Chapter 5

Integration

5.1 Definite integral

Definition 5.1 – Integrability

Suppose that f is defined on $[a,b]$ and let $x_0, x_1, \ldots, x_n$ be the points that divide $[a,b]$ into n intervals of length $\Delta x = \frac{b-a}{n}$. We say that f is integrable on $[a,b]$, if the limit

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^n f(x_k^*)\Delta x \tag{5.1}$$

exists and its value does not depend on which point x_k^* is chosen from each $[x_{k-1}, x_k]$.

- The sum that appears in definition (5.1) is also known as a Riemann sum. When f is positive, it gives the total area of the rectangles with height $f(x_k^*)$ and base Δx. The limit of their sum should be the area of the region that lies below the graph of f.

- There are very few functions for which the Riemann sums can be computed explicitly.

Example 5.2 We use the definition of integrability to show that every constant function is integrable. Indeed, suppose that $f(x) = c$ is constant for all x. Then equation (5.1) gives

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^n c\,\Delta x = \lim_{n\to\infty} \sum_{k=1}^n \frac{c(b-a)}{n} = \lim_{n\to\infty} n \cdot \frac{c(b-a)}{n} = c(b-a). \qquad \square$$

Example 5.3 We use the definition of integrability to show that

$$\int_0^b x\,dx = \frac{b^2}{2} \quad \text{for all } b > 0.$$

Consider the points $x_0, x_1, \ldots, x_n$ that divide $[0,b]$ into n intervals of equal length. These are given by $x_k = k\Delta x$ for each $0 \leq k \leq n$ and $\Delta x = b/n$, so it easily follows that

$$x_{k-1}\,\Delta x \leq x_k^*\,\Delta x \leq x_k\,\Delta x \quad \Longrightarrow \quad (k-1)\frac{b^2}{n^2} \leq x_k^*\,\Delta x \leq k\frac{b^2}{n^2}.$$

Adding up these inequalities over all possible values of $1 \leq k \leq n$, we conclude that

$$\frac{b^2}{n^2} \sum_{k=1}^{n} (k-1) \leq \sum_{k=1}^{n} x_k^* \,\Delta x \leq \frac{b^2}{n^2} \sum_{k=1}^{n} k. \tag{5.2}$$

The sum that appears in the right hand side of (5.2) is a standard sum whose value is

$$\sum_{k=1}^{n} k = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$

The sum that appears in the left hand side of (5.2) is practically the same because

$$\sum_{k=1}^{n} (k-1) = 0 + 1 + \ldots + (n-1) = \frac{(n-1)n}{2}.$$

In particular, one may simplify equation (5.2) to establish the estimates

$$\frac{b^2}{n^2} \cdot \frac{(n-1)n}{2} \leq \sum_{k=1}^{n} x_k^* \,\Delta x \leq \frac{b^2}{n^2} \cdot \frac{n(n+1)}{2}. \tag{5.3}$$

To compute the integral of $f(x) = x$, it remains to take the limit as $n \to \infty$. In this case,

$$\lim_{n\to\infty} \frac{b^2}{n^2} \cdot \frac{(n-1)n}{2} = \frac{b^2}{2} = \lim_{n\to\infty} \frac{b^2}{n^2} \cdot \frac{n(n+1)}{2},$$

so one may apply the squeeze theorem to find that $\int_0^b x\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} x_k^* \,\Delta x = \frac{b^2}{2}$. $\square$

Example 5.4 We show that f is not integrable on any interval $[a, b]$ when

$$f(x) = \left\{ \begin{array}{ll} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{array} \right\}.$$

Were f integrable on $[a, b]$, one would be able to express its integral in the form

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x$$

for an arbitrary choice of points $x_k^* \in [x_{k-1}, x_k]$. On the other hand, this interval contains both rational and irrational numbers. If we choose the points x_k^* to be irrational, then

$$f(x_k^*) = 0 \quad \Longrightarrow \quad \sum_{k=1}^{n} f(x_k^*)\,\Delta x = 0 \quad \Longrightarrow \quad \int_a^b f(x)\,dx = 0.$$

If we choose the points x_k^* to be rational, then we similarly get

$$f(x_k^*) = 1 \quad \Longrightarrow \quad \sum_{k=1}^{n} f(x_k^*)\,\Delta x = n\,\Delta x = b - a \quad \Longrightarrow \quad \int_a^b f(x)\,dx = b - a.$$

This gives two different values for the same integral, so f is not integrable on $[a, b]$. $\square$

5.2 Rules of integration

Theorem 5.5 – Linearity

If the functions f, g are integrable on $[a, b]$, then so is their sum, and one has

$$\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$

If a function f is integrable on $[a, b]$, then cf is integrable for any constant c and

$$\int_a^b cf(x)\, dx = c\int_a^b f(x)\, dx.$$

Theorem 5.6 – Integrals and inequalities

If the functions f, g are integrable on $[a, b]$ and $f(x) \leq g(x)$ for all $a \leq x \leq b$, then

$$\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx.$$

Theorem 5.7 – Continuous implies integrable

If a function f is continuous on $[a, b]$, then f is also integrable on $[a, b]$.

