} = \ln\frac{2}{\sqrt{2}} - \ln 1 = \ln\sqrt{2} = \frac{\ln 2}{2}. \qquad \square$$

5.5 Area, volume and arc length

Theorem 5.14 – Area between two graphs

Suppose that f, g are continuous on $[a, b]$ and $f(x) \leq g(x)$ for all $a \leq x \leq b$. Then the area of the region that lies between the graphs of the two functions is

$$\text{Area} = \int_a^b [g(x) - f(x)]\, dx.$$

Theorem 5.15 – Solids of revolution

Suppose that f is continuous on $[a, b]$ and consider the solid that is produced when the graph of f is rotated around the x-axis. The volume of the resulting solid is then

$$\text{Volume} = \int_a^b \pi f(x)^2\, dx.$$

Theorem 5.16 – Arc length

Suppose that f is differentiable on $[a, b]$ and f' is continuous on $[a, b]$. Then the length of the graph of f over the interval $[a, b]$ is given by

$$\text{Arc length} = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

Example 5.17 We find the area of the region that is enclosed by the graphs of $f(x) = x^2$ and $g(x) = 8 - x^2$. These are the graphs of two parabolas which intersect when

$$x^2 = 8 - x^2 \quad \Longleftrightarrow \quad 2x^2 = 8 \quad \Longleftrightarrow \quad x = \pm 2.$$

Since $f(x) \leq g(x)$ at all points $-2 \leq x \leq 2$, the area of the region is thus

$$\int_{-2}^2 [g(x) - f(x)]\, dx = \int_{-2}^2 \left[8 - 2x^2\right]\, dx = \left[8x - \frac{2x^3}{3}\right]_{-2}^2 = \frac{64}{3}. \qquad \square$$

Example 5.18 We compute the volume of a cone with radius r and height h. One may obtain such a cone by rotating a right triangle around the x-axis. Consider the triangle with vertices $(0, 0)$, $(h, 0)$ and (h, r). Its base has length h and its height has length r, so its hypotenuse has slope r/h and it is given by the line $f(x) = rx/h$. If we rotate the triangle around the x-axis, we get a cone of radius r and height h. The volume of the cone is thus

$$V = \int_0^h \pi f(x)^2\, dx = \frac{\pi r^2}{h^2} \int_0^h x^2\, dx = \frac{\pi r^2}{h^2} \left[\frac{x^3}{3}\right]_0^h = \frac{\pi r^2 h}{3}. \qquad \square$$

Example 5.19 We compute the circumference of a circle with radius $r = 1$. Let us only worry about the upper semicircle $f(x) = \sqrt{1-x^2}$, where $-1 \leq x \leq 1$. In this case,

$$f'(x) = \frac{(1-x^2)'}{2\sqrt{1-x^2}} = -\frac{x}{\sqrt{1-x^2}}$$

and the arc length is given by the integral of $\sqrt{1+f'(x)^2}$. Once we now simplify

$$1 + f'(x)^2 = 1 + \frac{x^2}{1-x^2} = \frac{1}{1-x^2},$$

we may take the square root of both sides to conclude that the arc length is

$$\int_{-1}^{1} \sqrt{1+f'(x)^2}\, dx = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \Big[\sin^{-1} x\Big]_{-1}^{1} = \frac{\pi}{2} + \frac{\pi}{2} = \pi. \qquad \square$$

Example 5.20 We compute the length of the graph of f in the case that

$$f(x) = \frac{x^3}{12} + \frac{1}{x}, \qquad 1 \leq x \leq 2.$$

This amounts to computing the integral of $\sqrt{1+f'(x)^2}$ and one can easily check that

$$\begin{aligned} f'(x) = \frac{x^2}{4} - \frac{1}{x^2} \quad &\Longrightarrow \quad f'(x)^2 + 1 = \frac{x^4}{16} + \frac{1}{x^4} - \frac{1}{2} + 1 \\ &\Longrightarrow \quad f'(x)^2 + 1 = \frac{x^8 + 16 + 8x^4}{16x^4} = \frac{(x^4+4)^2}{16x^4}. \end{aligned}$$

Taking the square root of both sides, we conclude that the length of the graph is

$$\int_1^2 \frac{x^4+4}{4x^2}\, dx = \int_1^2 \left(\frac{x^2}{4} + x^{-2}\right) dx = \left[\frac{x^3}{12} - \frac{1}{x}\right]_1^2 = \frac{13}{12}. \qquad \square$$

Example 5.21 We compute the volume of the solid obtained by rotating the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

around the x-axis. This solid is also known as an ellipsoid. Assuming that a is positive, we get $-a \leq x \leq a$ and one may rearrange terms to find that

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \quad \Longrightarrow \quad y^2 = b^2\left(1 - \frac{x^2}{a^2}\right).$$

The volume of the ellipsoid is the integral of $\pi f(x)^2 = \pi y^2$ and this is given by

$$\pi \int_{-a}^{a} y^2\, dx = b^2\pi \int_{-a}^{a} \left(1 - \frac{x^2}{a^2}\right) dx = b^2\pi \left[x - \frac{x^3}{3a^2}\right]_{-a}^{a} = \frac{4ab^2\pi}{3}. \qquad \square$$

5.6 Mass, centre of mass and work

Theorem 5.22 – Mass and centre of mass

Consider a thin rod which extends between the points $a \leq x \leq b$ and let $\delta(x)$ denote its density at the point x. The overall mass M of the rod is then

$$M = \int_a^b \delta(x)\,dx,$$

The centre of mass $\overline{x}$ is given by a similar formula and one has

$$\overline{x} = \frac{1}{M}\int_a^b x\delta(x)\,dx. \tag{5.4}$$

Definition 5.23 – Work

In physics, the amount of work that is required to move an object by d units using a constant force F in the direction of motion is defined as the product

$$\text{Work} = \text{Force} \cdot \text{Displacement} = F \cdot d.$$

Example 5.24 We compute the mass and the centre of mass for a thin rod with density

$$\delta(x) = 2x^2 + 3x + 4, \qquad 0 \leq x \leq 1.$$

The mass of the rod is given by the integral of its density function, namely

$$M = \int_0^1 (2x^2 + 3x + 4)\,dx = \left[\frac{2x^3}{3} + \frac{3x^2}{2} + 4x\right]_0^1 = \frac{2}{3} + \frac{3}{2} + 4 = \frac{37}{6}.$$

The centre of mass is given by equation (5.4) and one easily finds that

$$\overline{x} = \frac{1}{M}\int_0^1 x\delta(x)\,dx = \frac{6}{37}\int_0^1 (2x^3 + 3x^2 + 4x)\,dx = \frac{6}{37}\left[\frac{x^4}{2} + x^3 + 2x^2\right]_0^1 = \frac{21}{37}. \qquad \square$$

Example 5.25 A rectangular tank has length 4m, width 3m and height 2m. Suppose it is full with water of density 1000 kg/m^3. How much work does it take to pump out the water through a hole in the top? To find the answer, we consider a typical cross section of the tank. Assume that it has arbitrarily small height dx and lies x metres from the top. Then its volume is $V = 4 \cdot 3 \cdot dx$ and its mass is $m = (12\,dx) \cdot 1000$. The force needed to pump out this part is mass times acceleration, so $F = mg$ and the overall amount of work is

$$\text{Work} = \int F \cdot x = \int_0^2 12{,}000g \cdot x\,dx = 12{,}000g\left[\frac{x^2}{2}\right]_0^2 = 24{,}000g. \qquad \square$$

5.7 Improper integrals

- An improper integral is one that is not defined in the usual way because either the values of x or the values of $f(x)$ become infinite within the interval of integration.
- In these cases, one may integrate the given function over a generic interval $[a, b]$ and then compute the limit as the endpoints a, b approach the endpoints of interest.

