6.3 Reduction formulas

- A reduction formula expresses an integral I_n that depends on some integer n in terms of another integral I_m that involves a smaller integer m. If one repeatedly applies this formula, one may then express I_n in terms of a much simpler integral.

Example 6.10 We use integration by parts to establish the reduction formula

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cdot\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx. \tag{6.4}$$

If we take $dv = \sin x\,dx$, then we have $v = -\cos x$ and we may integrate by parts with

$$u = \sin^{n-1} x, \qquad du = (n-1)\sin^{n-2} x\cdot\cos x.$$

Using the fact that $\sin^2 x + \cos^2 x = 1$, one may thus conclude that

$$\begin{aligned}\int \sin^n x\,dx &= -\sin^{n-1} x\cdot\cos x + (n-1)\int \sin^{n-2} x\cdot\cos^2 x\,dx\\ &= -\sin^{n-1} x\cdot\cos x + (n-1)\int \sin^{n-2} x\cdot(1-\sin^2 x)\,dx\\ &= -\sin^{n-1} x\cdot\cos x + (n-1)\int \sin^{n-2} x\,dx + (1-n)\int \sin^n x\,dx.\end{aligned}$$

Here, the rightmost integral coincides with the original integral on the left. Once we now rearrange terms, we end up with n copies of the integral and equation (6.4) follows. □

Example 6.11 We use a reduction formula to compute the integral I_3 in the case that

$$I_n = \int x^n e^{2x}\,dx.$$

If we take $u = x^n$ and $dv = e^{2x}\,dx$, then $du = nx^{n-1}\,dx$ and $v = \frac{1}{2}e^{2x}$, so one has

$$I_n = \frac{1}{2}x^n e^{2x} - \frac{n}{2}\int x^{n-1}e^{2x}\,dx = \frac{1}{2}x^n e^{2x} - \frac{n}{2}\cdot I_{n-1}. \tag{6.5}$$

We now apply the last formula repeatedly to determine I_3. According to the formula,

$$\begin{aligned}I_3 &= \frac{1}{2}x^3e^{2x} - \frac{3}{2}\cdot I_2 = \frac{1}{2}x^3e^{2x} - \frac{3}{2}\cdot\left[\frac{1}{2}x^2e^{2x} - I_1\right]\\ &= \frac{1}{2}x^3e^{2x} - \frac{3}{2}\cdot\left[\frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{2}\cdot I_0\right]\\ &= \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{4}\int e^{2x}\,dx\\ &= \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{8}e^{2x} + C.\end{aligned}$$

□

Example 6.12 We use integration by parts to establish the reduction formula

$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x \cdot \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx. \tag{6.6}$$

In this case, we note that $(\tan x)' = \sec^2 x$ and we write the given integral as

$$\int \sec^n x\,dx = \int \sec^{n-2} x \cdot \sec^2 x\,dx.$$

If we take $dv = \sec^2 x\,dx$, then we have $v = \tan x$ and we may integrate by parts with

$$u = \sec^{n-2} x, \qquad du = (n-2)\sec^{n-3} x \cdot \sec x \tan x = (n-2)\sec^{n-2} x \cdot \tan x.$$

Using the fact that $1 + \tan^2 x = \sec^2 x$, one may thus establish the identity

$$\begin{aligned}\int \sec^n x\,dx &= \sec^{n-2} x \cdot \tan x - (n-2)\int \sec^{n-2} x \cdot \tan^2 x\,dx \\ &= \sec^{n-2} x \cdot \tan x - (n-2)\int \sec^{n-2} x \cdot (\sec^2 x - 1)\,dx \\ &= \sec^{n-2} x \cdot \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx.\end{aligned}$$

Since the integral on the left hand side also appears on the right hand side, this gives

$$(n-1)\int \sec^n x\,dx = \sec^{n-2} x \cdot \tan x + (n-2)\int \sec^{n-2} x\,dx.$$

In particular, the reduction formula (6.6) follows by dividing both sides with $n-1$. □