Definition 5.8 – Arbitrary limits of integration

The definition of integrability on $[a, b]$ implicitly assumes that $a < b$. However, one may extend this definition to any limits of integration using the formulas

$$\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx, \qquad \int_a^a f(x)\, dx = 0.$$

- The first two theorems are easy to prove, but the proof of Theorem 5.7 is difficult.
- Theorem 5.6 implies the triangle inequality for integrals which states that

$$\left|\int_a^b f(x)\, dx\right| \leq \int_a^b |f(x)|\, dx$$

for any continuous function f. In fact, one has $-|f(x)| \leq f(x) \leq |f(x)|$ and so

$$-\int_a^b |f(x)|\, dx \leq \int_a^b f(x)\, dx \leq \int_a^b |f(x)|\, dx.$$

Writing the last equation as $-A \leq B \leq A$, we conclude that $|B| \leq A$, as needed.

5.3 Fundamental theorem of calculus

Theorem 5.9 – Fundamental theorem of calculus, part 1

If f is continuous on $[a, b]$ and F is a function whose derivative is f, then

$$\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = F(b) - F(a).$$

Theorem 5.10 – Mean value theorem for integrals

If f is continuous on $[a, b]$, then there exists a point $a \leq c \leq b$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Theorem 5.11 – Fundamental theorem of calculus, part 2

Suppose that f is continuous on $[a, b]$ and let $F(x)$ denote its definite integral

$$F(x) = \int_a^x f(t)\,dt, \qquad a \leq x \leq b.$$

Then F is a function whose derivative is f. In other words, one has $F'(x) = f(x)$.

- An antiderivative of f is a function F whose derivative is f. It is denoted by

  $$F(x) = \int f(x)\,dx,$$

  an integral without limits, and it is also known as the indefinite integral of f.
- The difference $F(b) - F(a)$ is frequently denoted by $[F(x)]_a^b$. One may thus write

  $$\int_a^b F'(x)\,dx = F(b) - F(a) = [F(x)]_a^b\,.$$

Example 5.12 We use the fundamental theorem of calculus to compute the integral

$$I = \int_1^2 (3x^2 - 2)\,dx.$$

Since $F(x) = x^3 - 2x$ is such that $F'(x) = 3x^2 - 2$, one may apply Theorem 5.9 to get

$$I = \int_1^2 F'(x)\,dx = [F(x)]_1^2 = F(2) - F(1) = 4 - (-1) = 5. \qquad \square$$

5.4 Integrals of standard functions

- The following list includes the antiderivatives of some standard functions.
- In each case, the antiderivative is expressed in terms of an arbitrary constant C.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1 \qquad \int x^{-1}\,dx = \ln|x| + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \cot x\,dx = \ln|\sin x| + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x\,\tan x\,dx = \sec x + C \qquad \int \csc x\,\cot x\,dx = -\csc x + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C \qquad \int \csc x\,dx = -\ln|\csc x + \cot x| + C$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C \qquad \int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

$$\int e^{ax}\,dx = \frac{e^{ax}}{a} + C, \quad a \neq 0 \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C, \quad 0 < a \neq 1$$

- The formulas above can be verified using differentiation. For instance, one has

$$\Big[\ln|\sec x|\Big]' = \frac{(\sec x)'}{\sec x} = \frac{\sec x \tan x}{\sec x} = \tan x.$$

This makes $\ln|\sec x|$ an antiderivative of $\tan x$, so $\int \tan x\,dx = \ln|\sec x| + C$.

- The formula for the integral of $\sec x$ can be similarly verified by checking that

$$\Big[\ln|\sec x + \tan x|\Big]' = \frac{(\sec x + \tan x)'}{\sec x + \tan x} = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x.$$

Example 5.13 Using the formula for the antiderivative of x^n, one finds that

$$\int_0^2 x^2(\sqrt{x}+4)\,dx = \int_0^2 (x^{5/2} + 4x^2)\,dx = \left[\frac{2}{7}x^{7/2} + \frac{4x^3}{3}\right]_0^2 = \frac{16\sqrt{2}}{7} + \frac{32}{3}.$$

Using the formula for the antiderivative of tangent, one similarly finds that

$$\int_0^{\pi/4} \tan x\,dx = \Big[\ln|\sec x|\Big]_0^{\pi/4} = \ln\frac{2}{\sqrt{2}} - \ln 1 = \ln\sqrt{2} = \frac{\ln 2}{2}. \qquad \square$$

