Chapter 1

Functions

1.1 Domain and range

Definition 1.1 – Domain and range

A function $f\colon X \to Y$ is a rule or formula that assigns a particular value $f(x)$ to each admissible value of x. The set of all admissible values of x is called the domain of f, while the set of all possible values of $f(x)$ is called the range of f.

Definition 1.2 – Graph

The graph of a function f is the set of all points (x, y) such that $y = f(x)$. This is a curve in the xy-plane and every vertical line may intersect the curve at most once.

- The domain of a function is usually easy to determine. One excludes the values of x that lead to a zero denominator, the square root of a negative number, and so on.
- The range of a function is generally hard to determine. It can be obtained using the same approach, however, if the equation $y = f(x)$ can be solved for x.
- The interval that consists of all points between a and b is denoted by one of

$$(a, b), \qquad [a, b), \qquad (a, b], \qquad [a, b].$$

  In each case, a square bracket is used for endpoints that belong to the interval and a round bracket is used for those that do not. Thus, the interval $(a, b]$ consists of all points $a < x \leq b$ and the interval $[a, b]$ consists of all points $a \leq x \leq b$.
- The symbols $-\infty$ and $+\infty$ are used to denote infinite intervals such as

$$(-\infty, b), \qquad (-\infty, b], \qquad (a, +\infty), \qquad [a, +\infty).$$

  The leftmost interval consists of all points $x < b$ and the rightmost interval consists of all points $x \geq a$. Since the symbols $-\infty$ and $+\infty$ are not real numbers, they do not belong to any interval. Thus, a round bracket is always used for $-\infty$ and $+\infty$.

Example 1.3 We find the domain and the range of the function f defined by

$$f(x) = \frac{5x-1}{3x-4}.$$

The domain consists of all points $x \neq 4/3$, as the denominator cannot be zero. To find the range, we need to find the values that are attained by $y = f(x)$. For which values of y does this equation actually have a solution? In our case, it is easy to check that

$$y = \frac{5x-1}{3x-4} \quad \Longleftrightarrow \quad y(3x-4) = 5x-1 \quad \Longleftrightarrow \quad 3xy - 4y = 5x - 1.$$

Solving this equation for x, we bring all occurrences of x on one side and we get

$$3xy - 5x = 4y - 1 \quad \Longleftrightarrow \quad x(3y-5) = 4y-1 \quad \Longleftrightarrow \quad x = \frac{4y-1}{3y-5}.$$

The rightmost formula determines the value of x that satisfies $y = f(x)$. Since the formula makes sense for any number $y \neq 5/3$, the range consists of all numbers $y \neq 5/3$. □

Example 1.4 We find the domain and the range of the function f defined by

$$f(x) = \sqrt{\frac{1+x}{2-x}}.$$

When it comes to the domain, we need to have $x \neq 2$ to avoid a zero denominator and we need the fraction to be non-negative. In other words, the numerator and the denominator should have the same sign. If they are both non-negative, then one has

$$1 + x \geq 0, \qquad 2 - x \geq 0 \quad \Longrightarrow \quad -1 \leq x \leq 2.$$

If they are both non-positive, then one similarly has

$$1 + x \leq 0, \qquad 2 - x \leq 0 \quad \Longrightarrow \quad -1 \geq x \geq 2,$$

which is obviously absurd. In particular, only the former case may arise, so the domain is the interval $[-1, 2)$. To find the range, we note that $y = f(x)$ is non-negative and we solve this equation for x. Squaring both sides and rearranging terms, one finds that

$$\begin{aligned} y^2 = \frac{1+x}{2-x} \quad &\Longleftrightarrow \quad 2y^2 - xy^2 = 1 + x \quad \Longleftrightarrow \quad 2y^2 - 1 = xy^2 + x \\ &\Longleftrightarrow \quad 2y^2 - 1 = x(y^2+1) \quad \Longleftrightarrow \quad x = \frac{2y^2-1}{y^2+1}. \end{aligned}$$

The last expression is defined for all values of y, so it does not place any restrictions on y. However, we do have the restriction $y \geq 0$ which arose when we squared both sides of the original equation. This means that the range of the given function is $[0, +\infty)$. □

1.2 Injective, surjective and bijective

Definition 1.5 – Injective, surjective and bijective

A function $f\colon X \to Y$ is called injective or one-to-one, if it satisfies

$$f(x_1) = f(x_2) \quad \Longrightarrow \quad x_1 = x_2.$$

A function $f\colon X \to Y$ is called surjective or onto, if its range is equal to Y, and it is called bijective, if it is both injective and surjective.

- An injective function f maps distinct values of x to distinct values of $y = f(x)$. This means that no horizontal line may intersect the graph of f more than once.
- The function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is neither injective nor surjective.
- The function $f\colon [0,\infty) \to [0,\infty)$ defined by $f(x) = x^2$ is both injective and surjective.

Example 1.6 We show that the function $f\colon [0,1] \to \mathbb{R}$ is injective in the case that

$$f(x) = \frac{4x-3}{2x+1}.$$

First, we assume that $f(x_1) = f(x_2)$ and we clear denominators to write

$$\begin{aligned} \frac{4x_1-3}{2x_1+1} = \frac{4x_2-3}{2x_2+1} \quad &\Longrightarrow \quad (4x_1-3)(2x_2+1) = (4x_2-3)(2x_1+1) \\ &\Longrightarrow \quad 8x_1x_2 - 6x_2 + 4x_1 - 3 = 8x_1x_2 - 6x_1 + 4x_2 - 3. \end{aligned}$$

Once we now cancel the common terms, we may easily conclude that

$$-6x_2 + 4x_1 = -6x_1 + 4x_2 \quad \Longrightarrow \quad 10x_1 = 10x_2 \quad \Longrightarrow \quad x_1 = x_2. \qquad \square$$

Example 1.7 We show that the function $f\colon (0,1) \to (0,\infty)$ is surjective in the case that

$$f(x) = \frac{x}{1-x}.$$

Since $0 < x < 1$, both x and $1-x$ are positive, so the same is true for $y = f(x)$ and the range is contained in $(0,\infty)$. To show that the range is equal to $(0,\infty)$, we need to check that the equation $y = f(x)$ has a solution $0 < x < 1$ for each $y > 0$. Let us now note that

$$\begin{aligned} y = \frac{x}{1-x} \quad &\Longleftrightarrow \quad y - xy = x \quad \Longleftrightarrow \quad y = xy + x \\ &\Longleftrightarrow \quad y = x(y+1) \quad \Longleftrightarrow \quad x = \frac{y}{y+1}. \end{aligned}$$

The rightmost formula determines the value of x that satisfies $y = f(x)$. Since $y > 0$, it is easy to see that this value $x = y/(y+1)$ lies in the interval $(0,1)$, as needed. $\square$

1.3 Quadratic functions

Theorem 1.8 – Quadratic functions with real roots

Consider the quadratic $f(x) = ax^2 + bx + c$. If it happens that $b^2 - 4ac \geq 0$, then $f(x)$ can be factored as $f(x) = a(x - x_1)(x - x_2)$, where x_1, x_2 are its real roots

$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \qquad x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}. \tag{1.1}$$

Theorem 1.9 – Quadratic functions with no real roots

Consider the quadratic $f(x) = ax^2 + bx + c$. If it happens that $b^2 - 4ac < 0$, then $f(x)$ is not a product of real linear factors and thus $f(x)$ has no real roots.

- The expression $\Delta = b^2 - 4ac$ is called the discriminant of $f(x) = ax^2 + bx + c$.
- The theorems above can be established by completing the square to write

$$\frac{f(x)}{a} = \left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}, \tag{1.2}$$

where the right hand side is either the difference or the sum of two squares.

Example 1.10 Let $f(x) = 3x^2 - 4x + 1$. Then $\Delta = 16 - 4 \cdot 3 = 4$ is positive and

$$x_1 = \frac{4 - \sqrt{4}}{2 \cdot 3} = \frac{4 - 2}{6} = \frac{1}{3}, \qquad x_2 = \frac{4 + \sqrt{4}}{2 \cdot 3} = \frac{4 + 2}{6} = 1.$$

This means that $f(x)$ can be factored as $f(x) = 3(x - 1/3)(x - 1) = (3x - 1)(x - 1)$. □

Example 1.11 Let $f(x) = 3x^2 - 4x + 1$. To determine the range of f, we write

$$y = 3x^2 - 4x + 1$$

and then solve for x in terms of y. Using the quadratic formula (1.1), we get

$$3x^2 - 4x + (1 - y) = 0 \quad\Longrightarrow\quad x = \frac{4 \pm \sqrt{16 - 12(1 - y)}}{2 \cdot 3} = \frac{4 \pm \sqrt{12y + 4}}{6}.$$

This gives the restriction $12y \geq -4$, namely $y \geq -1/3$, so the range is $[-1/3, +\infty)$. □

Example 1.12 Let $f(x) = -2x^2 + 3x + 5$. Then $\Delta = 9 + 4 \cdot 10 = 49$ is positive and

$$x_1 = \frac{-3 - \sqrt{49}}{2 \cdot (-2)} = \frac{3 + 7}{4} = \frac{5}{2}, \qquad x_2 = \frac{-3 + \sqrt{49}}{2 \cdot (-2)} = \frac{3 - 7}{4} = -1.$$

In particular, $f(x)$ can be factored as $f(x) = -2(x - 5/2)(x + 1) = -(2x - 5)(x + 1)$. □

Example 1.13 Let $f(x) = -2x^2 + 3x + 5$. To determine the range of f, we note that

$$y = -2x^2 + 3x + 5 \quad \Longrightarrow \quad 2x^2 - 3x + (y-5) = 0.$$

Using the quadratic formula (1.1), one may then solve for x to conclude that

$$x = \frac{3 \pm \sqrt{9 - 8(y-5)}}{2 \cdot 2} = \frac{3 \pm \sqrt{49 - 8y}}{4}.$$

This gives the restriction $8y \leq 49$, namely $y \leq 49/8$, so the range is $(-\infty, 49/8]$. □

Example 1.14 We study the positivity of a quadratic polynomial with real roots, say

$$f(x) = a(x - x_1)(x - x_2), \quad \text{where } x_1 \leq x_2.$$

Assume that $a > 0$ first. Then $f(x) > 0$ if and only if $(x - x_1)(x - x_2) > 0$, hence if and only if the two factors have the same sign. The two factors are both positive when

$$x - x_1 > 0, \qquad x - x_2 > 0 \quad \Longleftrightarrow \quad x > x_2$$

and the two factors are both negative when

$$x - x_1 < 0, \qquad x - x_2 < 0 \quad \Longleftrightarrow \quad x < x_1.$$

Since the quadratic is positive only for those values, it is negative when $x_1 < x < x_2$, so it is negative between its two roots. This is true for the case $a > 0$ and the other case differs by a sign, so any quadratic with $a < 0$ must be positive between its two roots. □

Example 1.15 We determine the values of x for which $f(x) = 3x^2 + 5x - 2$ is negative. Since $\Delta = 25 + 4 \cdot 6 = 49$ is positive, the quadratic has two real roots which are given by

$$x_1 = \frac{-5 - \sqrt{49}}{2 \cdot 3} = \frac{-5 - 7}{6} = -2, \qquad x_2 = \frac{-5 + \sqrt{49}}{2 \cdot 3} = \frac{-5 + 7}{6} = \frac{1}{3}.$$

As in the previous example, this implies that $f(x) < 0$ if and only if $-2 < x < 1/3$. □

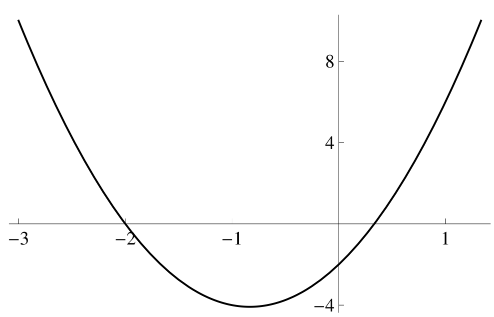


Figure 1.1: The graph of the quadratic $f(x) = 3x^2 + 5x - 2$.

1.4 Polynomial functions

Theorem 1.16 – Rational root theorem

Suppose that $f(x) = a_n x^n + \ldots + a_1 x + a_0$ has integer coefficients and $a_n \neq 0$. If f has a rational root x_0, then one can write $x_0 = p/q$ for some relatively prime integers p, q such that p divides a_0 and q divides a_n.

Theorem 1.17 – Factor theorem

A polynomial $f(x)$ that has x_0 as a root must have $x - x_0$ as a factor. In other words, one may factor $f(x)$ and write $f(x) = (x - x_0) \cdot g(x)$ for some polynomial $g(x)$.

- A rational number is the quotient p/q of two integers. To say that p, q are relatively prime is to say that p, q have no common integer factor other than ± 1.
- The rational root theorem is important because it gives a finite number of possible roots. One may then list the possible roots and check which of them are actual roots.

Example 1.18 We use division of polynomials to show that

$$x^3 + 2x^2 - 3 = (x - 1)(x^2 + 3x + 3). \tag{1.3}$$

The left hand side vanishes when $x = 1$ because $1 + 2 - 3 = 0$. This means that $x = 1$ is a root and that $x - 1$ is a factor. Let us now proceed to divide the two polynomials.

$$\begin{array}{r|lll} & \boldsymbol{x^2} & & \\ \hline \boldsymbol{x} - 1 & \boldsymbol{x^3} & +2x^2 & -3 \\ & x^3 & -x^2 & \\ \hline & & 3x^2 & -3 \end{array}$$

Division of polynomials is very similar to division of integers. The key idea is to look at the highest powers of x in each case. First, we start out with $x^3 + 2x^2 - 3$ and $x - 1$. Looking at the highest powers only, we get a quotient of $x^3/x = x^2$ and we include this term in our answer at the top. Next, we multiply x^2 by the whole divisor $x - 1$ and we subtract the result from the whole dividend $x^3 + 2x^2 - 3$. This yields a remainder of $3x^2 - 3$.