Example 6.13 Let $a \neq 0$ be some given constant and consider the integral

$$I_n = \int \frac{dx}{(x^2+a)^n} = \int (x^2+a)^{-n}\,dx.$$

If we take $u = (x^2+a)^{-n}$ and $dv = dx$, then we may integrate by parts to find that

$$I_n = x(x^2+a)^{-n} + n\int x(x^2+a)^{-n-1} \cdot 2x\,dx.$$

Let us now rearrange terms and express the last equation in the form

$$\begin{aligned}I_n &= x(x^2+a)^{-n} + 2n\int \frac{x^2+a-a}{(x^2+a)^{n+1}}\,dx \\ &= x(x^2+a)^{-n} + 2n\int \frac{dx}{(x^2+a)^n} - 2na\int \frac{dx}{(x^2+a)^{n+1}}.\end{aligned}$$

The integrals on the right hand side have the same form as the original integral, so

$$I_n = x(x^2+a)^{-n} + 2n \cdot I_n - 2na \cdot I_{n+1}.$$

Rearranging terms once again, one may thus establish the reduction formula

$$2na \cdot I_{n+1} = (2n-1) \cdot I_n + x(x^2+a)^{-n}.$$ □

6.4 Trigonometric integrals

Theorem 6.14 – Powers of sine and cosine

Consider the integral $\int \sin^m x \cdot \cos^n x \, dx$ for any non-negative integers m, n.

(a) When n is odd, one may compute this integral using the substitution $u = \sin x$.

(b) When m is odd, one may compute this integral using the substitution $u = \cos x$.

(c) When m, n are even, one may use the half-angle formulas to simplify the integral.

Theorem 6.15 – Powers of secant and tangent

Consider the integral $\int \sec^m x \cdot \tan^n x \, dx$ for any non-negative integers m, n.

(a) When n is odd, one may compute this integral using the substitution $u = \sec x$.

(b) When m is even, one may compute this integral using the substitution $u = \tan x$.

(c) When m is odd and n is even, one may reduce the integrand to powers of $\sec x$.

- The three cases that arise in Theorem 6.14 are closely related to the identities

$$(\sin x)' = \cos x, \qquad (\cos x)' = -\sin x, \qquad \sin^2 x + \cos^2 x = 1.$$

If one uses the substitution $u = \sin x$, then one may express any even power of cosine in terms of u^2, but also needs a copy of cosine for $du = \cos x \, dx$. This yields an odd number of cosines, so the substitution $u = \sin x$ will only help when n is odd.

- The last case that arises in Theorem 6.14 requires the half-angle formulas

$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}, \qquad \cos^2 \theta = \frac{1 + \cos(2\theta)}{2}. \tag{6.7}$$

These formulas are helpful for reducing the even powers of sine and cosine.

- The three cases that arise in Theorem 6.15 are closely related to the identities

$$(\sec x)' = \sec x \tan x, \qquad (\tan x)' = \sec^2 x, \qquad 1 + \tan^2 x = \sec^2 x.$$

These imply that an odd number of tangents is needed to substitute $u = \sec x$, while an even number of secants is needed to substitute $u = \tan x$.

Example 6.16 We use the substitution $u = \sin x$ to compute the integral

$$\int \sin^4 x \cdot \cos^5 x \, dx.$$

In this case, we have $du = \cos x\, dx$ and also $\sin^2 x + \cos^2 x = 1$, so

$$\begin{aligned}\int \sin^4 x \cdot \cos^5 x\, dx &= \int \sin^4 x \cdot (1 - \sin^2 x)^2 \cdot \cos x\, dx = \int u^4 (1-u^2)^2\, du \\ &= \int u^4(1 - 2u^2 + u^4)\, du = \int (u^4 - 2u^6 + u^8)\, du \\ &= \frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9} + C = \frac{\sin^5 x}{5} - \frac{2\sin^7 x}{7} + \frac{\sin^9 x}{9} + C. \qquad \square\end{aligned}$$

Example 6.17 We use the half-angle formulas to simplify and compute the integral

$$\int \sin^2 x \cdot \cos^2 x\, dx.$$

Since the exponents are both even, one needs to express the integrand in the form

$$\begin{aligned}\sin^2 x \cdot \cos^2 x &= \frac{1 - \cos(2x)}{2} \cdot \frac{1 + \cos(2x)}{2} = \frac{1}{4} \cdot \left[1 - \cos^2(2x)\right] \\ &= \frac{1}{4} \cdot \left[1 - \frac{1 + \cos(4x)}{2}\right] = \frac{1}{8} \cdot [1 - \cos(4x)]\,.\end{aligned}$$