Example 5.26 We use limits to compute the improper integral

$$I_1 = \int_0^1 \frac{dx}{\sqrt{x}}.$$

In this case, the integrand becomes infinite at $x = 0$, so we avoid this point and compute

$$\int_a^1 \frac{dx}{\sqrt{x}} = \int_a^1 x^{-1/2}\, dx = \left[\frac{x^{1/2}}{1/2}\right]_a^1 = 2 - 2\sqrt{a}.$$

Taking the limit as $a \to 0^+$, we conclude that the original integral is equal to $I_1 = 1$. □

Example 5.27 Let $c > 0$ be a given constant and consider the improper integral

$$I_2 = \int_0^\infty e^{-cx}\, dx.$$

Since the interval of integration is infinite, we start by computing the finite analogue

$$\int_0^L e^{-cx}\, dx = \left[-\frac{e^{-cx}}{c}\right]_0^L = -\frac{e^{-cL}}{c} + \frac{1}{c}.$$

Letting $L \to \infty$, we see that $cL \to \infty$ as well, so the original integral is equal to

$$I_2 = \lim_{L\to\infty} \int_0^L e^{-cx}\, dx = \lim_{L\to\infty} \frac{1 - e^{-cL}}{c} = \frac{1}{c}.$$ □

Example 5.28 Let $p > 1$ be a given constant and consider the improper integral

$$I_3 = \int_1^\infty x^{-p}\, dx.$$

Once again, one may compute this integral by considering its finite analogue

$$\int_1^L x^{-p}\, dx = \left[\frac{x^{1-p}}{1-p}\right]_1^L = \frac{L^{1-p}}{1-p} - \frac{1}{1-p}.$$

Since the exponent $1 - p$ is negative, one has $L^{1-p} \to 0$ as $L \to \infty$ and this implies that

$$I_3 = \lim_{L\to\infty} \int_1^L x^{-p}\, dx = -\frac{1}{1-p} = \frac{1}{p-1}.$$ □

Chapter 6

Techniques of integration

6.1 Integration by parts

Theorem 6.1 – Integration by parts

If f, g are differentiable functions and their derivatives f', g' are continuous, then

$$\int f(x) \cdot g'(x)\, dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x)\, dx. \tag{6.1}$$

- It is quite common, and also convenient, to express the last equation in the form

$$\int u\, dv = uv - \int v\, du. \tag{6.2}$$

  This alternative version arises by letting $u = f(x)$ and $v = g(x)$ for simplicity.

- Some typical integrals that may be computed using integration by parts are

$$\int p(x) \cdot e^{ax}\, dx, \qquad \int p(x) \cdot \sin(ax)\, dx, \qquad \int p(x) \cdot \cos(ax)\, dx,$$

  where $p(x)$ is a polynomial and $a \neq 0$ is a given constant. In each of these cases, one needs to integrate by parts n times, where n is the degree of the polynomial.

- Some other integrals for which integration by parts might be needed are those that involve $\sin^{-1}$, $\tan^{-1}$ or ln. These are functions whose derivatives are much simpler, so they are all natural choices for the variable u that appears in equation (6.2).

Example 6.2 We use integration by parts to compute the integral

$$\int xe^x\, dx.$$

Letting $u = x$ and $dv = e^x\, dx$, we find that $du = dx$ and $v = e^x$. It easily follows that

$$\int xe^x\, dx = uv - \int v\, du = xe^x - \int e^x\, dx = xe^x - e^x + C. \qquad \square$$

Example 6.3 We use integration by parts to compute the integral

$$\int x\cos(2x)\,dx.$$

In this case, we take $u = x$ and $dv = \cos(2x)\,dx$. Since $du = dx$ and $v = \frac{1}{2}\sin(2x)$, we get

$$\int x\cos(2x)\,dx = \frac{x}{2}\,\sin(2x) - \frac{1}{2}\int \sin(2x)\,dx = \frac{x}{2}\,\sin(2x) + \frac{1}{4}\,\cos(2x) + C. \qquad \square$$

Example 6.4 We use integration by parts to compute the integral

$$\int x^2 \ln x\,dx.$$

Letting $u = \ln x$ and $dv = x^2\,dx$, we find that $du = \frac{1}{x}\,dx$ and $v = \frac{x^3}{3}$. This implies that

$$\int x^2 \ln x\,dx = \frac{x^3}{3}\,\ln x - \int \frac{x^3}{3}\cdot\frac{1}{x}\,dx = \frac{x^3}{3}\ln x - \int \frac{x^2}{3}\,dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C. \qquad \square$$

Example 6.5 We use a double integration by parts to compute the integral

$$\int e^{ax}\sin(bx)\,dx, \qquad b \neq 0.$$

If we let $u = e^{ax}$ and $dv = \sin(bx)\,dx$, then $du = ae^{ax}\,dx$ and $v = -\frac{1}{b}\cos(bx)$, so

$$\int e^{ax}\sin(bx)\,dx = -\frac{1}{b}\,e^{ax}\cos(bx) + \frac{a}{b}\int e^{ax}\cos(bx)\,dx.$$

Next, we take $u = e^{ax}$ and $dv = \cos(bx)\,dx$. Since $du = ae^{ax}\,dx$ and $v = \frac{1}{b}\sin(bx)$, one has

$$\begin{aligned}\int e^{ax}\sin(bx)\,dx &= -\frac{1}{b}\,e^{ax}\cos(bx) + \frac{a}{b}\left[\frac{1}{b}\,e^{ax}\sin(bx) - \frac{a}{b}\int e^{ax}\sin(bx)\,dx\right]\\ &= -\frac{1}{b}\,e^{ax}\cos(bx) + \frac{a}{b^2}\,e^{ax}\sin(bx) - \frac{a^2}{b^2}\int e^{ax}\sin(bx)\,dx.\end{aligned}$$

Here, the rightmost integral is the same as the leftmost integral, so we actually have

$$\left(1 + \frac{a^2}{b^2}\right)\int e^{ax}\sin(bx)\,dx = -\frac{1}{b}\,e^{ax}\cos(bx) + \frac{a}{b^2}\,e^{ax}\sin(bx).$$

Once we now multiply the last equation by $b^2/(a^2+b^2)$, we may finally conclude that

$$\int e^{ax}\sin(bx)\,dx = -\frac{be^{ax}\cos(bx)}{a^2+b^2} + \frac{ae^{ax}\sin(bx)}{a^2+b^2} + C. \qquad \square$$

6.2 Integration by substitution

- A very useful tool for simplifying integrals is provided by the formula
$$\int g(f(x)) \cdot f'(x)\, dx = \int g(u)\, du. \tag{6.3}$$
- Here, the idea is to introduce a variable $u = f(x)$ to replace the leftmost integral by another integral which is much simpler and also expressible in terms of u alone.