Since the remainder $3x^2 - 3$ is quadratic and the divisor $x - 1$ is only linear, one may now proceed as before. The next term in the quotient is $3x^2/x = 3x$, so we insert that term in our answer, we multiply it by $x - 1$ and subtract. The new remainder is $3x - 3$.

$$\begin{array}{r|llll} & x^2 & +\boldsymbol{3x} & & \\ \hline \boldsymbol{x} - 1 & x^3 & +2x^2 & -3 & \\ & x^3 & -x^2 & & \\ \hline & & \boldsymbol{3x^2} & -3 & \\ & & 3x^2 & -3x & \\ \hline & & & 3x & -3 \end{array}$$

Needless to say, one may repeat this approach once again. The next term in the quotient will be $3x/x = 3$ and this is also the last term since $3(x - 1) = 3x - 3$. In particular, the polynomial division gives a quotient of $x^2 + 3x + 3$ and equation (1.3) follows. □

Example 1.19 We use the rational root theorem to factor the polynomial

$$f(x) = 3x^3 - 4x^2 - 5x + 2.$$

If there is a rational root, it must have the form p/q, where p divides 2 and q divides 3. The only possibilities are thus ± 1, ± 2, $\pm 1/3$, $\pm 2/3$. Checking the first few, one finds that

$$f(1) = -4, \qquad f(-1) = 0, \qquad f(2) = 0, \qquad f(-2) = -28.$$

This means that $x = -1$ and $x = 2$ are both roots, so $x + 1$ and $x - 2$ are both factors. To find the third factor of the cubic polynomial, one may proceed to check the other possible roots or else use division. Namely, $(x + 1)(x - 2) = x^2 - x - 2$ must divide $f(x)$ and it remains to determine the quotient $f(x)/(x^2 - x - 2)$.

$$\begin{array}{r|rrrr} & 3x & -1 & & \\ \cline{2-5} x^2 - x - 2 & 3x^3 & -4x^2 & -5x & +2 \\ & 3x^3 & -3x^2 & -6x & \\ \cline{2-5} & & -x^2 & +x & +2 \\ & & -x^2 & +x & +2 \\ \cline{3-5} & & & & 0 \end{array}$$

In view of this computation, the quotient is then $3x - 1$ and so $f(x)$ can be factored as

$$f(x) = (3x - 1)(x^2 - x - 2) = (3x - 1)(x + 1)(x - 2). \qquad \square$$

Example 1.20 We use the rational root theorem to factor the polynomial

$$f(x) = 2x^3 + x^2 + x - 1.$$

The only possible rational roots are ± 1, $\pm 1/2$ and one easily checks that

$$f(1) = 3, \qquad f(-1) = -3, \qquad f(1/2) = 0, \qquad f(-1/2) = -3/2.$$

This implies that $x = 1/2$ is a root and that $x - 1/2$ is a factor. In order to avoid dealing with fractions, we note that $2x - 1$ must also be a factor and proceed to use division.

$$\begin{array}{r|rrrr} & x^2 & +x & +1 & \\ \cline{2-5} 2x - 1 & 2x^3 & +x^2 & +x & -1 \\ & 2x^3 & -x^2 & & \\ \cline{2-5} & & +2x^2 & +x & -1 \\ & & +2x^2 & -x & \\ \cline{3-5} & & & +2x & -1 \\ & & & +2x & -1 \\ \cline{4-5} & & & & 0 \end{array}$$

This leads to the factorisation $f(x) = (2x - 1)(x^2 + x + 1)$ and the quadratic cannot be factored any further because its discriminant $\Delta = 1 - 4 \cdot 1 = -3$ is negative. $\square$

1.5 Trigonometric functions

Definition 1.21 – Trigonometric functions

The six trigonometric functions are defined in terms of Figure 1.2, where the angle θ is measured in radians and the point (x, y) is on the unit circle. The sine, cosine, tangent, cosecant, secant and cotangent of the angle θ are then defined as the ratios

$$\sin\theta = y/1, \qquad \cos\theta = x/1, \qquad \tan\theta = y/x,$$
$$\csc\theta = 1/y, \qquad \sec\theta = 1/x, \qquad \cot\theta = x/y.$$

Theorem 1.22 – Addition formulas for sine and cosine

Given any real numbers α and β, one has the addition formulas

$$\sin(\alpha \pm \beta) = \sin\alpha \cdot \cos\beta \pm \cos\alpha \cdot \sin\beta,$$
$$\cos(\alpha \pm \beta) = \cos\alpha \cdot \cos\beta \mp \sin\alpha \cdot \sin\beta.$$

- By definition, 1 radian is $180/\pi$ degrees and so 2π radians are 360 degrees.
- All trigonometric functions can be expressed in terms of sine and cosine, namely

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \csc\theta = \frac{1}{\sin\theta}. \tag{1.4}$$

- Using Pythagoras' theorem along with Definition 1.21, one easily finds that

$$\sin^2\theta + \cos^2\theta = 1, \qquad \tan^2\theta + 1 = \sec^2\theta, \qquad \cot^2\theta + 1 = \csc^2\theta. \tag{1.5}$$

  Here, the first identity holds by Pythagoras' theorem because $y^2 + x^2 = 1$. The other two identities follow from the first upon division with $\cos^2\theta$ and $\sin^2\theta$, respectively.

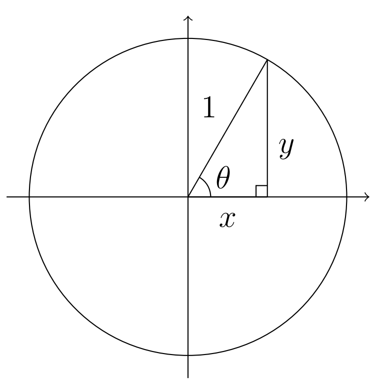


Figure 1.2: This triangle is used to define the six trigonometric functions.

Example 1.23 We use the addition formula for cosine to prove the half-angle formulas

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}, \qquad \cos^2\theta = \frac{1+\cos(2\theta)}{2}. \tag{1.6}$$

Since $\sin^2\theta + \cos^2\theta = 1$, the addition formula for cosine ensures that

$$\cos(2\theta) = \cos(\theta+\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta.$$

Rearranging terms, one obtains the first identity in (1.6). This also implies the second, as

$$\cos^2\theta = 1 - \sin^2\theta = 1 - \frac{1-\cos(2\theta)}{2} = \frac{1+\cos(2\theta)}{2}. \qquad \square$$

Example 1.24 We relate the sines and the cosines of any two angles θ_1, θ_2 whose sum is equal to π. Since $\theta_1 + \theta_2 = \pi$, we may use the addition formulas of Theorem 1.22 to get

$$\begin{aligned}\sin\theta_2 &= \sin(\pi-\theta_1) = \sin\pi\cdot\cos\theta_1 - \cos\pi\cdot\sin\theta_1,\\ \cos\theta_2 &= \cos(\pi-\theta_1) = \cos\pi\cdot\cos\theta_1 + \sin\pi\cdot\sin\theta_1.\end{aligned}$$

On the other hand, $\sin\pi = 0$ and $\cos\pi = -1$ by definition, so these equations reduce to

$$\sin\theta_2 = \sin\theta_1, \qquad \cos\theta_2 = -\cos\theta_1. \qquad \square$$

Example 1.25 We determine the angles $0 \leq \theta \leq 2\pi$ which satisfy $4\sin^2\theta + 8\sin\theta = 5$. If we let $x = \sin\theta$ to simplify the notation, then we have $4x^2 + 8x - 5 = 0$ and thus

$$x = \frac{-8 \pm \sqrt{64 + 4\cdot 20}}{2\cdot 4} = \frac{-8\pm\sqrt{144}}{8} = -1 \pm \frac{3}{2} \quad\Longrightarrow\quad x = \frac{1}{2}, -\frac{5}{2}.$$

Since $x = \sin\theta$ must lie between -1 and 1, the only relevant solution is $x = \sin\theta = \frac{1}{2}$. In view of the graph of the sine function, there should be two angles $0 \leq \theta \leq 2\pi$ that satisfy this condition. The first one is $\theta_1 = \frac{\pi}{6}$ and the second one is $\theta_2 = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$. $\square$

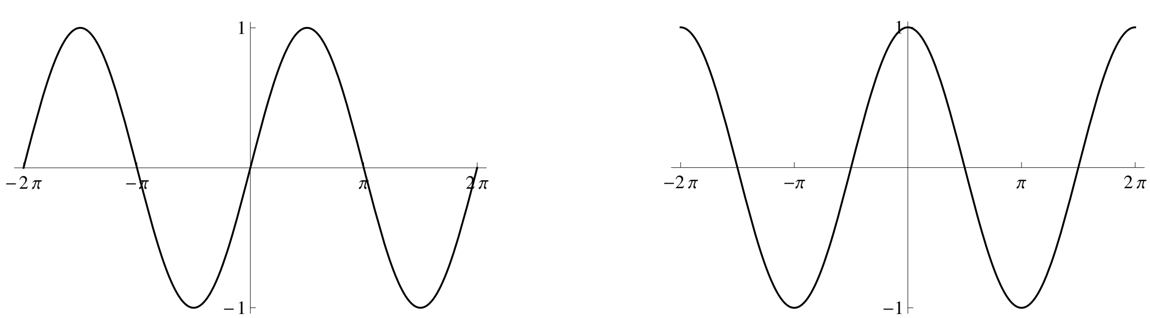


Figure 1.3: The graphs of $f(x) = \sin x$ and $f(x) = \cos x$, respectively.

1.6 Exponential functions

Definition 1.26 – Exponential function

The exponential function with base $a > 0$ is the function defined by $f(x) = a^x$. Its domain consists of all real numbers and its main properties are the following.

$$a^{x+y} = a^x \cdot a^y, \qquad (a^x)^y = a^{xy}, \qquad a^{-x} = 1/a^x,$$
$$a^{x-y} = a^x/a^y, \qquad (ab)^x = a^x b^x, \qquad a^0 = 1.$$

- When x is a positive integer, the power a^x is defined in the usual way as the product of x copies of a. The formula $a^{x+y} = a^x \cdot a^y$ is then easily seen to hold. When $y = 0$, this formula reduces to $a^x = a^x \cdot a^0$, so one is led to define $a^0 = 1$ and also

$$1 = a^0 = a^{x-x} = a^x \cdot a^{-x} \quad \Longrightarrow \quad a^{-x} = 1/a^x.$$

- When x is a rational number, the power a^x is defined in terms of roots. In fact, let n be a positive integer and consider the power $a^{1/n}$. To ensure that

$$\left(a^{1/n}\right)^n = a^{n/n} = a^1 = a,$$

one needs $a^{1/n}$ to be the nth root of a. This is the main reason that the base a is required to be positive. To compute rational powers of a, one may then argue that

$$9^{3/2} = (9^{1/2})^3 = 3^3 = 27, \qquad 8^{4/3} = (8^{1/3})^4 = 2^4 = 16.$$

- When x is an irrational number, a^x can be determined using approximations. In the case of $3^{\sqrt{2}}$ for instance, one notes that $\sqrt{2} \approx 1.4142$ and successively computes

$$3^{1.4} \approx 4.6555, \qquad 3^{1.41} \approx 4.7070, \qquad 3^{1.414} \approx 4.7277, \qquad 3^{1.4142} \approx 4.7287.$$

Since the latter powers involve rational exponents, those can be computed as before. In particular, one may obtain approximations of $3^{\sqrt{2}}$ to any level of accuracy.

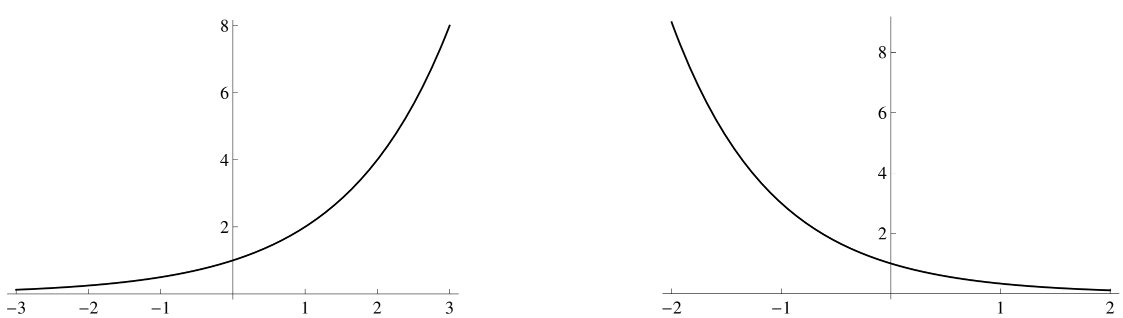


Figure 1.4: The graphs of $f(x) = 2^x$ and $f(x) = (\frac{1}{3})^x$, respectively.

1.7 Inverse functions

Theorem 1.27 – Inverse function

If the function $f\colon A \to B$ is bijective, then there exists a function $g\colon B \to A$ such that

$$g(f(x)) = x \text{ for all } x \text{ in } A, \qquad f(g(y)) = y \text{ for all } y \text{ in } B.$$

In fact, this function is unique, it is called the inverse of f and it is denoted by $g = f^{-1}$.

Definition 1.28 – Logarithmic function

Consider the exponential function defined by $f(x) = a^x$ for some $a > 0$. When $a \neq 1$, this is a bijection $f\colon \mathbb{R} \to (0, \infty)$ whose inverse is denoted by $\log_a$. Moreover, the usual properties for exponentials imply the following properties for logarithms.

$$\log_a a^x = x, \qquad \log_a(x \cdot y) = \log_a x + \log_a y, \qquad \log_a 1 = 0,$$
$$\log_a x^r = r \log_a x, \qquad \log_a(x/y) = \log_a x - \log_a y, \qquad a^{\log_a x} = x.$$

Definition 1.29 – Inverse trigonometric functions

The sine, cosine and tangent give bijective functions between the following sets.

$$\sin\colon \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1], \qquad \cos\colon [0,\pi] \to [-1,1], \qquad \tan\colon \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}.$$

We may thus define $\sin^{-1} x$, $\cos^{-1} x$ for each $-1 \leq x \leq 1$ and also $\tan^{-1} x$ for all x.