Once we now integrate both sides of this equation, we may easily conclude that

$$\int \sin^2 x \cdot \cos^2 x\, dx = \frac{1}{8}\left[x - \frac{\sin(4x)}{4}\right] + C = \frac{x}{8} - \frac{\sin(4x)}{32} + C. \qquad \square$$

Example 6.18 We use an appropriate substitution to compute the integral

$$\int \sec^4 x \cdot \tan^2 x\, dx.$$

If we let $u = \tan x$, then $du = \sec^2 x\, dx$ and also $\sec^2 x = 1 + \tan^2 x = 1 + u^2$, so one has

$$\begin{aligned}\int \sec^4 x \cdot \tan^2 x\, dx &= \int \sec^2 x \cdot \tan^2 x \cdot \sec^2 x\, dx = \int (1 + u^2) \cdot u^2\, du \\ &= \int (u^2 + u^4)\, du = \frac{u^3}{3} + \frac{u^5}{5} + C = \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C. \qquad \square\end{aligned}$$

Example 6.19 We use an appropriate substitution to compute the integral

$$\int \frac{\sin^3 x}{\cos^8 x}\, dx.$$

Since the cosine appears in the denominator, it is better to first simplify and write

$$\int \frac{\sin^3 x}{\cos^8 x}\, dx = \int \frac{\sin^3 x}{\cos^3 x} \cdot \frac{1}{\cos^5 x}\, dx = \int \tan^3 x \cdot \sec^5 x\, dx.$$

Let us take $u = \sec x$. Since $du = \sec x \tan x\, dx$ and also $u^2 = \sec^2 x = \tan^2 x + 1$, we get

$$\begin{aligned}\int \frac{\sin^3 x}{\cos^8 x}\, dx &= \int \tan^2 x \cdot \sec^4 x \cdot \sec x \tan x\, dx = \int (u^2 - 1) \cdot u^4\, du \\ &= \int (u^6 - u^4)\, du = \frac{u^7}{7} - \frac{u^5}{5} + C = \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C. \qquad \square\end{aligned}$$

6.5 Trigonometric substitutions

- Trigonometric substitutions are sometimes needed to simplify integrals that contain expressions of the form $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$ and $\sqrt{x^2+a^2}$ for some $a>0$. In each of these cases, one naturally seeks a substitution to simplify the square root.
- The three most common trigonometric substitutions are listed in the table below.

Expression	Substitution	Simplification	
$\sqrt{a^2-x^2}$	$x=a\sin\theta$	$\sqrt{a^2-x^2}=a\cos\theta$,	$dx=a\cos\theta\,d\theta$
$\sqrt{a^2+x^2}$	$x=a\tan\theta$	$\sqrt{a^2+x^2}=a\sec\theta$,	$dx=a\sec^2\theta\,d\theta$
$\sqrt{x^2-a^2}$	$x=a\sec\theta$	$\sqrt{x^2-a^2}=a\lvert\tan\theta\rvert$,	$dx=a\sec\theta\tan\theta\,d\theta$

- In the first case, one has $a^2-x^2=a^2-a^2\sin^2\theta=a^2\cos^2\theta$ and $\sqrt{a^2-x^2}=a\cos\theta$. This is because $\theta=\sin^{-1}(x/a)$ lies between $-\pi/2$ and $\pi/2$, so $\cos\theta$ is non-negative.

Example 6.20 We use a trigonometric substitution to compute the integral

$$\int\frac{dx}{\sqrt{a^2-x^2}},\qquad a>0.$$

If we let $x=a\sin\theta$, then $a^2-x^2=a^2-a^2\sin^2\theta=a^2\cos^2\theta$ and also $dx=a\cos\theta\,d\theta$, so

$$\int\frac{dx}{\sqrt{a^2-x^2}}=\int\frac{a\cos\theta\,d\theta}{a\cos\theta}=\int d\theta=\theta+C=\sin^{-1}\frac{x}{a}+C.\qquad\square$$