Example 6.6 We use an appropriate substitution to compute the integral
$$\int \frac{(\ln x)^3}{x}\, dx.$$
If we take $u = \ln x$, then we have $du = \frac{1}{x}\, dx$ and this is easily seen to imply that
$$\int \frac{(\ln x)^3}{x}\, dx = \int u^3\, du = \frac{1}{4}\, u^4 + C = \frac{1}{4}\, (\ln x)^4 + C. \qquad \square$$

Example 6.7 We use integration by substitution to compute the integral
$$\int x^2(x^3+6)^4\, dx.$$
In this case, the choice $u = x^3 + 6$ is suitable because $du = 3x^2\, dx$ and this gives
$$\int x^2(x^3+6)^4\, dx = \frac{1}{3} \int u^4\, du = \frac{1}{15}\, u^5 + C = \frac{1}{15}\, (x^3+6)^5 + C. \qquad \square$$

Example 6.8 We use an appropriate substitution to compute the integral
$$\int \frac{2x+7}{(x+2)^2}\, dx.$$
In this case, we take $u = x + 2$ to merely simplify the denominator. Since $du = dx$, we get
$$\begin{aligned} \int \frac{2x+7}{(x+2)^2}\, dx &= \int \frac{2(u-2)+7}{u^2}\, du = \int 2u^{-1}\, du + \int 3u^{-2}\, du \\ &= 2\ln|u| - 3u^{-1} + C = 2\ln|x+2| - \frac{3}{x+2} + C. \qquad \square \end{aligned}$$

Example 6.9 We use integration by substitution to compute the integral
$$\int \cos\sqrt{x}\, dx.$$
If we let $u = \sqrt{x}$ to simplify the square root, then $x = u^2$ and $dx = 2u\, du$, so
$$\int \cos\sqrt{x}\, dx = \int \cos u \cdot 2u\, du = 2\int u\cos u\, du.$$
Next, we need to integrate by parts. Letting $dv = \cos u\, du$ and $v = \sin u$, we find that
$$\begin{aligned} \int \cos\sqrt{x}\, dx &= 2u\sin u - 2\int \sin u\, du = 2u\sin u + 2\cos u + C \\ &= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C. \qquad \square \end{aligned}$$

6.3 Reduction formulas

- A reduction formula expresses an integral I_n that depends on some integer n in terms of another integral I_m that involves a smaller integer m. If one repeatedly applies this formula, one may then express I_n in terms of a much simpler integral.

Example 6.10 We use integration by parts to establish the reduction formula

$$\int \sin^n x\, dx = -\frac{1}{n}\sin^{n-1} x\cdot\cos x + \frac{n-1}{n}\int \sin^{n-2} x\, dx. \tag{6.4}$$

If we take $dv = \sin x\, dx$, then we have $v = -\cos x$ and we may integrate by parts with

$$u = \sin^{n-1} x, \qquad du = (n-1)\sin^{n-2} x\cdot\cos x.$$

Using the fact that $\sin^2 x + \cos^2 x = 1$, one may thus conclude that

$$\begin{aligned}\int \sin^n x\, dx &= -\sin^{n-1} x\cdot\cos x + (n-1)\int \sin^{n-2} x\cdot\cos^2 x\, dx\\ &= -\sin^{n-1} x\cdot\cos x + (n-1)\int \sin^{n-2} x\cdot(1-\sin^2 x)\, dx\\ &= -\sin^{n-1} x\cdot\cos x + (n-1)\int \sin^{n-2} x\, dx + (1-n)\int \sin^n x\, dx.\end{aligned}$$

Here, the rightmost integral coincides with the original integral on the left. Once we now rearrange terms, we end up with n copies of the integral and equation (6.4) follows. □

Example 6.11 We use a reduction formula to compute the integral I_3 in the case that

$$I_n = \int x^n e^{2x}\, dx.$$

If we take $u = x^n$ and $dv = e^{2x}\, dx$, then $du = nx^{n-1}\, dx$ and $v = \frac{1}{2}e^{2x}$, so one has

$$I_n = \frac{1}{2}x^n e^{2x} - \frac{n}{2}\int x^{n-1}e^{2x}\, dx = \frac{1}{2}x^n e^{2x} - \frac{n}{2}\cdot I_{n-1}. \tag{6.5}$$

We now apply the last formula repeatedly to determine I_3. According to the formula,

$$\begin{aligned}I_3 &= \frac{1}{2}x^3e^{2x} - \frac{3}{2}\cdot I_2 = \frac{1}{2}x^3e^{2x} - \frac{3}{2}\cdot\left[\frac{1}{2}x^2e^{2x} - I_1\right]\\ &= \frac{1}{2}x^3e^{2x} - \frac{3}{2}\cdot\left[\frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{2}\cdot I_0\right]\\ &= \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{4}\int e^{2x}\, dx\\ &= \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{8}e^{2x} + C.\end{aligned}$$

□

Example 6.12 We use integration by parts to establish the reduction formula

$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x \cdot \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx. \tag{6.6}$$

In this case, we note that $(\tan x)' = \sec^2 x$ and we write the given integral as

$$\int \sec^n x\,dx = \int \sec^{n-2} x \cdot \sec^2 x\,dx.$$

If we take $dv = \sec^2 x\,dx$, then we have $v = \tan x$ and we may integrate by parts with

$$u = \sec^{n-2} x, \qquad du = (n-2)\sec^{n-3} x \cdot \sec x \tan x = (n-2)\sec^{n-2} x \cdot \tan x.$$

Using the fact that $1 + \tan^2 x = \sec^2 x$, one may thus establish the identity

$$\begin{aligned}\int \sec^n x\,dx &= \sec^{n-2} x \cdot \tan x - (n-2)\int \sec^{n-2} x \cdot \tan^2 x\,dx \\ &= \sec^{n-2} x \cdot \tan x - (n-2)\int \sec^{n-2} x \cdot (\sec^2 x - 1)\,dx \\ &= \sec^{n-2} x \cdot \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx.\end{aligned}$$

Since the integral on the left hand side also appears on the right hand side, this gives

$$(n-1)\int \sec^n x\,dx = \sec^{n-2} x \cdot \tan x + (n-2)\int \sec^{n-2} x\,dx.$$

In particular, the reduction formula (6.6) follows by dividing both sides with $n-1$. □