- One may determine the inverse function $g = f^{-1}$ explicitly, if the equation $y = f(x)$ can be solved for x. This gives a unique solution $x = g(y)$ whenever f is bijective.
- Note that $\log_a x$ is only defined when $x > 0$. In particular, $\log_a 0$ is not defined.
- The values of the inverse trigonometric functions are frequently interpreted as angles. When it comes to $\theta = \sin^{-1} x$, for instance, θ is an angle whose sine is equal to x.

Example 1.30 We compute the inverse function f^{-1} in the case that

$$f(x) = 1 + \log_3(5 - 2x).$$

Consider the equation $y = 1 + \log_3(5 - 2x)$. Eliminating the logarithm, one finds that

$$y - 1 = \log_3(5 - 2x) \quad \Longleftrightarrow \quad 3^{y-1} = 5 - 2x \quad \Longleftrightarrow \quad 2x = 5 - 3^{y-1}.$$

Thus, the inverse function $x = f^{-1}(y)$ is the function defined by $f^{-1}(y) = \frac{1}{2}(5 - 3^{y-1})$. □

Example 1.31 We compute the inverse function f^{-1} in the case that

$$f(x) = \frac{2^x - 1}{2^x + 3}.$$

Proceeding as in the previous example, we first clear denominators to write

$$y = \frac{2^x - 1}{2^x + 3} \quad\Longleftrightarrow\quad 2^x y + 3y = 2^x - 1 \quad\Longleftrightarrow\quad 3y + 1 = 2^x(1 - y).$$

This implies that $2^x = \frac{3y+1}{1-y}$, so the inverse function is defined by $f^{-1}(y) = \log_2 \frac{3y+1}{1-y}$. □

Example 1.32 We derive the main properties of the logarithmic function. Since $\log_a$ is the inverse of the exponential function, one must clearly have

$$a^{\log_a x} = x, \qquad \log_a a^x = x, \qquad \log_a 1 = \log_a a^0 = 0.$$

This establishes three of the six properties, while the fourth one must hold because

$$x^r = \left(a^{\log_a x}\right)^r = a^{r \log_a x} \quad\Longrightarrow\quad \log_a x^r = \log_a a^{r \log_a x} = r \log_a x.$$

To prove the formula for the logarithm of a product, we start by noting that

$$a^{\log_a x + \log_a y} = a^{\log_a x} \cdot a^{\log_a y} = x \cdot y.$$

Taking the logarithm of both sides, we may thus deduce the fifth property

$$\log_a(x \cdot y) = \log_a a^{\log_a x + \log_a y} = \log_a x + \log_a y.$$

As for the remaining property about the logarithm of a quotient, this must hold because

$$\log_a(x/y) = \log_a(xy^{-1}) = \log_a x + \log_a y^{-1} = \log_a x - \log_a y.$$ □

Example 1.33 We simplify the expression $\tan(\sin^{-1} x)$. Note that $\theta = \sin^{-1} x$ is an angle whose sine is $\sin\theta = x$. When $x \geq 0$, this angle arises in a right triangle with an opposite side of length x and a hypotenuse of length 1. It follows by Pythagoras' theorem that the adjacent side has length $\sqrt{1 - x^2}$, so the definition of tangent gives

$$\tan(\sin^{-1} x) = \tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{x}{\sqrt{1 - x^2}}.$$

When $x \leq 0$, the last equation holds with $-x$ instead of x. This introduces a minus sign in each side of the equation, so the signs get to cancel and the same equation holds. □

Example 1.34 We simplify the expression $\sin(\tan^{-1} x)$. Note that $\theta = \tan^{-1} x$ is an angle whose tangent is $\tan\theta = x$. When $x \geq 0$, such an angle appears in a right triangle with an opposite side of length x and an adjacent side of length 1. The hypotenuse of the triangle must then have length $\sqrt{x^2 + 1}$, so the definition of sine gives

$$\sin(\tan^{-1} x) = \sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{x}{\sqrt{x^2 + 1}}.$$

When $x \leq 0$, the last equation holds with $-x$ instead of x. Once again, this changes each side of the equation by a minus sign, so the signs cancel and the same equation holds. □

Chapter 2

Limits and continuity

2.1 Introduction to limits

- The concept of a limit is a very technical, but extremely useful, concept in calculus. One studies the values $f(x)$ and their behaviour as x approaches a fixed point x_0. If the values $f(x)$ happen to approach some number L, then we say that L is the limit of $f(x)$ as x approaches x_0. In that case, we write $f(x) \to L$ as $x \to x_0$ or simply

$$\lim_{x \to x_0} f(x) = L.$$

- Note that the limit of $f(x)$ is obtained by looking at points x that approach x_0, so the value of the function at x_0 is irrelevant. In fact, we shall frequently need to study limits as x approaches x_0 for functions f which are not even defined at x_0.

- It may also happen that the values $f(x)$ do not approach any particular number as x approaches x_0. In that case, we say that the limit $\lim_{x \to x_0} f(x)$ does not exist.

Example 2.1 Consider a piecewise linear function such as the function defined by

$$f(x) = \left\{ \begin{array}{ll} 3x + 1 & \text{if } x \leq 2 \\ 5x - 3 & \text{if } x > 2 \end{array} \right\}.$$

We wish to study the limit of $f(x)$ as x approaches 2. Since the definition of $f(x)$ changes at that point, we need to consider two cases. If $x < 2$, then $f(x) = 3x + 1$ and the fact that x is approaching 2 suggests that $f(x)$ should be approaching $3 \cdot 2 + 1 = 7$. It may also happen that $x > 2$. In that case, $f(x) = 5x - 3$ is approaching $5 \cdot 2 - 3 = 7$. Thus, $f(x)$ is approaching 7 in either case and one may conclude that $\lim_{x \to 2} f(x) = 7$. □

Example 2.2 Consider a slightly different version of the previous example, namely

$$f(x) = \left\{ \begin{array}{ll} 3x + 1 & \text{if } x \leq 2 \\ 5x - 4 & \text{if } x > 2 \end{array} \right\}.$$

Repeating our previous approach, one finds that $f(x)$ is approaching 7 in the case $x < 2$, but approaching 6 in the case $x > 2$. This suggests that $\lim_{x \to 2} f(x)$ does not exist. □

Example 2.3 A very useful limit that involves trigonometric functions is the limit

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

We shall eventually prove this fact directly, but it is not very clear why the limit should be equal to 1. A common tool for developing some intuition is to make a table listing a few representative values of x and the corresponding values for $(\sin x)/x$.

x	± 0.1	± 0.01	± 0.001
$(\sin x)/x$	0.99833	0.999983333	0.9999999983

In our case, the table suggests that $(\sin x)/x$ is approaching 1 as x approaches 0. This is related to the fact that the graphs of $\sin x$ and x are almost identical near the origin. □

Example 2.4 A somewhat similar, yet more straightforward, example is provided by

$$\lim_{x\to 1} f(x), \qquad f(x) = \sqrt{\frac{x^3+x-2}{x-1}}.$$

This limit can be analysed more easily because it involves an algebraic function, one that is obtained using polynomials together with addition, subtraction, multiplication, division and roots. Once again, let us start by making a table with some representative values.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	1.9261	1.9925	1.99925	2.00075	2.0075	2.076

Although $f(x)$ is not defined at $x = 1$, it seems to approach the value 2 as x approaches 1. This can also be verified directly, if one proceeds to simplify $f(x)$ first. Since the numerator vanishes when $x = 1$, it contains a factor of $x - 1$ and division of polynomials gives

$$f(x) = \sqrt{\frac{x^3+x-2}{x-1}} = \sqrt{\frac{(x-1)(x^2+x+2)}{x-1}} = \sqrt{x^2+x+2}.$$

In our case, x is approaching 1, so this expression ought to approach $\sqrt{1+1+2} = 2$. □

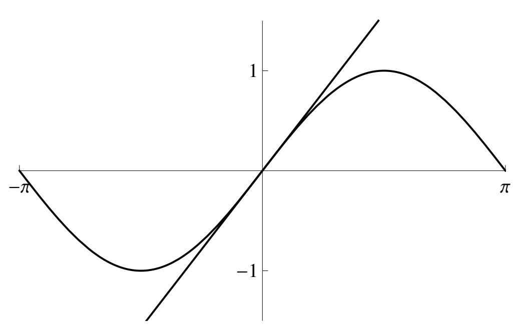


Figure 2.1: The graphs of $f(x) = \sin x$ and $g(x) = x$ near the origin.

2.2 Definition of limits

Definition 2.5 – Limit

We say that $f(x)$ approaches the limit L as x approaches x_0 and we write

$$\lim_{x \to x_0} f(x) = L$$

if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$0 \neq |x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - L| < \varepsilon. \tag{2.1}$$

- This definition asserts that $|f(x) - L|$ gets arbitrarily small, namely smaller than any positive number. Since the absolute value measures the distance between $f(x)$ and L, the definition is thus asserting that $f(x)$ gets arbitrarily close to L.
- In practice, the parameter δ is determined after a short computation. One uses the assumption $0 \neq |x - x_0| < \delta$ to estimate $|f(x) - L|$ and then specifies δ at the end. If it helps to simplify the computation, one may even assume that $\delta \leq 1$, for instance.

Example 2.6 We use the ε-δ definition to compute the limit of a constant function. Let us assume $f(x) = c$ for all x. We should then have $\lim_{x \to x_0} f(x) = c$ as well. To prove this formally using the definition, we let $\varepsilon > 0$ be given and we note that

$$|f(x) - L| = |f(x) - c| = |c - c| = 0 < \varepsilon.$$

This estimate holds for any choice of x, so equation (2.1) is satisfied for any $\delta > 0$. □

Example 2.7 We use the ε-δ definition to compute the limit of a linear function. Let us assume that $f(x) = ax + b$ and that $a \neq 0$. When x is approaching x_0, one expects $f(x)$ to be approaching $ax_0 + b$. In other words, one expects the limit to be

$$L = \lim_{x \to x_0} f(x) = ax_0 + b.$$

To prove this formally, we let $\varepsilon > 0$ be given and we estimate the difference

$$|f(x) - L| = |ax + b - ax_0 - b| = |ax - ax_0| = |a| \cdot |x - x_0|.$$

If we assume that $0 \neq |x - x_0| < \delta$, as in (2.1), we can then estimate this expression as

$$|f(x) - L| = |a| \cdot |x - x_0| < |a| \cdot \delta.$$

To ensure that the definition (2.1) holds, we need to ensure that $|f(x) - L| < \varepsilon$. This can be achieved by taking $|a| \cdot \delta = \varepsilon$, so an appropriate choice of δ would be $\delta = \varepsilon/|a|$. □

Example 2.8 We use the ε-δ definition to compute $L = \lim_{x \to 1} f(x)$ in the case that

$$f(x) = \left\{ \begin{array}{ll} 4 - 2x & \text{if } x \leq 1 \\ 5x - 3 & \text{if } x > 1 \end{array} \right\}.$$

Since x is approaching 1 and since $f(x)$ is either $4-2x$ or $5x-3$, one expects the limit to be $L=2$. Let us now fix some $\varepsilon>0$ and consider the expression

$$|f(x)-2| = \left\{ \begin{array}{ll} |2-2x| & \text{if } x\leq 1 \\ |5x-5| & \text{if } x>1 \end{array} \right\} = \left\{ \begin{array}{ll} 2|x-1| & \text{if } x\leq 1 \\ 5|x-1| & \text{if } x>1 \end{array} \right\}.$$

If we assume that $0\neq |x-1|<\delta$, as in (2.1), then we can easily argue that

$$|f(x)-2| \leq 5|x-1| < 5\delta.$$

Since our goal is to show that $|f(x)-2|<\varepsilon$, an appropriate choice of δ is $\delta=\varepsilon/5$. □

Example 2.9 We use division of polynomials to compute the limit

$$L = \lim_{x\to 2} \frac{2x^2-7x+6}{x-2}.$$

The function on the right hand side is not defined at $x=2$ since the denominator vanishes at that point. However, one may still study this expression as x approaches 2. In this case, it is easy to check that the numerator vanishes when $x=2$. Thus, it must have $x-2$ as a factor and the fraction can be simplified. Using division of polynomials, we now get

$$L = \lim_{x\to 2} \frac{2x^2-7x+6}{x-2} = \lim_{x\to 2} \frac{(x-2)(2x-3)}{x-2} = \lim_{x\to 2} (2x-3).$$

This is the limit of a linear function. In view of Example 2.7, the limit of a linear function as $x\to x_0$ can be obtained by letting $x=x_0$. We conclude that $L=2\cdot 2-3=1$. □

Example 2.10 We use the ε-δ definition to compute the limit of a quadratic function, say

$$L = \lim_{x\to 3} f(x), \qquad f(x) = 4x^2-7x+2.$$

In this case, we expect the limit to be $L=f(3)=17$. To prove this formally, we let $\varepsilon>0$ be given and we proceed to estimate the expression

$$|f(x)-L| = |f(x)-f(3)| = |4x^2-7x-15|.$$

The polynomial on the right hand side is $f(x)-f(3)$, so it obviously vanishes when $x=3$ and it must have $x-3$ as a factor. Using division of polynomials, one can then write

$$|f(x)-L| = |4x^2-7x-15| = |x-3|\cdot|4x+5|.$$

The factor $|x-3|$ is related to our usual assumption that $0\neq|x-3|<\delta$. To estimate the remaining factor $|4x+5|$, it is convenient to assume that $\delta\leq 1$, in which case

$$\begin{aligned} |x-3|<\delta\leq 1 \quad &\Longrightarrow \quad -1<x-3<1 \\ &\Longrightarrow \quad 2<x<4 \quad \Longrightarrow \quad 13<4x+5<21. \end{aligned}$$

Combining the estimates $|x-3|<\delta$ and $|4x+5|<21$, one may then conclude that

$$|f(x)-L| = |x-3|\cdot|4x+5| < 21\delta \leq \varepsilon,$$

as long as $\delta\leq\varepsilon/21$ and $\delta\leq 1$. An appropriate choice of δ is thus $\delta=\min(\varepsilon/21,1)$. □

2.3 One-sided limits

Definition 2.11 – Limit from the left

We say that $f(x)$ approaches L as x approaches x_0 from the left and we write

$$\lim_{x \to x_0^-} f(x) = L$$

if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$x_0 - \delta < x < x_0 \quad \Longrightarrow \quad |f(x) - L| < \varepsilon.$$

Definition 2.12 – Limit from the right

We say that $f(x)$ approaches L as x approaches x_0 from the right and we write

$$\lim_{x \to x_0^+} f(x) = L$$

if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$x_0 < x < x_0 + \delta \quad \Longrightarrow \quad |f(x) - L| < \varepsilon.$$

- Comparing the last two definitions with the original definition of limits, one has

$$\lim_{x \to x_0} f(x) = L \quad \Longleftrightarrow \quad \lim_{x \to x_0^-} f(x) = L = \lim_{x \to x_0^+} f(x).$$

- Although the ε-δ definitions are essential for proving some general facts about limits, we shall soon be able to compute limits without having to resort to these definitions.