Example 6.21 We use a trigonometric substitution to compute the integral

$$\int\frac{dx}{x^2+a^2},\qquad a>0.$$

If we let $x=a\tan\theta$, then $x^2+a^2=a^2\tan^2\theta+a^2=a^2\sec^2\theta$ and also $dx=a\sec^2\theta\,d\theta$, so

$$\int\frac{dx}{x^2+a^2}=\int\frac{a\sec^2\theta\,d\theta}{a^2\sec^2\theta}=\frac{1}{a}\int d\theta=\frac{1}{a}\theta+C=\frac{1}{a}\tan^{-1}\frac{x}{a}+C.\qquad\square$$

Example 6.22 We use a trigonometric substitution to compute the integral

$$\int\frac{x^2\,dx}{\sqrt{4-x^2}}.$$

If we let $x=2\sin\theta$, then $4-x^2=4-4\sin^2\theta=4\cos^2\theta$ and also $dx=2\cos\theta\,d\theta$, so

$$\int\frac{x^2\,dx}{\sqrt{4-x^2}}=\int\frac{4\sin^2\theta\cdot2\cos\theta\,d\theta}{2\cos\theta}=\int4\sin^2\theta\,d\theta=2\int[1-\cos(2\theta)]\,d\theta$$
$$=2\theta-\sin(2\theta)+C=2\theta-2\sin\theta\cdot\cos\theta+C.$$

It remains to express this equation in terms of $x=2\sin\theta$. Since $\theta=\sin^{-1}\frac{x}{2}$, we get

$$\int\frac{x^2\,dx}{\sqrt{4-x^2}}=2\sin^{-1}\frac{x}{2}-2\cdot\frac{x}{2}\cdot\sqrt{1-\frac{x^2}{4}}+C=2\sin^{-1}\frac{x}{2}-\frac{x}{2}\sqrt{4-x^2}+C.\qquad\square$$

6.6 Partial fractions

Definition 6.23 – Proper rational function

A proper rational function is a quotient of two polynomials $P(x)/Q(x)$ such that the degree of the numerator $P(x)$ is smaller than the degree of the denominator $Q(x)$.

Theorem 6.24 – Partial fractions

Suppose that $f(x)$ is a proper rational function whose denominator is the product of relatively prime polynomials. Then $f(x)$ can be expressed as a sum of proper rational functions whose denominators are these relatively prime polynomials.

- Two polynomials are relatively prime, if they do not have any common divisor other than constant factors. For instance, $(x+1)(x-1)$ and $x^2(x+3)$ are relatively prime, whereas $x(x-1)$ and $x^2(x+3)$ have a non-constant factor in common.

Example 6.25 We use partial fractions to compute the integral

$$\int \frac{x^2+3x-4}{(x^2+1)(x+1)}\,dx.$$

According to the last theorem, the integrand can be expressed in the form

$$\frac{x^2+3x-4}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1}$$

for some constants A, B, C that need to be determined. Clearing denominators gives

$$x^2+3x-4 = (Ax+B)(x+1) + C(x^2+1)$$

and one may look at some suitable choices of x to find that

$$x=-1,0,1 \quad\Longrightarrow\quad -6=2C, \qquad -4=B+C, \qquad 0=2A+2B+2C.$$

Solving these equations, we now get $C=-3$, $B=-1$ and $A=4$, which means that

$$\int \frac{x^2+3x-4}{(x^2+1)(x+1)}\,dx = \int \frac{4x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx - \int \frac{3}{x+1}\,dx.$$

The two rightmost integrals are rather easy to compute, and so is the integral

$$\int \frac{4x}{x^2+1}\,dx = \int \frac{2\,du}{u} = 2\ln|u| + C = 2\ln(x^2+1) + C,$$

if one substitutes $u = x^2+1$. In view of the last two equations, we must thus have

$$\int \frac{x^2+3x-4}{(x^2+1)(x+1)}\,dx = 2\ln(x^2+1) - \tan^{-1}x - 3\ln|x+1| + C. \qquad \square$$

Example 6.26 We use partial fractions to compute the integral

$$\int \frac{x^3+3x^2+5}{x(x-1)}\,dx.$$

This rational function is not proper because its numerator is cubic and its denominator is only quadratic. Thus, one needs to first use division of polynomials to write

$$\frac{x^3+3x^2+5}{x(x-1)} = \frac{x^3+3x^2+5}{x^2-x} = x+4+\frac{4x+5}{x(x-1)}.$$