Example 6.13 Let $a \neq 0$ be some given constant and consider the integral

$$I_n = \int \frac{dx}{(x^2+a)^n} = \int (x^2+a)^{-n}\,dx.$$

If we take $u = (x^2+a)^{-n}$ and $dv = dx$, then we may integrate by parts to find that

$$I_n = x(x^2+a)^{-n} + n\int x(x^2+a)^{-n-1} \cdot 2x\,dx.$$

Let us now rearrange terms and express the last equation in the form

$$\begin{aligned}I_n &= x(x^2+a)^{-n} + 2n\int \frac{x^2+a-a}{(x^2+a)^{n+1}}\,dx \\ &= x(x^2+a)^{-n} + 2n\int \frac{dx}{(x^2+a)^n} - 2na\int \frac{dx}{(x^2+a)^{n+1}}.\end{aligned}$$

The integrals on the right hand side have the same form as the original integral, so

$$I_n = x(x^2+a)^{-n} + 2n \cdot I_n - 2na \cdot I_{n+1}.$$

Rearranging terms once again, one may thus establish the reduction formula

$$2na \cdot I_{n+1} = (2n-1) \cdot I_n + x(x^2+a)^{-n}.$$ □

6.4 Trigonometric integrals

Theorem 6.14 – Powers of sine and cosine

Consider the integral $\int \sin^m x \cdot \cos^n x \, dx$ for any non-negative integers m, n.

(a) When n is odd, one may compute this integral using the substitution $u = \sin x$.

(b) When m is odd, one may compute this integral using the substitution $u = \cos x$.

(c) When m, n are even, one may use the half-angle formulas to simplify the integral.

Theorem 6.15 – Powers of secant and tangent

Consider the integral $\int \sec^m x \cdot \tan^n x \, dx$ for any non-negative integers m, n.

(a) When n is odd, one may compute this integral using the substitution $u = \sec x$.

(b) When m is even, one may compute this integral using the substitution $u = \tan x$.

(c) When m is odd and n is even, one may reduce the integrand to powers of $\sec x$.

- The three cases that arise in Theorem 6.14 are closely related to the identities

$$(\sin x)' = \cos x, \qquad (\cos x)' = -\sin x, \qquad \sin^2 x + \cos^2 x = 1.$$

  If one uses the substitution $u = \sin x$, then one may express any even power of cosine in terms of u^2, but also needs a copy of cosine for $du = \cos x \, dx$. This yields an odd number of cosines, so the substitution $u = \sin x$ will only help when n is odd.

- The last case that arises in Theorem 6.14 requires the half-angle formulas

$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}, \qquad \cos^2 \theta = \frac{1 + \cos(2\theta)}{2}. \tag{6.7}$$

  These formulas are helpful for reducing the even powers of sine and cosine.

- The three cases that arise in Theorem 6.15 are closely related to the identities

$$(\sec x)' = \sec x \tan x, \qquad (\tan x)' = \sec^2 x, \qquad 1 + \tan^2 x = \sec^2 x.$$

  These imply that an odd number of tangents is needed to substitute $u = \sec x$, while an even number of secants is needed to substitute $u = \tan x$.

Example 6.16 We use the substitution $u = \sin x$ to compute the integral

$$\int \sin^4 x \cdot \cos^5 x \, dx.$$

In this case, we have $du = \cos x\,dx$ and also $\sin^2 x + \cos^2 x = 1$, so

$$\begin{aligned}\int \sin^4 x \cdot \cos^5 x\,dx &= \int \sin^4 x \cdot (1-\sin^2 x)^2 \cdot \cos x\,dx = \int u^4(1-u^2)^2\,du \\ &= \int u^4(1-2u^2+u^4)\,du = \int (u^4 - 2u^6 + u^8)\,du \\ &= \frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9} + C = \frac{\sin^5 x}{5} - \frac{2\sin^7 x}{7} + \frac{\sin^9 x}{9} + C. \qquad \square\end{aligned}$$

Example 6.17 We use the half-angle formulas to simplify and compute the integral

$$\int \sin^2 x \cdot \cos^2 x\,dx.$$

Since the exponents are both even, one needs to express the integrand in the form

$$\begin{aligned}\sin^2 x \cdot \cos^2 x &= \frac{1-\cos(2x)}{2} \cdot \frac{1+\cos(2x)}{2} = \frac{1}{4} \cdot \left[1-\cos^2(2x)\right] \\ &= \frac{1}{4} \cdot \left[1 - \frac{1+\cos(4x)}{2}\right] = \frac{1}{8} \cdot [1-\cos(4x)]\,.\end{aligned}$$

Once we now integrate both sides of this equation, we may easily conclude that

$$\int \sin^2 x \cdot \cos^2 x\,dx = \frac{1}{8}\left[x - \frac{\sin(4x)}{4}\right] + C = \frac{x}{8} - \frac{\sin(4x)}{32} + C. \qquad \square$$

Example 6.18 We use an appropriate substitution to compute the integral

$$\int \sec^4 x \cdot \tan^2 x\,dx.$$

If we let $u = \tan x$, then $du = \sec^2 x\,dx$ and also $\sec^2 x = 1 + \tan^2 x = 1 + u^2$, so one has

$$\begin{aligned}\int \sec^4 x \cdot \tan^2 x\,dx &= \int \sec^2 x \cdot \tan^2 x \cdot \sec^2 x\,dx = \int (1+u^2)\cdot u^2\,du \\ &= \int (u^2+u^4)\,du = \frac{u^3}{3} + \frac{u^5}{5} + C = \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C. \qquad \square\end{aligned}$$

Example 6.19 We use an appropriate substitution to compute the integral

$$\int \frac{\sin^3 x}{\cos^8 x}\,dx.$$

Since the cosine appears in the denominator, it is better to first simplify and write

$$\int \frac{\sin^3 x}{\cos^8 x}\,dx = \int \frac{\sin^3 x}{\cos^3 x} \cdot \frac{1}{\cos^5 x}\,dx = \int \tan^3 x \cdot \sec^5 x\,dx.$$

Let us take $u = \sec x$. Since $du = \sec x \tan x\,dx$ and also $u^2 = \sec^2 x = \tan^2 x + 1$, we get

$$\begin{aligned}\int \frac{\sin^3 x}{\cos^8 x}\,dx &= \int \tan^2 x \cdot \sec^4 x \cdot \sec x \tan x\,dx = \int (u^2-1)\cdot u^4\,du \\ &= \int (u^6-u^4)\,du = \frac{u^7}{7} - \frac{u^5}{5} + C = \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C. \qquad \square\end{aligned}$$

6.5 Trigonometric substitutions

- Trigonometric substitutions are sometimes needed to simplify integrals that contain expressions of the form $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$ and $\sqrt{x^2+a^2}$ for some $a>0$. In each of these cases, one naturally seeks a substitution to simplify the square root.

- The three most common trigonometric substitutions are listed in the table below.