Example 2.13 We compute the limit $L = \lim_{x \to 2} f(x)$ in the case that

$$f(x) = \left\{ \begin{array}{cl} 9 - 3x & \text{if } x < 2 \\ 5 & \text{if } x = 2 \\ 4x - 5 & \text{if } x > 2 \end{array} \right\}.$$

Since the given function is linear on the interval $(-\infty, 2)$, its limit from the left is

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (9 - 3x) = 9 - 3 \cdot 2 = 3.$$

The same argument applies for the interval $(2, +\infty)$, so the limit from the right is

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (4x - 5) = 4 \cdot 2 - 5 = 3.$$

This shows that $f(x)$ approaches the same value as x approaches 2 from either side. In particular, the given function approaches a limit as $x \to 2$ and the limit is $L = 3$. $\square$

2.4 Properties of limits

Theorem 2.14 – Limits of sums, products and quotients

The limit of a sum is equal to the sum of the limits, namely

$$\lim_{x\to x_0} f(x) = L, \qquad \lim_{x\to x_0} g(x) = M \quad\Longrightarrow\quad \lim_{x\to x_0} [f(x)+g(x)] = L+M.$$

The limit of a product is equal to the product of the limits, namely

$$\lim_{x\to x_0} f(x) = L, \qquad \lim_{x\to x_0} g(x) = M \quad\Longrightarrow\quad \lim_{x\to x_0} [f(x)\cdot g(x)] = LM.$$

When defined, the limit of a quotient is equal to the quotient of the limits, namely

$$\lim_{x\to x_0} f(x) = L, \qquad \lim_{x\to x_0} g(x) = M \neq 0 \quad\Longrightarrow\quad \lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

Theorem 2.15 – Limits of polynomials and rational functions

Suppose that f is either a polynomial or a quotient of polynomials which is defined at the point x_0. Then the limit of $f(x)$ as x approaches x_0 agrees with the value of f at that point. In other words, the limit of $f(x)$ is given by

$$\lim_{x\to x_0} f(x) = f(x_0).$$

- A function which is a quotient of polynomials is also known as a rational function.
- Rational functions are only defined at points at which their denominator is nonzero.

Example 2.16 According to the last theorem, one may easily compute limits such as

$$\lim_{x\to 2} (x^3-3x+4) = 2^3 - 3\cdot 2 + 4 = 6, \qquad \lim_{x\to 3} \frac{x^3-x+1}{x^2-2} = \frac{3^3-3+1}{3^2-2} = \frac{25}{7}. \qquad \square$$

Example 2.17 We use the last theorem and division of polynomials to find the limit

$$L = \lim_{x\to -2} \frac{5x^3+12x^2-8}{x+2}.$$

Note that the theorem does not apply directly, as the given function is not defined at the point $x=-2$. Nevertheless, it is easy to check that the numerator vanishes when $x=-2$, so the numerator contains a factor of $x+2$ and division of polynomials gives

$$L = \lim_{x\to -2} \frac{(x+2)(5x^2+2x-4)}{x+2} = \lim_{x\to -2} (5x^2+2x-4).$$

This means that L is really the limit of a polynomial function, so we easily get

$$L = \lim_{x\to -2} (5x^2+2x-4) = 5(-2)^2 + 2(-2) - 4 = 20 - 4 - 4 = 12. \qquad \square$$

2.5 Definition of continuity

Definition 2.18 – Continuity

We say that a function f is continuous at the point x_0, if it happens that

$$\lim_{x\to x_0} f(x) = f(x_0). \tag{2.2}$$

Thus, f is continuous at x_0 if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$|x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - f(x_0)| < \varepsilon. \tag{2.3}$$

Theorem 2.19 – Examples of continuous functions

The polynomial, trigonometric and exponential functions are all continuous throughout their domains. The same is true for the square root function defined by $f(x) = \sqrt{x}$ and any sum, product or quotient of continuous functions is continuous.

Theorem 2.20 – Squeeze theorem

Suppose that $f(x) \leq g(x) \leq h(x)$ in some interval around the point x_0 and

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = L.$$

Then the function g must also attain the same limit, namely $\lim_{x\to x_0} g(x) = L$ as well.

- In practice, the condition in (2.2) is usually sufficient for checking the continuity of a given function. The condition in (2.3) is equivalent, but it is mostly needed for proofs.
- The continuity condition (2.2) can also be stated in terms of one-sided limits as

$$\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x) = f(x_0).$$

Example 2.21 We show that the function f is discontinuous at the point $x = 3$ when

$$f(x) = \left\{ \begin{array}{ll} 3x^2 - 2x + 1 & \text{if } x \leq 3 \\ x^3 - 2x^2 + 5 & \text{if } x > 3 \end{array} \right\}.$$

Since this function is a polynomial on the interval $(-\infty, 3)$, its limit from the left is

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (3x^2 - 2x + 1) = 3 \cdot 9 - 2 \cdot 3 + 1 = 22.$$

The same argument applies for the interval $(3, +\infty)$, so the limit from the right is

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (x^3 - 2x^2 + 5) = 27 - 2 \cdot 9 + 5 = 14.$$

Thus, the one-sided limits are not equal and f is not continuous at the point $x = 3$. □

Example 2.22 We examine the continuity of the function f at the point $x = 3$ when

$$f(x) = \left\{ \begin{array}{cc} x^2 + 2x + a & \text{if } x < 3 \\ 2a - b & \text{if } x = 3 \\ x^2 - bx + 1 & \text{if } x > 3 \end{array} \right\}.$$

Since f is a polynomial on the intervals $(-\infty, 3)$ and $(3, +\infty)$, one easily finds that

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2 + 2x + a) = a + 15,$$
$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x^2 - bx + 1) = 10 - 3b.$$

In particular, f is continuous at the given point if and only if

$$a + 15 = 10 - 3b = 2a - b.$$

Solving this system of equations, one obtains a unique solution which is given by

$$-5 - 3b = a = 5 - b \implies 2b = -10 \implies b = -5 \implies a = 10.$$

We conclude that f is continuous at the given point if and only if $a = 10$ and $b = -5$. □

Example 2.23 We show that the function f is discontinuous at all points when

$$f(x) = \left\{ \begin{array}{cl} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{array} \right\}.$$

This is one of the few cases for which condition (2.3) becomes useful. Suppose that f is continuous at some point x_0. We can then take $\varepsilon = 1$ to find some $\delta > 0$ such that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < 1.$$

Rearranging terms in the last equation, one may also express it in the form

$$x_0 - \delta < x < x_0 + \delta \implies f(x_0) - 1 < f(x) < f(x_0) + 1.$$

If it happens that x_0 is rational, then $f(x_0) = 1$ and our conclusion above reads

$$x_0 - \delta < x < x_0 + \delta \implies 0 < f(x) < 2.$$

In view of the definition of f, this gives $f(x) = 1$ for all points in $(x_0 - \delta, x_0 + \delta)$, so this interval does not contain any irrational numbers and we have reached a contradiction. If it happens that x_0 is irrational, then $f(x_0) = 0$ and our conclusion above reads

$$x_0 - \delta < x < x_0 + \delta \implies -1 < f(x) < 1.$$

Once again, this gives $f(x) = 0$ for all points in $(x_0 - \delta, x_0 + \delta)$, so this interval does not contain any rational numbers and another contradiction is reached. We conclude that the condition (2.3) does not hold at any point, so f is discontinuous at all points. □

2.6 Properties of continuous functions

Theorem 2.24 – Continuity and positivity

Suppose that the function f is continuous at the point x_0.

(a) If $f(x_0) > 0$, then there exists $\delta > 0$ such that $f(x) > 0$ for all $x_0 - \delta < x < x_0 + \delta$.

(b) If $f(x_0) < 0$, then there exists $\delta > 0$ such that $f(x) < 0$ for all $x_0 - \delta < x < x_0 + \delta$.

Theorem 2.25 – Composition of continuous functions

If the function f is continuous at the point x_0 and the function g is continuous at the point $f(x_0)$, then the composition $g \circ f$ is continuous at the point x_0.

- Plainly stated, Theorem 2.24 asserts that a continuous function which is positive at a point must be positive in a whole interval around that point.
- The composition $g \circ f$ of two functions is defined by the formula $(g \circ f)(x) = g(f(x))$.

Example 2.26 The proof of Theorem 2.24 is fairly short. Let us assume that $f(x_0) > 0$, as the other case is similar. If we take $\varepsilon = f(x_0)$, then we can find some $\delta > 0$ such that

$$x_0 - \delta < x < x_0 + \delta \quad \Longrightarrow \quad f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon \quad \Longrightarrow \quad f(x) > 0. \qquad \square$$

Example 2.27 The result of Theorem 2.25 is commonly used to compute limits such as

$$\lim_{x \to 3} \sqrt{\frac{x^3 - 2x^2 - 7}{x^2 - 2x + 5}} = \sqrt{\frac{27 - 2 \cdot 9 - 7}{9 - 2 \cdot 3 + 5}} = \sqrt{\frac{2}{8}} = \frac{1}{2}.$$

Since the composition of continuous functions is known to be continuous, the limit on the left hand side is that of a continuous function, so one may simply substitute $x = 3$. $\square$

Example 2.28 Given an arbitrary function f and a continuous function g, one has

$$\lim_{x \to x_0} g(f(x)) = g\left(\lim_{x \to x_0} f(x) \right). \tag{2.4}$$

For instance, the limit of a square root is the square root of the limit. The proof of this identity relies on the ε-δ definition, but the general idea is simple. Let $L = \lim_{x \to x_0} f(x)$ for simplicity. Our assumption that g is continuous can be expressed in the form

$$\lim_{z \to z_0} g(z) = g(z_0).$$

In our case, we have $x \to x_0$ and $f(x) \to L$, so we must also have $g(f(x)) \to g(L)$ by the last equation. This suggests that the limit of $g(f(x))$ should be $g(L)$, as needed. $\square$

2.7 Intermediate value theorem

Theorem 2.29 – Bolzano's theorem

Suppose that f is continuous on $[a, b]$ and $f(a), f(b)$ have opposite sign. Then f has a root in (a, b). In other words, there exists a point $a < x < b$ such that $f(x) = 0$.

Theorem 2.30 – Intermediate Value Theorem

If a function f is continuous on $[a, b]$, then f attains all values between $f(a)$ and $f(b)$.

- The function f must be continuous throughout $[a, b]$ for these theorems to hold. For instance, $f(x) = 1/x$ attains both positive and negative values, but it is never zero.
- The Intermediate Value Theorem appears more general than Bolzano's theorem, but the two are actually equivalent. One may use either of them to prove the other.

Example 2.31 We use Bolzano's theorem to show that there exists a real number x such that $\cos x = x$. Consider the function f defined by $f(x) = \cos x - x$. Being the difference of continuous functions, f is obviously continuous and it also satisfies

$$f(0) = \cos 0 = 1, \qquad f(\pi/2) = \cos(\pi/2) - \pi/2 = -\pi/2.$$

Since $f(0)$ and $f(\pi/2)$ have opposite signs, we find that $f(x) = 0$ for some $0 < x < \pi/2$. □

Example 2.32 We use Bolzano's theorem to locate the roots of the polynomial

$$f(x) = 2x^3 - 6x + 1.$$

To show that f has three roots in the interval $(-2, 2)$, we note that f is continuous with

$$f(-2) = -3, \qquad f(-1) = 5, \qquad f(0) = 1, \qquad f(1) = -3, \qquad f(2) = 5.$$

Since the values $f(-2)$ and $f(-1)$ have opposite signs, f has a root that lies in $(-2, -1)$. The same argument yields a second root in $(0, 1)$ and also a third root in $(1, 2)$. □

Example 2.33 We approximate the root $0 < x < 1$ that appears in the previous example. As we already know, the function f changes sign on the interval $[0, 1]$. Consider the points that one obtains by dividing this interval into 100 smaller subintervals, say

$$x_k = k/100, \qquad 0 \leq k \leq 100.$$

Since f changes sign on the interval $[0, 1]$, it must change sign on a subinterval $[x_k, x_{k+1}]$. We now compute the values $f(x_k)$ and find a change of sign between 0.16 and 0.17. This gives an approximation of the root. If one wishes to obtain a better approximation, then one may simply introduce a larger number of points and subintervals. □

2.8 Infinite limits

Definition 2.34 – Infinite limits

If a function f is defined near the point x_0 and its values $f(x)$ become arbitrarily large as x approaches x_0 from the left, then we say that the limit of $f(x)$ from the left is

$$\lim_{x\to x_0^-} f(x) = +\infty. \tag{2.5}$$

Moreover, there are similar definitions that one may introduce for limits of the form

$$\lim_{x\to x_0^-} f(x) = -\infty, \qquad \lim_{x\to x_0^+} f(x) = +\infty, \qquad \lim_{x\to x_0^+} f(x) = -\infty.$$

- It is also possible to give a rigourous definition that resembles the ε-δ definition for finite limits. To say that (2.5) holds, for instance, is to say that $f(x)$ becomes larger than any positive number. Given any $N > 0$, there should thus exist $\delta > 0$ such that
$$0 \neq |x - x_0| < \delta \quad \Longrightarrow \quad f(x) > N.$$
- Infinite limits usually arise when a denominator becomes zero. In that case, one has to determine the sign of the denominator in order to find the value of the limit.