Since the rightmost fraction is proper, one may use partial fractions to express it as

$$\frac{4x+5}{x(x-1)} = \frac{A}{x}+\frac{B}{x-1} \quad\Longrightarrow\quad 4x+5 = A(x-1)+Bx.$$

Setting $x=0$ gives $5=-A$ and setting $x=1$ gives $9=B$. It easily follows that

$$\frac{x^3+3x^2+5}{x(x-1)} = x+4+\frac{A}{x}+\frac{B}{x-1} = x+4-\frac{5}{x}+\frac{9}{x-1}.$$

Once we now integrate this equation term by term, we may finally conclude that

$$\int \frac{x^3+3x^2+5}{x(x-1)}\,dx = \frac{x^2}{2}+4x-5\ln|x|+9\ln|x-1|+C. \qquad \square$$

Example 6.27 We use a substitution and partial fractions to compute the integral

$$\int \frac{e^{5x}\,dx}{e^{2x}-1}.$$

If we take $u=e^x$, then $du=e^x\,dx$ and the given integral takes the form

$$\int \frac{e^{5x}\,dx}{e^{2x}-1} = \int \frac{e^{4x}\cdot e^x\,dx}{e^{2x}-1} = \int \frac{u^4\,du}{u^2-1}.$$

This is not a proper rational function, so one needs to first use division to write

$$\int \frac{e^{5x}\,dx}{e^{2x}-1} = \int \frac{u^4-1+1}{u^2-1}\,du = \int \left(u^2+1+\frac{1}{u^2-1}\right)du. \tag{6.8}$$

Let us merely focus on the proper rational function. Using partial fractions, we get

$$\frac{1}{u^2-1} = \frac{A}{u-1}+\frac{B}{u+1} \quad\Longrightarrow\quad 1 = A(u+1)+B(u-1).$$

When $u=1$, this gives $1=2A$. When $u=-1$, it gives $1=-2B$. In particular, one has

$$u^2+1+\frac{1}{u^2-1} = u^2+1+\frac{A}{u-1}+\frac{B}{u+1} = u^2+1+\frac{1/2}{u-1}-\frac{1/2}{u+1}$$

and each of these terms can be easily integrated. Returning to (6.8), we conclude that

$$\begin{aligned}\int \frac{e^{5x}\,dx}{e^{2x}-1} &= \frac{1}{3}\,u^3+u+\frac{1}{2}\,\ln|u-1|-\frac{1}{2}\,\ln|u+1|+C\\ &= \frac{1}{3}\,e^{3x}+e^x+\frac{1}{2}\,\ln|e^x-1|-\frac{1}{2}\,\ln(e^x+1)+C. \qquad \square\end{aligned}$$

Chapter 7

Sequences and series

7.1 Convergence of sequences

Definition 7.1 – Convergence of sequences

A sequence is a function that is defined on the set $\mathbb{N}$ of natural numbers. Its values are usually denoted by writing a_n for each $n \in \mathbb{N}$. We say that the sequence $\{a_n\}$ converges, if a_n approaches a finite limit as $n \to \infty$. Otherwise, we say that $\{a_n\}$ diverges.

Definition 7.2 – Monotonicity

A sequence $\{a_n\}$ is called monotonic, if it is either increasing, in which case $a_n \leq a_{n+1}$ for each $n \in \mathbb{N}$, or else decreasing, in which case $a_n \geq a_{n+1}$ for each $n \in \mathbb{N}$.

Theorem 7.3 – Monotonic and bounded

If a sequence is monotonic and bounded, then the sequence is also convergent.

- When a precise formula for a_n is known, one may use that formula to compute the limit of a_n and prove convergence. However, a precise formula is not always available.