Expression	Substitution	Simplification	
$\sqrt{a^2-x^2}$	$x=a\sin\theta$	$\sqrt{a^2-x^2}=a\cos\theta$,	$dx=a\cos\theta\,d\theta$
$\sqrt{a^2+x^2}$	$x=a\tan\theta$	$\sqrt{a^2+x^2}=a\sec\theta$,	$dx=a\sec^2\theta\,d\theta$
$\sqrt{x^2-a^2}$	$x=a\sec\theta$	$\sqrt{x^2-a^2}=a\lvert\tan\theta\rvert$,	$dx=a\sec\theta\tan\theta\,d\theta$

- In the first case, one has $a^2-x^2=a^2-a^2\sin^2\theta=a^2\cos^2\theta$ and $\sqrt{a^2-x^2}=a\cos\theta$. This is because $\theta=\sin^{-1}(x/a)$ lies between $-\pi/2$ and $\pi/2$, so $\cos\theta$ is non-negative.

Example 6.20 We use a trigonometric substitution to compute the integral

$$\int\frac{dx}{\sqrt{a^2-x^2}},\qquad a>0.$$

If we let $x=a\sin\theta$, then $a^2-x^2=a^2-a^2\sin^2\theta=a^2\cos^2\theta$ and also $dx=a\cos\theta\,d\theta$, so

$$\int\frac{dx}{\sqrt{a^2-x^2}}=\int\frac{a\cos\theta\,d\theta}{a\cos\theta}=\int d\theta=\theta+C=\sin^{-1}\frac{x}{a}+C.\qquad\square$$

Example 6.21 We use a trigonometric substitution to compute the integral

$$\int\frac{dx}{x^2+a^2},\qquad a>0.$$

If we let $x=a\tan\theta$, then $x^2+a^2=a^2\tan^2\theta+a^2=a^2\sec^2\theta$ and also $dx=a\sec^2\theta\,d\theta$, so

$$\int\frac{dx}{x^2+a^2}=\int\frac{a\sec^2\theta\,d\theta}{a^2\sec^2\theta}=\frac{1}{a}\int d\theta=\frac{1}{a}\,\theta+C=\frac{1}{a}\tan^{-1}\frac{x}{a}+C.\qquad\square$$

Example 6.22 We use a trigonometric substitution to compute the integral

$$\int\frac{x^2\,dx}{\sqrt{4-x^2}}.$$

If we let $x=2\sin\theta$, then $4-x^2=4-4\sin^2\theta=4\cos^2\theta$ and also $dx=2\cos\theta\,d\theta$, so

$$\begin{aligned}\int\frac{x^2\,dx}{\sqrt{4-x^2}}&=\int\frac{4\sin^2\theta\cdot 2\cos\theta\,d\theta}{2\cos\theta}=\int 4\sin^2\theta\,d\theta=2\int\left[1-\cos(2\theta)\right]\,d\theta\\&=2\theta-\sin(2\theta)+C=2\theta-2\sin\theta\cdot\cos\theta+C.\end{aligned}$$

It remains to express this equation in terms of $x=2\sin\theta$. Since $\theta=\sin^{-1}\frac{x}{2}$, we get

$$\int\frac{x^2\,dx}{\sqrt{4-x^2}}=2\sin^{-1}\frac{x}{2}-2\cdot\frac{x}{2}\cdot\sqrt{1-\frac{x^2}{4}}+C=2\sin^{-1}\frac{x}{2}-\frac{x}{2}\sqrt{4-x^2}+C.\qquad\square$$

6.6 Partial fractions

Definition 6.23 – Proper rational function

A proper rational function is a quotient of two polynomials $P(x)/Q(x)$ such that the degree of the numerator $P(x)$ is smaller than the degree of the denominator $Q(x)$.

Theorem 6.24 – Partial fractions

Suppose that $f(x)$ is a proper rational function whose denominator is the product of relatively prime polynomials. Then $f(x)$ can be expressed as a sum of proper rational functions whose denominators are these relatively prime polynomials.

- Two polynomials are relatively prime, if they do not have any common divisor other than constant factors. For instance, $(x+1)(x-1)$ and $x^2(x+3)$ are relatively prime, whereas $x(x-1)$ and $x^2(x+3)$ have a non-constant factor in common.

Example 6.25 We use partial fractions to compute the integral

$$\int \frac{x^2+3x-4}{(x^2+1)(x+1)}\,dx.$$

According to the last theorem, the integrand can be expressed in the form

$$\frac{x^2+3x-4}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1}$$

for some constants A, B, C that need to be determined. Clearing denominators gives

$$x^2+3x-4 = (Ax+B)(x+1) + C(x^2+1)$$

and one may look at some suitable choices of x to find that

$$x=-1,0,1 \quad\Longrightarrow\quad -6=2C, \qquad -4=B+C, \qquad 0=2A+2B+2C.$$

Solving these equations, we now get $C=-3$, $B=-1$ and $A=4$, which means that

$$\int \frac{x^2+3x-4}{(x^2+1)(x+1)}\,dx = \int \frac{4x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx - \int \frac{3}{x+1}\,dx.$$

The two rightmost integrals are rather easy to compute, and so is the integral

$$\int \frac{4x}{x^2+1}\,dx = \int \frac{2\,du}{u} = 2\ln|u| + C = 2\ln(x^2+1) + C,$$

if one substitutes $u = x^2+1$. In view of the last two equations, we must thus have

$$\int \frac{x^2+3x-4}{(x^2+1)(x+1)}\,dx = 2\ln(x^2+1) - \tan^{-1} x - 3\ln|x+1| + C. \qquad \square$$

Example 6.26 We use partial fractions to compute the integral

$$\int \frac{x^3+3x^2+5}{x(x-1)}\,dx.$$

This rational function is not proper because its numerator is cubic and its denominator is only quadratic. Thus, one needs to first use division of polynomials to write

$$\frac{x^3+3x^2+5}{x(x-1)} = \frac{x^3+3x^2+5}{x^2-x} = x+4+\frac{4x+5}{x(x-1)}.$$

Since the rightmost fraction is proper, one may use partial fractions to express it as

$$\frac{4x+5}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \qquad \Longrightarrow \qquad 4x+5 = A(x-1)+Bx.$$

Setting $x=0$ gives $5=-A$ and setting $x=1$ gives $9=B$. It easily follows that

$$\frac{x^3+3x^2+5}{x(x-1)} = x+4+\frac{A}{x}+\frac{B}{x-1} = x+4-\frac{5}{x}+\frac{9}{x-1}.$$

Once we now integrate this equation term by term, we may finally conclude that

$$\int \frac{x^3+3x^2+5}{x(x-1)}\,dx = \frac{x^2}{2}+4x-5\ln|x|+9\ln|x-1|+C. \qquad \square$$

Example 6.27 We use a substitution and partial fractions to compute the integral

$$\int \frac{e^{5x}\,dx}{e^{2x}-1}.$$

If we take $u=e^x$, then $du=e^x\,dx$ and the given integral takes the form

$$\int \frac{e^{5x}\,dx}{e^{2x}-1} = \int \frac{e^{4x}\cdot e^x\,dx}{e^{2x}-1} = \int \frac{u^4\,du}{u^2-1}.$$