Example 2.35 We consider two rational functions and compute their limits

$$L_1 = \lim_{x\to 2^-} \frac{x^2 - 3x + 5}{x - 2}, \qquad L_2 = \lim_{x\to 2^+} \frac{3x^2 - 4x + 6}{2x^2 - 5x + 2}.$$

When it comes to the first limit, the numerator is nonzero when $x = 2$ and this gives

$$L_1 = \lim_{x\to 2^-} \frac{4 - 6 + 5}{x - 2} = \lim_{x\to 2^-} \frac{3}{x - 2} = -\infty$$

because $x - 2$ is arbitrarily small but negative. When it comes to the second limit, we have

$$L_2 = \lim_{x\to 2^+} \frac{3x^2 - 4x + 6}{2x^2 - 5x + 2} = \lim_{x\to 2^+} \frac{12 - 8 + 6}{2x^2 - 5x + 2} = \lim_{x\to 2^+} \frac{10}{2x^2 - 5x + 2}.$$

The denominator becomes zero when $x = 2$, so we need to determine its sign. This can be done more easily, if one factors the denominator. Using division of polynomials, we get

$$L_2 = \lim_{x\to 2^+} \frac{10}{(x-2)(2x-1)} = \lim_{x\to 2^+} \frac{10}{(x-2)\cdot 3} = +\infty. \qquad \square$$

Example 2.36 We consider the tangent function and compute its limit

$$L_3 = \lim_{x\to \frac{\pi}{2}^-} \tan x = \lim_{x\to \frac{\pi}{2}^-} \frac{\sin x}{\cos x}.$$

Since $\sin(\pi/2) = 1$ and $\cos(\pi/2) = 0$, the numerator can be treated easily and one has

$$L_3 = \lim_{x\to \frac{\pi}{2}^-} \frac{\sin x}{\cos x} = \lim_{x\to \frac{\pi}{2}^-} \frac{1}{\cos x}.$$

To determine the sign of the denominator, we recall that $\cos x > 0$ whenever $0 < x < \pi/2$. Since x is approaching $\pi/2$ from the left, $\cos x$ is thus positive and the limit is $+\infty$. $\square$

2.9 Limits at infinity

Theorem 2.37 – Limits of rational functions

The following statements hold for each real number $p > 0$ and each integer $n > 0$.

$$\lim_{x\to+\infty} x^p = +\infty, \qquad \lim_{x\to-\infty} x^n = \left\{ \begin{array}{ll} -\infty & \text{if } n \text{ is odd} \\ +\infty & \text{if } n \text{ is even} \end{array} \right\}, \qquad \lim_{x\to\pm\infty} \frac{1}{x^n} = 0.$$

Moreover, the limit of a polynomial function f can be determined using the formula

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \big(a_n x^n + \ldots + a_1 x + a_0\big) = \lim_{x\to\pm\infty} a_n x^n. \tag{2.6}$$

Theorem 2.38 – Limits of exponentials and logarithms

The exponential and logarithmic functions with base $a > 1$ are such that

$$\lim_{x\to+\infty} a^x = +\infty, \qquad \lim_{x\to-\infty} a^x = 0, \qquad \lim_{x\to+\infty} \log_a x = +\infty, \qquad \lim_{x\to0^+} \log_a x = -\infty.$$

The remaining case $0 < a < 1$ is slightly different and the corresponding limits are

$$\lim_{x\to+\infty} a^x = 0, \qquad \lim_{x\to-\infty} a^x = +\infty, \qquad \lim_{x\to+\infty} \log_a x = -\infty, \qquad \lim_{x\to0^+} \log_a x = +\infty.$$

- Equation (2.6) asserts that every polynomial behaves like its highest-order term for large values of x. This is because the lower-order terms are considerably smaller.
- One may use equation (2.6) to find the limit of any rational function as $x \to \pm\infty$.

Example 2.39 We consider two rational functions and compute their limits

$$L_1 = \lim_{x\to\pm\infty} \frac{4x^3 - 3x + 2}{5x^3 - 2x^2 + 1}, \qquad L_2 = \lim_{x\to\pm\infty} \frac{2x^2 - 7x + 1}{5x^3 - 4x + 3}.$$

When it comes to the first limit, one may easily argue that

$$L_1 = \lim_{x\to\pm\infty} \frac{4x^3 - 3x + 2}{5x^3 - 2x^2 + 1} = \lim_{x\to\pm\infty} \frac{4x^3}{5x^3} = \frac{4}{5}.$$

When it comes to the second limit, one similarly has

$$L_2 = \lim_{x\to\pm\infty} \frac{2x^2 - 7x + 1}{5x^3 - 4x + 3} = \lim_{x\to\pm\infty} \frac{2x^2}{5x^3} = \lim_{x\to\pm\infty} \frac{2}{5x} = 0. \qquad \square$$

Example 2.40 We use Theorem 2.38 in order to compute the limit

$$L_3 = \lim_{x\to+\infty} \frac{3^x + 4^x}{4^x + 5^x}.$$

In this case, one may argue that 3^x is much smaller compared to 4^x which is much smaller compared to 5^x. Let us then isolate the dominant terms and write

$$L_3 = \lim_{x\to+\infty} \frac{3^x + 4^x}{4^x + 5^x} = \lim_{x\to+\infty} \frac{4^x}{5^x} \cdot \frac{(3/4)^x + 1}{(4/5)^x + 1}.$$

According to the theorem, both $(3/4)^x$ and $(4/5)^x$ must approach zero as $x \to +\infty$, so

$$L_3 = \lim_{x\to+\infty} \frac{4^x}{5^x} \cdot \frac{0+1}{0+1} = \lim_{x\to+\infty} \left(\frac{4}{5}\right)^x = 0. \qquad \square$$

Example 2.41 We use limits to analyse the rational function f which is defined by

$$f(x) = \frac{4x-2}{x-3}.$$

First of all, we note that the domain consists of all points $x \neq 3$. Although f is not defined at the point $x = 3$, it is defined at all nearby points and one easily finds that

$$\lim_{x\to3^-} f(x) = \lim_{x\to3^-} \frac{4x-2}{x-3} = \lim_{x\to3^-} \frac{10}{x-3} = -\infty,$$
$$\lim_{x\to3^+} f(x) = \lim_{x\to3^+} \frac{4x-2}{x-3} = \lim_{x\to3^+} \frac{10}{x-3} = +\infty.$$

These limits describe the behaviour of the function near the missing point $x = 3$. We can also determine the behaviour of the function as $x \to \pm\infty$. Using equation (2.6), we get

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{4x-2}{x-3} = \lim_{x\to\pm\infty} \frac{4x}{x} = 4.$$

Note that this value is approached for large enough x, but it is never attained because

$$f(x) = \frac{4x-2}{x-3} = \frac{4(x-3)+10}{x-3} = 4 + \frac{10}{x-3}$$

is never equal to 4. A precise graph of the given function appears in the figure below. $\square$

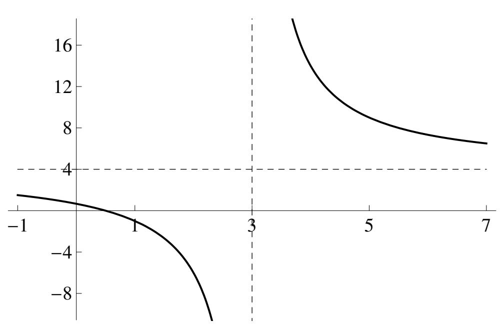


Figure 2.2: The graph of $f(x) = \frac{4x-2}{x-3}$.

Chapter 3

Differentiation

3.1 Definition of derivative

Definition 3.1 – Average rate of change

The average rate of change of a function f over the interval $[x_0, x_1]$ is defined as

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Definition 3.2 – Derivative

We say that the function f is differentiable at the point x_0, if the limit

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{3.1}$$

exists. When this limit does exist, we call it the derivative of f at the point x_0.

- The derivative $f'(x_0)$ gives the rate at which the function f changes at the point x_0. It is also known as the instantaneous rate of change at that point.
- A function which is differentiable at the point x_0 must be continuous at x_0 because

$$\lim_{x \to x_0} [f(x) - f(x_0)] = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) = f'(x_0) \cdot 0 = 0.$$

- However, a function which is continuous at x_0 need not be differentiable at x_0. For instance, $f(x) = |x|$ is continuous at $x = 0$, but it is not differentiable at $x = 0$.

Example 3.3 To show that $f(x) = ax + b$ is differentiable at all points, we note that

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{ax + b - (ax_0 + b)}{x - x_0} = \lim_{x \to x_0} \frac{a(x - x_0)}{x - x_0} = a.$$

This actually proves that linear functions have the same rate of change at all points. □

Example 3.4 To show that $f(x) = x^2$ is differentiable at all points, we note that

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x \to x_0} \frac{(x - x_0)(x + x_0)}{x - x_0}.$$

Once we now cancel the factor $x - x_0$, we obtain the limit of a linear function, so

$$f'(x_0) = \lim_{x \to x_0} (x + x_0) = 2x_0.$$

□

Example 3.5 We show that $f(x) = |x|$ is not differentiable at the point $x_0 = 0$. Since the absolute value is defined as $|x| = x$ whenever $x > 0$, one easily finds that

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1.$$

On the other hand, one has $|x| = -x$ for the remaining values $x < 0$, and this implies

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1.$$

In particular, the one-sided limits are not equal, so the limit (3.1) does not exist. □

Example 3.6 We compute the derivative of $f(x) = 1/x$ at any point $x_0 \neq 0$. Since

$$f(x) - f(x_0) = \frac{1}{x} - \frac{1}{x_0} = \frac{x_0 - x}{xx_0},$$

one may use the definition (3.1) of the derivative to conclude that

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{-1}{xx_0} = -\frac{1}{x_0^2}.$$

□

Example 3.7 We compute the derivative of $f(x) = \sqrt{x}$. In this case, one has

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0} = \frac{1}{\sqrt{x} + \sqrt{x_0}}$$

and the square root function is known to be continuous, so the last equation gives

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{1}{\sqrt{x} + \sqrt{x_0}} = \frac{1}{2\sqrt{x_0}}.$$

This proves that f is differentiable at any point $x_0 > 0$. When $x_0 = 0$, one finds that

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{\sqrt{x}}{x} = \lim_{x \to 0^+} \frac{1}{\sqrt{x}} = +\infty.$$

In particular, the graph of the given function is virtually vertical at the point $x_0 = 0$. □

3.2 Rules of differentiation

Theorem 3.8 – Sums and constant multiples

Suppose that the functions f, g are differentiable at x and c is a fixed constant. Then both $f+g$ and $c \cdot f$ are differentiable at x and their derivatives are given by

$$[f(x)+g(x)]' = f'(x)+g'(x), \qquad [c \cdot f(x)]' = c \cdot f'(x).$$

Theorem 3.9 – Product rule

Suppose that the functions f, g are differentiable at x. Then their product $f \cdot g$ is also differentiable at x and its derivative is given by

$$[f(x) \cdot g(x)]' = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

Theorem 3.10 – Quotient rule

Suppose that the functions f, g are differentiable at x and suppose that $g(x) \neq 0$. Then the quotient f/g is also differentiable at x and its derivative is given by

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}.$$

Example 3.11 We use the product rule and induction to prove the power rule

$$(x^n)' = nx^{n-1} \quad \text{for each positive integer } n. \tag{3.2}$$

Suppose first that $n = 1$. Then $x^n = x$ and so $(x^n)' = 1 = 1x^0$. Thus, the given formula does hold when $n = 1$. If we now assume that the formula holds for some n, then

$$(x^{n+1})' = (x^n \cdot x)' = nx^{n-1} \cdot x + x^n \cdot 1 = (n+1)x^n.$$

In particular, the formula holds for $n+1$ as well and the result follows by induction. □

Example 3.12 One may use the formula from the previous example to differentiate any given polynomial. Since the derivative of a sum is the sum of the derivatives, one has

$$(x^4 + 3x^2 + 2x)' = (x^4)' + 3(x^2)' + 2(x)' = 4x^3 + 3 \cdot 2x + 2 \cdot 1 = 4x^3 + 6x + 2. \qquad \square$$

Example 3.13 Using formula (3.2) along with the quotient rule, one finds that

$$\left(\frac{x^2-1}{3x+1}\right)' = \frac{2x \cdot (3x+1) - 3 \cdot (x^2-1)}{(3x+1)^2} = \frac{6x^2+2x-3x^2+3}{(3x+1)^2} = \frac{3x^2+2x+3}{(3x+1)^2}. \qquad \square$$

3.3 Derivatives of standard functions

Theorem 3.14 – Derivatives of trigonometric functions

The trigonometric functions are differentiable and their derivatives are given by

$$(\sin x)' = \cos x, \qquad (\sec x)' = \sec x \tan x, \qquad (\tan x)' = \sec^2 x,$$
$$(\cos x)' = -\sin x, \qquad (\csc x)' = -\csc x \cot x, \qquad (\cot x)' = -\csc^2 x.$$