Example 7.4 We show that each of the following sequences converges.

$$a_n = \sqrt{\frac{8n^2+3}{2n^2+5}}, \qquad b_n = \frac{3+\sin n}{n^2}, \qquad c_n = \left(1+\frac{1}{n}\right)^n.$$

Since the limit of a square root is the square root of the limit, it should be clear that

$$\lim_{n\to\infty} \frac{8n^2+3}{2n^2+5} = \lim_{n\to\infty} \frac{8n^2}{2n^2} = 4 \quad \Longrightarrow \quad \lim_{n\to\infty} a_n = \sqrt{4} = 2.$$

The limit of the second sequence is zero because $2/n^2 \leq b_n \leq 4/n^2$ for each $n \geq 1$. This means that b_n is squeezed between two sequences that converge to zero. Finally, one has

$$c_n = \left(1+\frac{1}{n}\right)^n \quad \Longrightarrow \quad \ln c_n = n \cdot \ln\left(1+\frac{1}{n}\right) = \frac{\ln(1+1/n)}{1/n}.$$

This is a limit of the form 0/0, so one may use L'Hôpital's rule to conclude that

$$\lim_{n\to\infty} \ln c_n = \lim_{n\to\infty} \frac{(1+1/n)^{-1}\cdot(1/n)'}{(1/n)'} = 1 \quad\Longrightarrow\quad \lim_{n\to\infty} c_n = e^1 = e. \qquad \square$$

Example 7.5 There are two different ways of checking that $a_n = n/(n+1)$ is increasing. First of all, one may use derivatives. If we define $f(x) = x/(x+1)$ for each $x \geq 1$, then

$$f'(x) = \frac{(x+1)-x}{(x+1)^2} = \frac{1}{(x+1)^2} > 0.$$

This makes $f(x)$ increasing for all $x \geq 1$ and thus a_n is increasing for all $n \geq 1$. It is also possible to check this directly. To show that a_n is increasing, one needs to show that

$$a_n \leq a_{n+1} \quad\Longleftrightarrow\quad \frac{n}{n+1} \leq \frac{n+1}{n+2} \quad\Longleftrightarrow\quad n^2+2n \leq n^2+2n+1.$$

Since the rightmost inequality is obviously true, the leftmost inequality holds as well. $\square$

Example 7.6 We show that $a_n = \frac{2^n}{n!}$ is decreasing for all $n \geq 1$. In this case, we have

$$a_n \geq a_{n+1} \quad\Longleftrightarrow\quad \frac{2^n}{n!} \geq \frac{2^{n+1}}{(n+1)!} \quad\Longleftrightarrow\quad n+1 \geq 2.$$

Since the rightmost inequality is obviously true, the leftmost inequality holds as well. $\square$

Example 7.7 We find the limit of the sequence $\{a_n\}$ which is defined by $a_1 = 1$ and

$$a_{n+1} = \sqrt{2a_n} \quad \text{for each } n \geq 1.$$

To show that this sequence converges, we shall first show that

$$1 \leq a_n \leq a_{n+1} \leq 2 \quad \text{for each } n \geq 1. \tag{7.1}$$

When $n = 1$, this statement asserts that $1 \leq 1 \leq \sqrt{2} \leq 2$, so it is certainly true. Suppose that it is true for some n. Multiplying by 2 and taking square roots, we then find that

$$\begin{aligned} 2 \leq 2a_n \leq 2a_{n+1} \leq 4 \quad &\Longrightarrow \quad \sqrt{2} \leq \sqrt{2a_n} \leq \sqrt{2a_{n+1}} \leq 2 \\ &\Longrightarrow \quad 1 \leq a_{n+1} \leq a_{n+2} \leq 2. \end{aligned}$$

In particular, the statement holds for $n+1$ as well, so it actually holds for all $n \in \mathbb{N}$. This shows that the given sequence is monotonic and bounded, hence also convergent; denote its limit by L. Using the definition of the sequence, one may then argue that

$$a_{n+1} = \sqrt{2a_n} \quad\Longrightarrow\quad \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2a_n} \quad\Longrightarrow\quad L = \sqrt{2L}.$$

This gives $L^2 = 2L$, so either $L = 0$ or else $L = 2$. On the other hand, we must also have

$$1 \leq a_n \leq 2 \quad\Longrightarrow\quad 1 \leq \lim_{n\to\infty} a_n \leq 2 \quad\Longrightarrow\quad 1 \leq L \leq 2$$

because of equation (7.1). We conclude that the limit of the sequence is $L = 2$. $\square$