This is not a proper rational function, so one needs to first use division to write

$$\int \frac{e^{5x}\,dx}{e^{2x}-1} = \int \frac{u^4-1+1}{u^2-1}\,du = \int \left(u^2+1+\frac{1}{u^2-1}\right)du. \tag{6.8}$$

Let us merely focus on the proper rational function. Using partial fractions, we get

$$\frac{1}{u^2-1} = \frac{A}{u-1}+\frac{B}{u+1} \qquad \Longrightarrow \qquad 1 = A(u+1)+B(u-1).$$

When $u=1$, this gives $1=2A$. When $u=-1$, it gives $1=-2B$. In particular, one has

$$u^2+1+\frac{1}{u^2-1} = u^2+1+\frac{A}{u-1}+\frac{B}{u+1} = u^2+1+\frac{1/2}{u-1}-\frac{1/2}{u+1}$$

and each of these terms can be easily integrated. Returning to (6.8), we conclude that

$$\begin{aligned}\int \frac{e^{5x}\,dx}{e^{2x}-1} &= \frac{1}{3}\,u^3+u+\frac{1}{2}\,\ln|u-1|-\frac{1}{2}\,\ln|u+1|+C\\ &= \frac{1}{3}\,e^{3x}+e^x+\frac{1}{2}\,\ln|e^x-1|-\frac{1}{2}\,\ln(e^x+1)+C. \qquad \square\end{aligned}$$

Chapter 7

Sequences and series

7.1 Convergence of sequences

Definition 7.1 – Convergence of sequences

A sequence is a function that is defined on the set $\mathbb{N}$ of natural numbers. Its values are usually denoted by writing a_n for each $n \in \mathbb{N}$. We say that the sequence $\{a_n\}$ converges, if a_n approaches a finite limit as $n \to \infty$. Otherwise, we say that $\{a_n\}$ diverges.

Definition 7.2 – Monotonicity

A sequence $\{a_n\}$ is called monotonic, if it is either increasing, in which case $a_n \leq a_{n+1}$ for each $n \in \mathbb{N}$, or else decreasing, in which case $a_n \geq a_{n+1}$ for each $n \in \mathbb{N}$.

Theorem 7.3 – Monotonic and bounded

If a sequence is monotonic and bounded, then the sequence is also convergent.

- When a precise formula for a_n is known, one may use that formula to compute the limit of a_n and prove convergence. However, a precise formula is not always available.

Example 7.4 We show that each of the following sequences converges.

$$a_n = \sqrt{\frac{8n^2+3}{2n^2+5}}, \qquad b_n = \frac{3+\sin n}{n^2}, \qquad c_n = \left(1+\frac{1}{n}\right)^n.$$

Since the limit of a square root is the square root of the limit, it should be clear that

$$\lim_{n\to\infty} \frac{8n^2+3}{2n^2+5} = \lim_{n\to\infty} \frac{8n^2}{2n^2} = 4 \quad \Longrightarrow \quad \lim_{n\to\infty} a_n = \sqrt{4} = 2.$$

The limit of the second sequence is zero because $2/n^2 \leq b_n \leq 4/n^2$ for each $n \geq 1$. This means that b_n is squeezed between two sequences that converge to zero. Finally, one has

$$c_n = \left(1+\frac{1}{n}\right)^n \quad \Longrightarrow \quad \ln c_n = n \cdot \ln\left(1+\frac{1}{n}\right) = \frac{\ln(1+1/n)}{1/n}.$$

This is a limit of the form 0/0, so one may use L'Hôpital's rule to conclude that

$$\lim_{n\to\infty} \ln c_n = \lim_{n\to\infty} \frac{(1+1/n)^{-1}\cdot(1/n)'}{(1/n)'} = 1 \quad\Longrightarrow\quad \lim_{n\to\infty} c_n = e^1 = e. \qquad \square$$

Example 7.5 There are two different ways of checking that $a_n = n/(n+1)$ is increasing. First of all, one may use derivatives. If we define $f(x) = x/(x+1)$ for each $x \geq 1$, then

$$f'(x) = \frac{(x+1)-x}{(x+1)^2} = \frac{1}{(x+1)^2} > 0.$$

This makes $f(x)$ increasing for all $x \geq 1$ and thus a_n is increasing for all $n \geq 1$. It is also possible to check this directly. To show that a_n is increasing, one needs to show that

$$a_n \leq a_{n+1} \quad\Longleftrightarrow\quad \frac{n}{n+1} \leq \frac{n+1}{n+2} \quad\Longleftrightarrow\quad n^2+2n \leq n^2+2n+1.$$

Since the rightmost inequality is obviously true, the leftmost inequality holds as well. $\square$

Example 7.6 We show that $a_n = \frac{2^n}{n!}$ is decreasing for all $n \geq 1$. In this case, we have

$$a_n \geq a_{n+1} \quad\Longleftrightarrow\quad \frac{2^n}{n!} \geq \frac{2^{n+1}}{(n+1)!} \quad\Longleftrightarrow\quad n+1 \geq 2.$$

Since the rightmost inequality is obviously true, the leftmost inequality holds as well. $\square$

Example 7.7 We find the limit of the sequence $\{a_n\}$ which is defined by $a_1 = 1$ and

$$a_{n+1} = \sqrt{2a_n} \quad \text{for each } n \geq 1.$$

To show that this sequence converges, we shall first show that

$$1 \leq a_n \leq a_{n+1} \leq 2 \quad \text{for each } n \geq 1. \tag{7.1}$$

When $n = 1$, this statement asserts that $1 \leq 1 \leq \sqrt{2} \leq 2$, so it is certainly true. Suppose that it is true for some n. Multiplying by 2 and taking square roots, we then find that

$$\begin{aligned} 2 \leq 2a_n \leq 2a_{n+1} \leq 4 \quad &\Longrightarrow\quad \sqrt{2} \leq \sqrt{2a_n} \leq \sqrt{2a_{n+1}} \leq 2 \\ &\Longrightarrow\quad 1 \leq a_{n+1} \leq a_{n+2} \leq 2. \end{aligned}$$

In particular, the statement holds for $n+1$ as well, so it actually holds for all $n \in \mathbb{N}$. This shows that the given sequence is monotonic and bounded, hence also convergent; denote its limit by L. Using the definition of the sequence, one may then argue that

$$a_{n+1} = \sqrt{2a_n} \quad\Longrightarrow\quad \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2a_n} \quad\Longrightarrow\quad L = \sqrt{2L}.$$

This gives $L^2 = 2L$, so either $L = 0$ or else $L = 2$. On the other hand, we must also have

$$1 \leq a_n \leq 2 \quad\Longrightarrow\quad 1 \leq \lim_{n\to\infty} a_n \leq 2 \quad\Longrightarrow\quad 1 \leq L \leq 2$$

because of equation (7.1). We conclude that the limit of the sequence is $L = 2$. $\square$

7.2 Convergence of series

Definition 7.8 – Partial sums

Given a sequence $\{a_n\}$, we define the sequence of its partial sums $\{s_n\}$ by

$$s_n = a_1 + a_2 + \ldots + a_n \quad \text{for each } n \geq 1.$$

Definition 7.9 – Infinite series

Let $\{a_n\}$ be a given sequence and let $\{s_n\}$ be the sequence of its partial sums. In the case that $\{s_n\}$ happens to converge, one may introduce the series

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} s_N$$

and we say that the series converges. Otherwise, we say that the series diverges.