Theorem 3.15 – Derivative of exponential functions

Every exponential function is differentiable and the derivative of $f(x) = a^x$ is given by

$$(a^x)' = C_a \cdot a^x, \qquad C_a = \lim_{z \to 0} \frac{a^z - 1}{z}.$$

The simplest version of this formula arises for a number $e > 1$ which satisfies

$$(e^x)' = e^x, \qquad e = \lim_{z \to 0} (1 + z)^{1/z}.$$

Definition 3.16 – Natural logarithm

The inverse of the exponential $f(x) = e^x$ is called the natural logarithm $g(x) = \ln x$. One may express every logarithmic function in terms of $\ln x$ by writing

$$\log_a x = \frac{\ln x}{\ln a}.$$

Example 3.17 According to the quotient rule, the derivative of $\tan x$ is given by

$$(\tan x)' = \left(\frac{\sin x}{\cos x} \right)' = \frac{(\sin x)' \cdot \cos x - (\cos x)' \cdot \sin x}{\cos^2 x}.$$

Since $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$, one may simplify the last equation to get

$$(\tan x)' = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x. \qquad \square$$

Example 3.18 Using Theorem 3.14 along with the product rule, one finds that

$$(x^3 \cdot \sec x)' = (x^3)' \cdot \sec x + x^3 \cdot (\sec x)' = 3x^2 \sec x + x^3 \sec x \tan x.$$

Using Theorem 3.14 along with the quotient rule, one similarly finds that

$$\left(\frac{\sin x}{x^2} \right)' = \frac{(\sin x)' \cdot x^2 - (x^2)' \cdot \sin x}{x^4} = \frac{x^2 \cos x - 2x \sin x}{x^4} = \frac{x \cos x - 2 \sin x}{x^3}. \qquad \square$$

3.4 Derivatives of inverse functions

Theorem 3.19 – Derivative of inverse function

Suppose that $f\colon A \to B$ is a bijective differentiable function and let $g\colon B \to A$ be the inverse function. Then g is differentiable at all points x with $f'(g(x)) \neq 0$ and

$$g'(x) = \frac{1}{f'(g(x))}.$$

Theorem 3.20 – Derivatives of inverse trigonometric functions

The inverse trigonometric functions are differentiable and their derivatives are given by

$$\left(\sin^{-1} x\right)' = \frac{1}{\sqrt{1-x^2}}, \qquad \left(\cos^{-1} x\right)' = -\frac{1}{\sqrt{1-x^2}}, \qquad \left(\tan^{-1} x\right)' = \frac{1}{x^2+1}.$$

Theorem 3.21 – Derivative of logarithmic function

The logarithmic function is differentiable and its derivative is given by

$$(\ln x)' = \frac{1}{x} \quad \text{for all } x > 0.$$

Example 3.22 To verify the formula for the derivative of $\sin^{-1} x$, we consider the case

$$f(x) = \sin x, \qquad g(x) = \sin^{-1} x.$$

Since g is the inverse function of f, one may apply Theorem 3.19 to find that

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\cos(g(x))} = \frac{1}{\cos(\sin^{-1} x)}.$$

It remains to simplify the right hand side. Letting $\theta = \sin^{-1} x$ for simplicity, we get

$$\sin\theta = x \quad \Longrightarrow \quad \cos^2\theta = 1 - \sin^2\theta = 1 - x^2.$$

On the other hand, $\theta = \sin^{-1} x$ is between $-\pi/2$ and $\pi/2$ by definition, so $\cos\theta \geq 0$ and

$$\cos\theta = \sqrt{1-x^2} \quad \Longrightarrow \quad g'(x) = \frac{1}{\cos(\sin^{-1} x)} = \frac{1}{\cos\theta} = \frac{1}{\sqrt{1-x^2}}.$$ □

Example 3.23 To verify the formula for the derivative of $\ln x$, we consider the case

$$f(x) = e^x, \qquad g(x) = \ln x.$$

Since $f'(x) = f(x)$ and g is the inverse function, one may apply Theorem 3.19 to get

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{f(g(x))} = \frac{1}{x}.$$ □

3.5 Chain rule

Theorem 3.24 – Chain rule

Suppose that f is differentiable at x and suppose that g is differentiable at $f(x)$. Then the composition $g \circ f$ is differentiable at x and its derivative is given by

$$[g(f(x))]' = g'(f(x)) \cdot f'(x). \tag{3.3}$$

- The chain rule is easier to express in terms of the Leibniz notation for derivatives. This amounts to writing $\frac{dy}{dx}$ for the derivative of y with respect to x. When $y = g(u)$ depends on some variable u and $u = f(x)$ depends on x, one has $y = g(f(x))$ and

$$\frac{dy}{dx} = [g(f(x))]' = g'(f(x)) \cdot f'(x) = g'(u) \cdot f'(x) = \frac{dy}{du} \cdot \frac{du}{dx}. \tag{3.4}$$

- To differentiate compositions of three or more functions, one may simply apply the chain rule repeatedly. When it comes to three functions, one finds that

$$[h(g(f(x)))]' = h'(g(f(x))) \cdot g'(f(x)) \cdot f'(x).$$

Example 3.25 A typical application of the chain rule (3.3) gives the formula

$$[f(x)^n]' = nf(x)^{n-1} \cdot f'(x). \tag{3.5}$$

This allows us to differentiate the powers of any given function. For instance, one has

$$\begin{aligned} y = (\sin x + 4x^2)^3 \quad &\implies \quad y' = 3(\sin x + 4x^2)^2 \cdot (\sin x + 4x^2)' \\ &\implies \quad y' = 3(\sin x + 4x^2)^2 \cdot (\cos x + 8x). \end{aligned}$$

□

Example 3.26 Another typical application of the chain rule (3.3) gives the formula

$$[\sin f(x)]' = \cos f(x) \cdot f'(x). \tag{3.6}$$

This allows us to differentiate the sine of any given function. For instance, one has

$$\begin{aligned} y = \sin(2x^3 + \tan x) \quad &\implies \quad y' = \cos(2x^3 + \tan x) \cdot (2x^3 + \tan x)' \\ &\implies \quad y' = \cos(2x^3 + \tan x) \cdot (6x^2 + \sec^2 x). \end{aligned}$$

□

Example 3.27 We use the chain rule (3.3) to compute the derivative of

$$y = \tan(e^{2x}).$$

Since $(\tan x)' = \sec^2 x$ and $(e^x)' = e^x$, we may apply the chain rule twice to find that

$$y' = \sec^2(e^{2x}) \cdot (e^{2x})' = \sec^2(e^{2x}) \cdot e^{2x} \cdot (2x)' = 2e^{2x} \sec^2(e^{2x}).$$

□

Example 3.28 We use the chain rule (3.3) to compute the derivative of

$$y = \ln(\cos(x^2)).$$

In this case, we have $(\ln x)' = 1/x$ and $(\cos x)' = -\sin x$, so the chain rule gives

$$y' = \frac{1}{\cos(x^2)} \cdot \left[\cos(x^2)\right]' = -\frac{1}{\cos(x^2)} \cdot \sin(x^2) \cdot (x^2)' = -2x\tan(x^2). \qquad \square$$

Example 3.29 We use the chain rule (3.3) to compute the derivative of

$$y = \tan^{-1}(e^{ax})$$

for any given constant a. Since $(\tan^{-1} x)' = \frac{1}{x^2+1}$, we may apply the chain rule to get

$$y' = \frac{1}{(e^{ax})^2 + 1} \cdot (e^{ax})' = \frac{1}{e^{2ax}+1} \cdot e^{ax} \cdot (ax)' = \frac{ae^{ax}}{e^{2ax}+1}. \qquad \square$$

Example 3.30 The Leibniz form of the chain rule (3.4) is often useful when one needs to deal with several variables at the same time. In those cases, the notation y' is ambiguous, so the Leibniz notation $\frac{dy}{dx}$ is naturally preferable. Suppose, for instance, that

$$y = \sin u, \qquad u = \sec w, \qquad w = \tan x.$$

Since y is expressed in terms of u, one may compute $\frac{dy}{du}$ directly by simply differentiating the relevant equation. The same is true for $\frac{du}{dw}$ and $\frac{dw}{dx}$. To compute some other derivative, however, one needs to resort to the chain rule (3.4). When it comes to $\frac{dy}{dw}$, one has

$$\frac{dy}{dw} = \frac{dy}{du} \cdot \frac{du}{dw} = \cos u \cdot \sec w \tan w.$$

When it comes to $\frac{dy}{dx}$, the exact same argument applies to show that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dw} \cdot \frac{dw}{dx} = \cos u \cdot \sec w \tan w \cdot \sec^2 x. \qquad \square$$

Example 3.31 We use the Leibniz form of the chain rule (3.4) to compute $\frac{dy}{d\theta}$ when

$$y = u^4, \qquad u = \frac{x-1}{x+1}, \qquad x = \sin\theta.$$

Differentiating the given equations, one may easily determine the derivatives

$$\frac{dy}{du} = 4u^3, \qquad \frac{du}{dx} = \frac{x+1-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}, \qquad \frac{dx}{d\theta} = \cos\theta.$$

To express the derivative $\frac{dy}{d\theta}$ in terms of those, we now use the chain rule (3.4) to get

$$\frac{dy}{d\theta} = \frac{dy}{du} \cdot \frac{du}{dx} \cdot \frac{dx}{d\theta} = \frac{8u^3\cos\theta}{(x+1)^2}. \qquad \square$$

3.6 Implicit differentiation

- When a variable y depends on a variable x, one usually has an explicit formula which expresses $y = f(x)$ as a function of x. In that case, the derivative $y' = f'(x)$ can be determined directly using the standard rules of differentiation.
- Suppose, more generally, that the variables x, y are related by some equation which does not necessarily have the form $y = f(x)$. One may then differentiate both sides of the equation and determine y' without having to solve for y in terms of x.

Example 3.32 Consider the coordinates x, y of a point that lies on the hyperbola

$$y^2 - x^2 = 1. \tag{3.7}$$

If one uses this equation to solve for y in terms of x, then one finds that

$$y^2 = x^2 + 1 \quad \Longrightarrow \quad y = \pm\sqrt{x^2+1} \quad \Longrightarrow \quad y' = \pm\frac{2x}{2\sqrt{x^2+1}} = \frac{x}{y}.$$

However, it is also possible to differentiate (3.7) directly without having to solve for y first. Differentiating with respect to x, one has $(y^2)' = 2yy'$ because of the chain rule, so

$$y^2 - x^2 = 1 \quad \Longrightarrow \quad 2yy' - 2x = 0 \quad \Longrightarrow \quad yy' = x \quad \Longrightarrow \quad y' = \frac{x}{y}. \qquad \square$$

Example 3.33 Suppose that the variables x, y are related by the equation

$$y + \cos y = x^3.$$

In this case, it is impossible to solve for y in terms of x. Let us then differentiate directly. Using the chain rule to differentiate $\cos y$, we find that

$$y' - (\sin y)y' = 3x^2 \quad \Longrightarrow \quad y'(1 - \sin y) = 3x^2 \quad \Longrightarrow \quad y' = \frac{3x^2}{1 - \sin y}. \qquad \square$$

Example 3.34 We compute the derivative $y' = \frac{dy}{dx}$ in the case that

$$x^3 + y = \sin(x^2 y).$$

Differentiating both sides of this equation, we use the chain rule to get

$$3x^2 + y' = \cos(x^2y) \cdot (x^2y)'.$$

The derivative on the right hand side is the derivative of a product, so we actually have

$$3x^2 + y' = \cos(x^2y) \cdot (2x \cdot y + x^2 \cdot y').$$

It remains to solve this equation for y'. Expanding the right hand side gives

$$3x^2 + y' = 2xy\cos(x^2y) + x^2y'\cos(x^2y)$$

and we may finally collect the terms which contain y' to conclude that

$$y' - x^2y'\cos(x^2y) = 2xy\cos(x^2y) - 3x^2 \quad \Longrightarrow \quad y' = \frac{2xy\cos(x^2y) - 3x^2}{1 - x^2\cos(x^2y)}. \qquad \square$$

Example 3.35 We use implicit differentiation to compute the derivative of

$$y = \sqrt{e^{3x} + \sin^2 x + 4}.$$

Even though y can be differentiated directly, it is simpler to start by writing

$$y^2 = e^{3x} + \sin^2 x + 4.$$

This allows us to avoid the square root. Differentiating both sides of the equation, we get

$$2yy' = 3e^{3x} + 2\sin x\cos x \quad\Longrightarrow\quad y' = \frac{3e^{3x} + 2\sin x\cos x}{2y}. \qquad \square$$

Example 3.36 Suppose that the variables x, y are related by the equation

$$\cos(e^y) = e^x y.$$

Using both the chain rule and the product rule, one finds that

$$-\sin(e^y)\cdot(e^y)' = e^x y + e^x y' \quad\Longrightarrow\quad -\sin(e^y)\cdot e^y y' = e^x y + e^x y'.$$

Once we now rearrange terms and solve for y', we arrive at

$$-\left(e^y\sin(e^y) + e^x\right)\cdot y' = e^x y \quad\Longrightarrow\quad y' = -\frac{e^x y}{e^y\sin(e^y) + e^x}. \qquad \square$$

Example 3.37 We compute the derivative $y' = \frac{dy}{dx}$ in the case that

$$\sin(x/y) = e^{2x} + y.$$

First, we differentiate both sides with respect to x. In view of the chain rule, one has

$$\cos(x/y)\cdot(x/y)' = 2e^{2x} + y'.$$

The derivative on the left hand side is the derivative of a quotient, so we actually have

$$\cos(x/y)\cdot\frac{y - xy'}{y^2} = 2e^{2x} + y'.$$

To solve this equation for y', we now expand the left hand side and write

$$\cos(x/y)\cdot\frac{1}{y} - \cos(x/y)\cdot\frac{xy'}{y^2} = 2e^{2x} + y'.$$

Collecting the terms which contain y' and clearing denominators, we conclude that

$$\left(\frac{x\cos(x/y)}{y^2} + 1\right)\cdot y' = \frac{\cos(x/y)}{y} - 2e^{2x} \quad\Longrightarrow\quad y' = \frac{y\cos(x/y) - 2y^2e^{2x}}{x\cos(x/y) + y^2}. \qquad \square$$

3.7 Logarithmic differentiation

- Logarithms are useful for simplifying products, quotients and exponents because

$$\ln(x \cdot y) = \ln x + \ln y, \qquad \ln \frac{x}{y} = \ln x - \ln y, \qquad \ln x^r = r \ln x.$$

- If a function involves products, quotients and exponents, one may thus introduce the logarithm of the function in order to simplify it before differentiating.
- This approach is called logarithmic differentiation and it relies on the formula

$$\big[\ln |x|\big]' = \frac{1}{x}, \qquad \big[\ln |f(x)|\big]' = \frac{f'(x)}{f(x)}. \tag{3.8}$$

Example 3.38 We use logarithmic differentiation to compute the derivative of

$$f(x) = (x^2+1)^3 \cdot (x^4+5x^2+3)^6 \cdot e^{8x}.$$

Since $f(x)$ is positive in this case, its logarithm $\ln f(x)$ is defined for all x and

$$\begin{aligned} \ln f(x) &= \ln(x^2+1)^3 + \ln(x^4+5x^2+3)^6 + \ln e^{8x} \\ &= 3\ln(x^2+1) + 6\ln(x^4+5x^2+3) + 8x. \end{aligned}$$

Differentiating both sides of this equation, one may now use the chain rule to get

$$\frac{f'(x)}{f(x)} = \frac{3 \cdot 2x}{x^2+1} + \frac{6(4x^3+10x)}{x^4+5x^2+3} + 8.$$

In other words, the derivative of the given function is

$$f'(x) = f(x) \cdot \left(\frac{6x}{x^2+1} + \frac{12(2x^3+5x)}{x^4+5x^2+3} + 8 \right). \qquad \square$$

Example 3.39 We use logarithmic differentiation to establish the power rule

$$(x^n)' = nx^{n-1} \quad \text{for any real number } n.$$

Since $f(x) = x^n$ is not necessarily positive, let us introduce absolute values and write

$$|f(x)| = |x^n| = |x|^n \quad \Longrightarrow \quad \ln |f(x)| = \ln |x|^n = n \cdot \ln |x|.$$

Differentiating both sides of the rightmost equation, we may thus conclude that

$$\frac{f'(x)}{f(x)} = \frac{n}{x} \quad \Longrightarrow \quad f'(x) = \frac{nf(x)}{x} = \frac{nx^n}{x} = nx^{n-1}. \qquad \square$$

Example 3.40 We use logarithmic differentiation to compute the derivative of $y = a^x$ for any given base $a > 0$. Since a is constant, the same is true for $\ln a$ and one has

$$y = a^x \quad\Longrightarrow\quad \ln y = \ln a^x = x \ln a \quad\Longrightarrow\quad \frac{y'}{y} = \ln a \quad\Longrightarrow\quad y' = y \ln a.$$

This proves the formula $(a^x)' = a^x \ln a$ which is closely related to Theorem 3.15. □

Example 3.41 We use logarithmic differentiation to compute the derivative of

$$f(x) = x^x, \qquad x > 0.$$

It is a common mistake to argue that $f'(x) = x \cdot x^{x-1}$, but this is not correct because the power rule $(x^n)' = nx^{n-1}$ is only valid when the exponent n is a constant. Let us write

$$\ln f(x) = \ln x^x = x \cdot \ln x$$

and then differentiate both sides of the equation. Using the product rule, we get

$$\begin{aligned} \frac{f'(x)}{f(x)} = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1 \quad &\Longrightarrow\quad f'(x) = f(x) \cdot (\ln x + 1) \\ &\Longrightarrow\quad f'(x) = x^x \cdot (\ln x + 1). \end{aligned}$$ □

Example 3.42 We use logarithmic differentiation to compute $f'(0)$ in the case that

$$f(x) = \frac{(x^2 + 3x + 1)^5 \cdot \sqrt{3x^2 + 4\cos x}}{e^{2x} - 3x}.$$

Since the given expression is somewhat messy, it is better to simplify first. Let us write

$$|f(x)| = |x^2 + 3x + 1|^5 \cdot |3x^2 + 4\cos x|^{1/2} \cdot |e^{2x} - 3x|^{-1}$$

and then take logarithms of both sides to find that

$$\begin{aligned} \ln |f(x)| &= \ln |x^2 + 3x + 1|^5 + \ln |3x^2 + 4\cos x|^{1/2} + \ln |e^{2x} - 3x|^{-1} \\ &= 5 \ln |x^2 + 3x + 1| + \frac{1}{2} \ln |3x^2 + 4\cos x| - \ln |e^{2x} - 3x|. \end{aligned}$$

Using the chain rule and formula (3.8), in particular, we conclude that

$$\frac{f'(x)}{f(x)} = \frac{5(2x + 3)}{x^2 + 3x + 1} + \frac{6x - 4\sin x}{2(3x^2 + 4\cos x)} - \frac{2e^{2x} - 3}{e^{2x} - 3x}.$$

It remains to evaluate this expression at the point $x = 0$. Since $f(0) = \sqrt{4} = 2$, we get

$$\frac{f'(0)}{2} = 5 \cdot 3 + 0 - (2 - 3) = 15 + 1 = 16 \quad\Longrightarrow\quad f'(0) = 32.$$ □

3.8 Mean value theorem

Theorem 3.43 – Extreme value theorem

If a function f is continuous on a finite interval $[a, b]$, then f attains both a minimum and a maximum value on $[a, b]$. That is, there exist points $x_1, x_2 \in [a, b]$ such that

$$f(x_1) \leq f(x) \leq f(x_2) \quad \text{for all } x \in [a, b].$$

Theorem 3.44 – Rolle's theorem

If a function f is continuous on $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$, then there exists a point c in the interval (a, b) such that $f'(c) = 0$.

Theorem 3.45 – Mean value theorem

Suppose that a function f is just continuous on $[a, b]$ and differentiable on (a, b). Then there exists a point c in the interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

- Rolle's theorem can be used to relate the roots of f with those of f'. If f has two roots, then its derivative f' must have a root that lies between them. If f has $n > 1$ roots, then its derivative f' must have $n - 1$ roots that lie between them.
- The mean value theorem asserts that the instantaneous rate of change is equal to the average rate of change at some point. For instance, a car that travels at an average speed of 50 km/h must be travelling at exactly that speed at some point.

Example 3.46 We show that the polynomial $f(x) = x^3 + 3x + 1$ has a unique real root. To prove existence using Bolzano's theorem, we note that f is continuous with

$$f(-1) = -1 - 3 + 1 = -3, \qquad f(0) = 1.$$

Since $f(-1)$ and $f(0)$ have opposite signs, f must have a root that lies in $(-1, 0)$. To prove uniqueness, suppose that f has two roots $x_1 < x_2$. Then $f(x_1) = f(x_2) = 0$ and one may use Rolle's theorem to get $f'(x) = 0$ for some $x_1 < x < x_2$. This is a contradiction since

$$f'(x) = 3x^2 + 3 = 3(x^2 + 1)$$

does not have any real roots. Thus, the given polynomial has only one real root. □

Example 3.47 We show that the polynomial $f(x) = 2x^3 + x^2 - 8x + 2$ has exactly two roots in the interval $(0, 2)$. To prove that these roots exist, we note that f is continuous with

$$f(0) = 2, \qquad f(1) = 2 + 1 - 8 + 2 = -3, \qquad f(2) = 16 + 4 - 16 + 2 = 6.$$

In view of Bolzano's theorem, f must then have a root in $(0,1)$ and another root in $(1,2)$, so it has two roots in $(0,2)$. Suppose that it has three roots in $(0,2)$. Then f' must have two roots in this interval by Rolle's theorem. On the other hand, it is easy to check that

$$f'(x) = 6x^2 + 2x - 8 = 2(3x^2 + x - 4) = 2(3x+4)(x-1).$$

Since f' has only one root in $(0,2)$, we conclude that f has only two roots in $(0,2)$. □

Example 3.48 We use the mean value theorem to prove the inequality

$$|\sin a - \sin b| \leq |a-b| \quad \text{for all } a,b \in \mathbb{R}.$$

When $a = b$, both sides are equal to zero, so the inequality certainly holds. Suppose now that $a < b$, as the case $b < a$ is similar. Using the mean value theorem, one may write

$$\frac{f(a)-f(b)}{a-b} = \frac{f(b)-f(a)}{b-a} = f'(c) \quad \text{for some } a < c < b.$$

In this case, we have $f(x) = \sin x$ and $f'(x) = \cos x$, so the mean value theorem gives

$$\frac{|\sin a - \sin b|}{|a-b|} = |\cos c| \leq 1 \quad \Longrightarrow \quad |\sin a - \sin b| \leq |a-b|.$$ □

Example 3.49 We use the mean value theorem to prove the inequality

$$|\tan^{-1} a - \tan^{-1} b| \leq |a-b| \quad \text{for all } a,b \in \mathbb{R}.$$

Once again, the result is clear when $a = b$, so it suffices to treat the case $a \neq b$. Using the mean value theorem with $f(x) = \tan^{-1} x$, one finds a point c such that

$$\frac{|f(a)-f(b)|}{|a-b|} = |f'(c)| = \frac{1}{1+c^2} \leq 1 \quad \Longrightarrow \quad |f(a)-f(b)| \leq |a-b|.$$ □

Example 3.50 We use the mean value theorem to establish the approximation

$$7 + \frac{1}{8} < \sqrt{51} < 7 + \frac{1}{7}.$$

Consider the function f that is defined by $f(x) = \sqrt{x+49}$. This satisfies $f(0) = \sqrt{49} = 7$ and we wish to approximate $f(2) = \sqrt{51}$. According to the mean value theorem, one has

$$\frac{f(2)-f(0)}{2-0} = f'(c) = \frac{1}{2\sqrt{c+49}}$$

for some $0 < c < 2$. To estimate the expression on the right hand side, we note that

$$0 < c < 2 \quad \Longrightarrow \quad 49 < c + 49 < 51 < 64 \quad \Longrightarrow \quad 7 < \sqrt{c+49} < 8.$$

Once we now combine the last two equations, we may easily conclude that

$$\frac{1}{8} < \frac{1}{\sqrt{c+49}} < \frac{1}{7} \quad \Longrightarrow \quad \frac{1}{8} < \sqrt{51} - 7 < \frac{1}{7}.$$ □

Chapter 4

Applications of derivatives

4.1 L'Hôpital's rule

Theorem 4.1 – L'Hôpital's rule

Suppose that f, g are differentiable functions such that

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = L, \tag{4.1}$$

where L is either 0 or $\pm\infty$ and x_0 is any real number. In those cases, one has

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}, \tag{4.2}$$

as long as the rightmost limit in (4.2) exists. The exact same statement is also true, if the four limits above are replaced by either one-sided limits or limits at infinity.

- L'Hôpital's rule may only be used for limits of the form $0/0$ and ∞/∞. These are called indeterminate forms because x/x is not necessarily 1 when x is zero or infinite.
- Some other indeterminate forms are $0 \cdot \infty$, 0^0, ∞^0 and 1^∞. These can all be reduced to the forms $0/0$ and ∞/∞ for which L'Hôpital's rule becomes applicable.

Example 4.2 Using L'Hôpital's rule (4.2) for the case $0/0$, one easily finds that

$$\lim_{x\to 1} \frac{x^3 + 5x - 6}{2x - 2} = \lim_{x\to 1} \frac{3x^2 + 5}{2} = \frac{8}{2} = 4. \qquad \square$$

Example 4.3 Using L'Hôpital's rule (4.2) for the case ∞/∞, one similarly gets

$$\lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0. \qquad \square$$

Example 4.4 We apply L'Hôpital's rule (4.2) for the case 0/0 to show that

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1.$$

Since both $e^x - 1$ and x approach zero as $x \to 0$, L'Hôpital's rule is applicable and so

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \frac{e^x}{1} = e^0 = 1. \qquad \square$$

Example 4.5 To compute a limit of the form $0 \cdot \infty$, one may rearrange terms and express the product as a quotient. Let us carry out this idea to compute the limit

$$\lim_{x\to 0^+} x \ln x.$$

The factor x is approaching zero and the factor $\ln x$ is approaching $-\infty$, so the given limit has the form $0 \cdot \infty$. We move one of the factors in the denominator and we write

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x}.$$

This is now a limit of the form ∞/∞, so L'Hôpital's rule is applicable and we get

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{(\ln x)'}{(1/x)'} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0. \qquad \square$$

Example 4.6 Limits involving non-constant exponents are usually easier to treat, if one introduces logarithms to eliminate the exponent. A typical example is the limit

$$L = \lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x, \tag{4.3}$$

where a is a given constant. First of all, we take logarithms of both sides to write

$$\ln L = \ln \lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x = \lim_{x\to\infty} \ln \left(1 + \frac{a}{x}\right)^x = \lim_{x\to\infty} x \cdot \ln \left(1 + \frac{a}{x}\right).$$

In this case, the factor x approaches ∞ and the factor $\ln(1 + a/x)$ approaches $\ln 1 = 0$. We thus have a limit of the form $0 \cdot \infty$ and we need to rearrange terms to get

$$\ln L = \lim_{x\to\infty} \frac{\ln(1 + a/x)}{1/x}.$$

This is now a limit of the form 0/0, so one may use L'Hôpital's rule to conclude that

$$\ln L = \lim_{x\to\infty} \frac{\ln(1 + a/x)}{1/x} = \lim_{x\to\infty} \frac{(1 + a/x)^{-1} \cdot (-a/x^2)}{-1/x^2} = a \cdot \lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^{-1}.$$

It easily follows that $\ln L = a$, so the original limit L is equal to $L = e^{\ln L} = e^a$. $\square$

4.2 Monotonicity

Definition 4.7 – Monotonicity

We say that a function f is increasing on some interval I, if

$$f(a) < f(b) \quad \text{for all points } a < b \text{ in } I. \tag{4.4}$$

Similarly, we say that f is decreasing on some interval I, if

$$f(a) > f(b) \quad \text{for all points } a < b \text{ in } I. \tag{4.5}$$

A function which is either increasing or decreasing is also known as monotonic.