Theorem 7.10 – *n*th term test

If the series $\sum_{n=1}^{\infty} a_n$ converges, then its nth term must satisfy $\lim_{n\to\infty} a_n = 0$.

Theorem 7.11 – Geometric series

The geometric series $\sum_{n=1}^{\infty} x^n$ converges if and only if $|x| < 1$, in which case

$$\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}, \qquad \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

- There are very few series whose exact value can be determined explicitly. Thus, we shall mainly worry about the convergence of a series and not its exact value.
- The nth term test implies the divergence of the series $\sum_{n=1}^{\infty} a_n$ whenever a_n fails to approach zero, but it does not provide any conclusions when a_n approaches zero.

Example 7.12 In view of the nth term test, each of the following series is divergent.

$$\sum_{n=1}^{\infty} \frac{n}{n+1}, \qquad \sum_{n=1}^{\infty} \frac{2n+1}{3n+2}, \qquad \sum_{n=1}^{\infty} (-1)^n, \qquad \sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n. \qquad \square$$

Example 7.13 We use the formula for a geometric series to compute the sum

$$S = \sum_{n=1}^{\infty} \frac{2^{n+2}}{3^{2n+1}}.$$

If we first isolate the part of the exponent that depends on n, then we can simplify to get

$$S = \frac{4}{3} \cdot \sum_{n=1}^{\infty} \frac{2^n}{3^{2n}} = \frac{4}{3} \cdot \sum_{n=1}^{\infty} \left(\frac{2}{9}\right)^n = \frac{4}{3} \cdot \frac{2/9}{1-2/9} = \frac{8}{21}. \qquad \square$$

7.3 Integral test

Theorem 7.14 – Integral test

Suppose $f(x)$ is continuous, non-negative and decreasing for large x. Then the series $\sum_{n=1}^\infty f(n)$ converges if and only if the integral $\int_1^n f(x)\,dx$ is bounded for all $n \in \mathbb{N}$.

Theorem 7.15 – Convergence of p-series

The series $\sum_{n=1}^\infty \frac{1}{n^p}$ converges when $p > 1$ and it diverges when $p \leq 1$.

- The integral test is not commonly needed, but it is useful for proving Theorem 7.15.
- The p-series $\sum_{n=1}^\infty \frac{1}{n^p}$ and the geometric series $\sum_{n=1}^\infty x^n$ are the two most important series. They can be used to address the convergence of other series by comparison.

Example 7.16 We give the proof of Theorem 7.15. If it happens that $p \leq 0$, then the nth term n^{-p} does not approach zero and thus the series diverges. Suppose now that $p > 0$. Since $f(x) = x^{-p}$ is continuous, positive and decreasing for each $x > 0$, the integral test is applicable. If we assume that $p \neq 1$, then convergence is determined by the integral

$$\int_1^n f(x)\,dx = \int_1^n x^{-p}\,dx = \left[\frac{x^{1-p}}{1-p}\right]_1^n = \frac{n^{1-p}-1}{1-p}.$$

When $p > 1$, the exponent $1-p$ is negative, so this expression is bounded for all n and the series converges. When $p < 1$, on the other hand, n^{1-p} becomes arbitrarily large and the series diverges. When $p = 1$, finally, convergence is determined by the integral

$$\int_1^n f(x)\,dx = \int_1^n x^{-1}\,dx = \Big[\ln x\Big]_1^n = \ln n.$$

Since $\ln n$ becomes arbitrarily large, the series diverges when $p = 1$ as well. □

Example 7.17 We use the integral test to study the convergence of the series

$$\sum_{n=1}^\infty \frac{1}{n^2+1}.$$

In this case, $f(x) = 1/(x^2+1)$ is continuous, positive and decreasing for each $x > 0$, as a larger choice of x leads to a larger denominator. Let us then compute the integral

$$\int_1^n f(x)\,dx = \int_1^n \frac{dx}{x^2+1} = \tan^{-1} n - \tan^{-1} 1 = \tan^{-1} n - \frac{\pi}{4}.$$

In view of the definition of the inverse tangent, one has $\tan^{-1} n \leq \pi/2$ for all $n \in \mathbb{N}$. This means that the integral is bounded for all $n \in \mathbb{N}$ and that the series converges. □

7.4 Comparison tests

Theorem 7.18 – Comparison test

Consider two non-negative series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ which satisfy

$$a_n \leq b_n \quad \text{for all large enough } n.$$

If the series $\sum_{n=1}^{\infty} b_n$ happens to converge, then the series $\sum_{n=1}^{\infty} a_n$ must also converge. If the series $\sum_{n=1}^{\infty} a_n$ happens to diverge, then the series $\sum_{n=1}^{\infty} b_n$ must also diverge.

Theorem 7.19 – Limit comparison test

Suppose that $\{a_n\}$ and $\{b_n\}$ are non-negative for large enough n and that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1. \tag{7.2}$$

Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{n=1}^{\infty} b_n$ converges.

- The comparison test asserts that any series which is smaller than a convergent series is convergent and that any series which is bigger than a divergent series is divergent.
- The limit comparison test is actually applicable whenever the limit (7.2) is nonzero. This test is especially useful when the terms a_n are given by a rational function.

Example 7.20 As a typical application of the comparison test, one may argue that

$$\sum_{n=1}^{\infty} \frac{2n}{n^3 + e^n} \leq \sum_{n=1}^{\infty} \frac{2n}{n^3} = \sum_{n=1}^{\infty} \frac{2}{n^2}.$$

Since the rightmost series is a convergent p-series, the leftmost series converges as well. □

Example 7.21 We use the limit comparison test to address the convergence of

$$\sum_{n=1}^{\infty} \frac{2n^2 + n + 1}{n^3 + 2}.$$

When it comes to large values of n, the numerator behaves like $2n^2$ and the denominator behaves like n^3. We thus expect the limit comparison test to be applicable with

$$a_n = \frac{2n^2 + n + 1}{n^3 + 2}, \qquad b_n = \frac{2n^2}{n^3} = \frac{2}{n}.$$

To verify that the condition (7.2) is satisfied, we note that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2 + n + 1}{n^3 + 2} \cdot \frac{n}{2} = \lim_{n\to\infty} \frac{2n^3}{2n^3} = 1.$$

Since $\sum_{n=1}^{\infty} b_n$ is a divergent p-series, we conclude that $\sum_{n=1}^{\infty} a_n$ diverges as well. □

7.5 Ratio test

Theorem 7.22 – Ratio test for non-negative series

Consider a non-negative series $\sum_{n=1}^{\infty} a_n$ together with the limit

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}. \tag{7.3}$$

If $L < 1$, then the given series converges. If $L > 1$, then the given series diverges.