Theorem 4.8 – Monotonicity test

Consider a function f which is differentiable on some interval I.

(a) If $f'(x) > 0$ for all $x \in I$, then f is increasing on I.

(b) If $f'(x) < 0$ for all $x \in I$, then f is decreasing on I.

- Plainly stated, condition (4.4) asserts that larger values of x give rise to larger values of $f(x)$. This condition is frequently needed to justify statements such as

  $$a < b \implies e^a < e^b \quad \text{for all } a, b \in \mathbb{R}.$$

  Here, the inequality is preserved because $(e^x)' = e^x$ is positive and e^x is increasing.

- In a similar fashion, one may use condition (4.5) to justify statements such as

  $$a < b \implies -2a > -2b \quad \text{for all } a, b \in \mathbb{R}.$$

  Here, the inequality is reversed since $(-2x)' = -2$ is negative and $-2x$ is decreasing.

Example 4.9 We determine the intervals on which $f(x) = x^4 - 2x^2 + 3$ is increasing. In view of the last theorem, we need to ensure that $f'(x) > 0$. Let us then compute

$$f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x-1)(x+1).$$

To determine the sign of $f'(x)$, we use the factorisation above and worry about each factor separately. The points at which $f'(x)$ is zero are the points $x = 0, 1, -1$. We list those in order along the first row of a table and list the factors of $f'(x)$ in the following rows.

	-1	0	1	
$4x$	$-$	$-$	$+$	$+$
$x-1$	$-$	$-$	$-$	$+$
$x+1$	$-$	$+$	$+$	$+$
$f'(x)$	$-$	$+$	$-$	$+$

First, consider the factor $4x$. This is positive when $x > 0$ and it is negative when $x < 0$. We may thus complete the row for the factor $4x$ by inserting a plus sign when $x > 0$ and a minus sign when $x < 0$. The rows for the other factors are completed in a similar way. For instance, the factor $x - 1$ is positive when $x > 1$, but it is negative when $x < 1$.

The last row corresponds to $f'(x)$ which is the product of the three factors. This row is filled at the end by reading the signs vertically. When $x < -1$, we have three factors of negative sign, so their product $f'(x)$ is negative. When $-1 < x < 0$, we have two negative factors and one positive factor, so the product $f'(x)$ is positive. Proceeding in this manner, one obtains the sign of $f'(x)$ for all values of x and one finds that

$$\begin{aligned} f'(x) < 0 \quad &\Longleftrightarrow \quad x \in (-\infty, -1) \cup (0, 1), \\ f'(x) > 0 \quad &\Longleftrightarrow \quad x \in (-1, 0) \cup (1, +\infty). \end{aligned}$$

In particular, f is decreasing on $(-\infty, -1) \cup (0, 1)$ and increasing on $(-1, 0) \cup (1, +\infty)$. □

Example 4.10 We determine the intervals on which $f(x)$ is increasing in the case that

$$f(x) = x^2 \cdot e^{3x}.$$

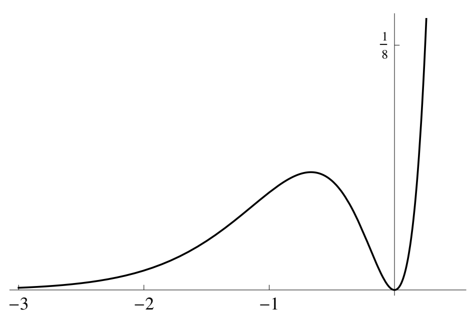
Using both the product rule and the chain rule to differentiate this function, we get

$$f'(x) = 2x \cdot e^{3x} + x^2 \cdot 3e^{3x} = xe^{3x} \cdot (2 + 3x).$$

Since e^{3x} is always positive, the points at which $f'(x)$ is zero are the points $x = 0, -2/3$. As in the previous example, we order those along the first row of a table and list the factors of $f'(x)$ in the following rows. The resulting table appears in Figure 4.1 together with the graph of f. The table lists the signs of the factors xe^{3x} and $2 + 3x$ which also determine the sign of their product $f'(x)$. According to the table, one has

$$\begin{aligned} f'(x) < 0 \quad &\Longleftrightarrow \quad x \in (-2/3, 0), \\ f'(x) > 0 \quad &\Longleftrightarrow \quad x \in (-\infty, -2/3) \cup (0, +\infty). \end{aligned}$$

In particular, f is decreasing on $(-2/3, 0)$ and increasing on $(-\infty, -2/3) \cup (0, +\infty)$. □



	$-2/3$	0	
xe^{3x}	$-$	$-$	$+$
$2 + 3x$	$-$	$+$	$+$
$f'(x)$	$+$	$-$	$+$

Figure 4.1: The graph of $f(x) = x^2 e^{3x}$.

Example 4.11 We study the monotonicity of $f(x) = x^2 \ln x$. The domain of this function consists of all points $x > 0$ and one may use the product rule to find that

$$f'(x) = 2x \ln x + x^2 \cdot \frac{1}{x} = 2x \ln x + x = x(2 \ln x + 1).$$

Since $x > 0$ by above, the sign of $f'(x)$ coincides with the sign of $2 \ln x + 1$ and

$$f'(x) > 0 \quad \Longleftrightarrow \quad 2 \ln x + 1 > 0 \quad \Longleftrightarrow \quad \ln x > -1/2 \quad \Longleftrightarrow \quad x > e^{-1/2}.$$

In other words, f is increasing on $(e^{-1/2}, \infty)$ and decreasing on $(0, e^{-1/2})$. □

Example 4.12 We determine the intervals on which $f(x)$ is increasing in the case that

$$f(x) = \frac{4x - 1}{x^2 + 5}.$$

First of all, we use the quotient rule to compute its derivative

$$f'(x) = \frac{4(x^2 + 5) - 2x(4x - 1)}{(x^2 + 5)^2} = \frac{-4x^2 + 2x + 20}{(x^2 + 5)^2} = -\frac{2(2x^2 - x - 10)}{(x^2 + 5)^2}.$$

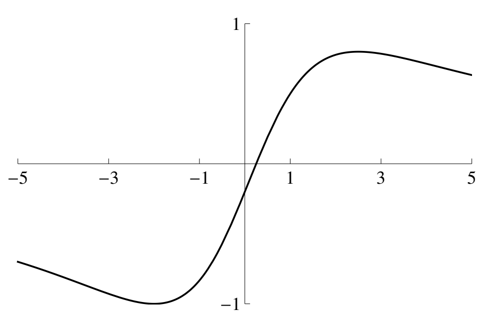
The quadratic in the numerator has roots $x_1 = -2$ and $x_2 = 5/2$, so one may factor to write

$$f'(x) = -\frac{4(x - x_1)(x - x_2)}{(x^2 + 5)^2} = -\frac{4(x + 2)(x - 5/2)}{(x^2 + 5)^2}.$$

Since the denominator is obviously positive, the sign of $f'(x)$ coincides with the sign of its numerator. One may determine this sign, as in the table of Figure 4.2, by looking at each of the factors separately. The overall conclusion of the table is that

$$\begin{aligned} f'(x) < 0 \quad &\Longleftrightarrow \quad x \in (-\infty, -2) \cup (5/2, +\infty), \\ f'(x) > 0 \quad &\Longleftrightarrow \quad x \in (-2, 5/2). \end{aligned}$$

In other words, f is decreasing on $(-\infty, -2) \cup (5/2, +\infty)$ and increasing on $(-2, 5/2)$. □



	-2	$5/2$	
$-4(x+2)$	$+$	$-$	$-$
$x - 5/2$	$-$	$-$	$+$
$f'(x)$	$-$	$+$	$-$

Figure 4.2: The graph of $f(x) = \dfrac{4x - 1}{x^2 + 5}$.

4.3 Concavity

Definition 4.13 – Concavity

We say that a function f is concave up on some interval I, if

$$f(x) < \frac{f(b)-f(a)}{b-a} \cdot (x-a) + f(a) \quad \text{for all points } a < x < b \text{ in } I. \tag{4.6}$$

Similarly, we say that f is concave down on some interval I, if

$$f(x) > \frac{f(b)-f(a)}{b-a} \cdot (x-a) + f(a) \quad \text{for all points } a < x < b \text{ in } I. \tag{4.7}$$

Definition 4.14 – Inflection point

A function f has an inflection point at x_0, if the concavity of f changes at that point, namely if f is concave up on one side of x_0 and concave down on the other.

Theorem 4.15 – Concavity test

Consider a function f which is twice differentiable on some interval I.

(a) If $f''(x) > 0$ for all $x \in I$, then f is concave up on I.

(b) If $f''(x) < 0$ for all $x \in I$, then f is concave down on I.

- The condition (4.6) for a function f to be concave up requires the graph of f to lie below the line that connects the points $(a, f(a))$ and $(b, f(b))$. Intuitively speaking, this condition holds on intervals on which the graph of f has a shape like $\cup$.
- The condition (4.7) for a function f to be concave down has a similar interpretation and it holds on intervals on which the graph of f has a shape like $\cap$.

Example 4.16 Consider the quadratic $f(x) = ax^2 + bx + c$, where a, b, c are some given constants and $a \neq 0$. In this case, one has $f'(x) = 2ax + b$ and $f''(x) = 2a$. When $a > 0$, the function is concave up at all points. When $a < 0$, it is concave down at all points. □

Example 4.17 We determine the intervals on which $f(x) = xe^{-x}$ is concave up. In view of the last theorem, we need to ensure that $f''(x) > 0$. Let us then compute

$$\begin{aligned} f'(x) &= e^{-x} + x(e^{-x})' = e^{-x} - xe^{-x}, \\ f''(x) &= -e^{-x} - e^{-x} - x(e^{-x})' = -2e^{-x} + xe^{-x} = (x-2)e^{-x}. \end{aligned}$$

Since the exponential factor e^{-x} is always positive, one finds that

$$\begin{aligned} f''(x) < 0 \quad &\Longleftrightarrow \quad x < 2, \\ f''(x) > 0 \quad &\Longleftrightarrow \quad x > 2. \end{aligned}$$

In other words, f is concave down on $(-\infty, 2)$ and concave up on $(2, +\infty)$. The graph of this function appears in Figure 4.3 and it includes an inflection point at $x = 2$. □

Example 4.18 To study the concavity of the cubic $f(x) = x^3 - 3x^2$, we compute

$$f'(x) = 3x^2 - 6x \quad \Longrightarrow \quad f''(x) = 6x - 6 = 6(x-1).$$

This implies that f is concave down on $(-\infty, 1)$ and concave up on $(1, +\infty)$. The graph of this function appears in Figure 4.3 and it includes an inflection point at $x = 1$. □

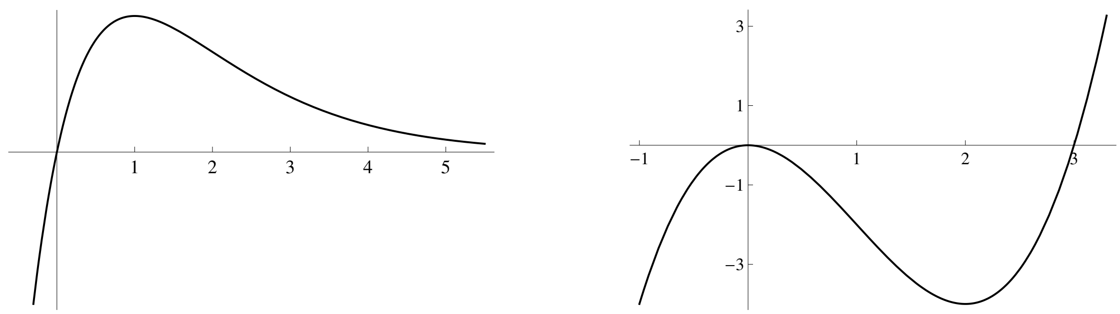


Figure 4.3: The graphs of $f(x) = xe^{-x}$ and $f(x) = x^3 - 3x^2$, respectively.

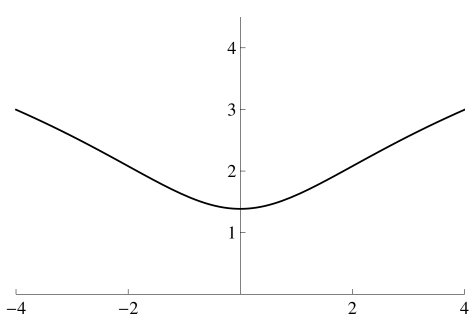
Example 4.19 Consider the function f which is defined by $f(x) = \ln(x^2 + 4)$. Since

$$f'(x) = \frac{2x}{x^2 + 4}$$

by the chain rule, one may use the quotient rule to compute the second derivative

$$f''(x) = \frac{2(x^2+4) - 2x \cdot 2x}{(x^2+4)^2} = \frac{8 - 2x^2}{(x^2+4)^2} = \frac{2(2-x)(2+x)}{(x^2+4)^2}.$$

The sign of this expression can be determined using the table in Figure 4.4. According to the table, f is concave up on $(-2, 2)$ and concave down on $(-\infty, -2) \cup (2, +\infty)$. □



		-2	2
$2 - x$	$+$	$+$	$-$
$2 + x$	$-$	$+$	$+$
$f''(x)$	$-$	$+$	$-$

Figure 4.4: The graph of $f(x) = \ln(x^2 + 4)$.