- The ratio test is especially useful when a_n involves either exponents or factorials.
- The ratio test does not provide any conclusions for the remaining case $L = 1$.

Example 7.23 Consider the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$. In this case, we have

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{2^n}{2^{n+1}} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}.$$

Since this limit is strictly smaller than 1, the given series converges by the ratio test. □

Example 7.24 Consider the series $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$. In this case, we have

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{e^{n+1}}{e^n} \cdot \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{en^2}{n^2} = e.$$

Since this limit is strictly larger than 1, the given series diverges by the ratio test. □

Example 7.25 Consider the series $\sum_{n=1}^{\infty} \frac{\ln n}{e^n}$. In this case, we have

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \cdot \frac{e^n}{e^{n+1}} = \lim_{n\to\infty} \frac{\ln(n+1)}{e \ln n}.$$

The last limit has the form ∞/∞, so L'Hôpital's rule is applicable and one finds that

$$L = \lim_{n\to\infty} \frac{1/(n+1)}{e/n} = \lim_{n\to\infty} \frac{n}{e(n+1)} = \lim_{n\to\infty} \frac{n}{en} = \frac{1}{e}.$$

Since this limit is strictly smaller than 1, the given series converges by the ratio test. □

Example 7.26 Consider the series $\sum_{n=1}^{\infty} \frac{n \cdot 3^n}{2^n}$. In this case, we have

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{3^{n+1}}{3^n} \cdot \frac{2^n}{2^{n+1}} = \lim_{n\to\infty} \frac{3(n+1)}{2n} = \frac{3}{2}.$$

Since this limit is strictly larger than 1, the given series diverges by the ratio test. □

7.6 Absolute convergence

Definition 7.27 – Absolute convergence

A series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent, if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Theorem 7.28 – Absolute convergence implies convergence

If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then the series $\sum_{n=1}^{\infty} a_n$ converges as well.

Theorem 7.29 – Ratio test for arbitrary series

Consider an arbitrary series $\sum_{n=1}^{\infty} a_n$ together with the limit

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

If $L < 1$, then the given series converges. If $L > 1$, then the given series diverges.

Example 7.30 To show that the series $\sum_{n=1}^{\infty} \frac{n \sin n}{n^3+1}$ converges absolutely, we note that

$$\sum_{n=1}^{\infty} \left| \frac{n \sin n}{n^3+1} \right| = \sum_{n=1}^{\infty} \frac{n|\sin n|}{n^3+1} \leq \sum_{n=1}^{\infty} \frac{n}{n^3+1} \leq \sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Since the rightmost series is a convergent p-series, the leftmost series converges as well. □

Example 7.31 Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{2^n}$. Using the ratio test, we get

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \cdot \frac{n+1}{n} \cdot \frac{2^n}{2^{n+1}} \right| = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}.$$

Since this limit is strictly smaller than 1, the given series converges by the ratio test. □

Example 7.32 To show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^4+3n+1}$ converges absolutely, we let

$$a_n = \frac{n}{n^4+3n+1}, \qquad b_n = \frac{n}{n^4} = \frac{1}{n^3}.$$

Note that these terms are comparable for large enough n, as one can easily check that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^4}{n^4+3n+1} = \lim_{n\to\infty} \frac{n^4}{n^4} = 1.$$

Thus, the limit comparison test is applicable. Since $\sum_{n=1}^{\infty} b_n$ is a convergent p-series, the series $\sum_{n=1}^{\infty} a_n$ converges as well and the original series converges absolutely. □

7.7 Alternating series test

Theorem 7.33 – Alternating series test

Suppose $\{a_n\}$ is non-negative and decreasing for large enough n with $\lim_{n\to\infty} a_n = 0$. Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ is convergent.

Example 7.34 According to the last theorem, the following series are all convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{e^n}.$$

This is easy to see because the denominators n, n^2, e^n are all increasing and they become arbitrarily large, so the fractions $1/n$, $1/n^2$, $1/e^n$ are all decreasing towards zero. □

Example 7.35 We use the alternating series test to establish the convergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, n}{n^2+1}. \tag{7.4}$$

In this case, it is clear that $f(x) = x/(x^2+1)$ is positive for all $x > 0$ and we also have

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{x^2+1} = \lim_{x\to\infty} \frac{x}{x^2} = 0.$$

To check that $f(x)$ is decreasing for large x, one may use the quotient rule to verify that

$$f'(x) = \frac{x^2+1-2x\cdot x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} < 0$$

for all $x > 1$. This already implies that the alternating series (7.4) is convergent. □

Example 7.36 We use the alternating series test to establish the convergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\ln n}{n}. \tag{7.5}$$

In this case, the function $f(x) = \frac{\ln x}{x}$ is positive for all $x > 1$ and we also have

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$$

by L'Hôpital's rule. To check that $f(x)$ is decreasing for large x, we note that

$$f'(x) = \frac{\frac{1}{x}\cdot x - \ln x}{x^2} = \frac{1-\ln x}{x^2} < 0$$

for all $x > e$. Using the last theorem, we conclude that the series (7.5) converges. □

7.8 Power series

Definition 7.37 – Power series

A power series is a sum of powers of x such as

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{n=0}^{\infty} a_n x^n.$$

This expression is only defined for the values of x for which the series converges.

Definition 7.38 – Radius of convergence

If a power series converges when $|x| < R$ and diverges when $|x| > R$, then we call R the radius of convergence. This radius could be any positive number or even $+\infty$.

Theorem 7.39 – Differentiation of power series

Every power series may be differentiated term by term. More precisely, suppose that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad g(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}.$$

If $f(x)$ converges when $|x| < R$, then the same is true for $g(x)$, and $f'(x) = g(x)$.

- To determine the radius of convergence, one usually resorts to the ratio test.
- The differentiation of infinite sums is not always justified. Namely, it is not true that

$$f(x) = \sum_{n=0}^{\infty} f_n(x) \quad \Longrightarrow \quad f'(x) = \sum_{n=0}^{\infty} f_n'(x).$$

Example 7.40 Consider the power series $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n+1}$. In this case, we have

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|x|^{n+1}}{|x|^n} \cdot \frac{n+1}{n+2} = |x| \cdot \lim_{n\to\infty} \frac{n+1}{n+2} = |x|.$$

In particular, the given power series converges when $|x| < 1$ and it diverges when $|x| > 1$, so its radius of convergence is equal to $R = 1$. □

Example 7.41 Consider the power series $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. In this case, we have

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|x|^{n+1}}{|x|^n} \cdot \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0.$$

Since $L < 1$ for any value of x, the series converges for all x and the radius is $R = +\infty$. □