5.5 Area, volume and arc length

Theorem 5.14 – Area between two graphs

Suppose that f, g are continuous on $[a, b]$ and $f(x) \leq g(x)$ for all $a \leq x \leq b$. Then the area of the region that lies between the graphs of the two functions is

$$\text{Area} = \int_a^b [g(x) - f(x)]\, dx.$$

Theorem 5.15 – Solids of revolution

Suppose that f is continuous on $[a, b]$ and consider the solid that is produced when the graph of f is rotated around the x-axis. The volume of the resulting solid is then

$$\text{Volume} = \int_a^b \pi f(x)^2\, dx.$$

Theorem 5.16 – Arc length

Suppose that f is differentiable on $[a, b]$ and f' is continuous on $[a, b]$. Then the length of the graph of f over the interval $[a, b]$ is given by

$$\text{Arc length} = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

Example 5.17 We find the area of the region that is enclosed by the graphs of $f(x) = x^2$ and $g(x) = 8 - x^2$. These are the graphs of two parabolas which intersect when

$$x^2 = 8 - x^2 \quad \Longleftrightarrow \quad 2x^2 = 8 \quad \Longleftrightarrow \quad x = \pm 2.$$

Since $f(x) \leq g(x)$ at all points $-2 \leq x \leq 2$, the area of the region is thus

$$\int_{-2}^2 [g(x) - f(x)]\, dx = \int_{-2}^2 \left[8 - 2x^2\right]\, dx = \left[8x - \frac{2x^3}{3}\right]_{-2}^2 = \frac{64}{3}. \qquad \square$$

Example 5.18 We compute the volume of a cone with radius r and height h. One may obtain such a cone by rotating a right triangle around the x-axis. Consider the triangle with vertices $(0, 0)$, $(h, 0)$ and (h, r). Its base has length h and its height has length r, so its hypotenuse has slope r/h and it is given by the line $f(x) = rx/h$. If we rotate the triangle around the x-axis, we get a cone of radius r and height h. The volume of the cone is thus

$$V = \int_0^h \pi f(x)^2\, dx = \frac{\pi r^2}{h^2} \int_0^h x^2\, dx = \frac{\pi r^2}{h^2} \left[\frac{x^3}{3}\right]_0^h = \frac{\pi r^2 h}{3}. \qquad \square$$

Example 5.19 We compute the circumference of a circle with radius $r = 1$. Let us only worry about the upper semicircle $f(x) = \sqrt{1 - x^2}$, where $-1 \leq x \leq 1$. In this case,

$$f'(x) = \frac{(1 - x^2)'}{2\sqrt{1 - x^2}} = -\frac{x}{\sqrt{1 - x^2}}$$

and the arc length is given by the integral of $\sqrt{1 + f'(x)^2}$. Once we now simplify

$$1 + f'(x)^2 = 1 + \frac{x^2}{1 - x^2} = \frac{1}{1 - x^2},$$

we may take the square root of both sides to conclude that the arc length is

$$\int_{-1}^{1} \sqrt{1 + f'(x)^2}\, dx = \int_{-1}^{1} \frac{dx}{\sqrt{1 - x^2}} = \Big[\sin^{-1} x\Big]_{-1}^{1} = \frac{\pi}{2} + \frac{\pi}{2} = \pi. \qquad \square$$

Example 5.20 We compute the length of the graph of f in the case that

$$f(x) = \frac{x^3}{12} + \frac{1}{x}, \qquad 1 \leq x \leq 2.$$

This amounts to computing the integral of $\sqrt{1 + f'(x)^2}$ and one can easily check that

$$\begin{aligned} f'(x) = \frac{x^2}{4} - \frac{1}{x^2} \quad &\Longrightarrow \quad f'(x)^2 + 1 = \frac{x^4}{16} + \frac{1}{x^4} - \frac{1}{2} + 1 \\ &\Longrightarrow \quad f'(x)^2 + 1 = \frac{x^8 + 16 + 8x^4}{16x^4} = \frac{(x^4 + 4)^2}{16x^4}. \end{aligned}$$

Taking the square root of both sides, we conclude that the length of the graph is

$$\int_{1}^{2} \frac{x^4 + 4}{4x^2}\, dx = \int_{1}^{2} \left(\frac{x^2}{4} + x^{-2}\right) dx = \left[\frac{x^3}{12} - \frac{1}{x}\right]_{1}^{2} = \frac{13}{12}. \qquad \square$$

Example 5.21 We compute the volume of the solid obtained by rotating the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

around the x-axis. This solid is also known as an ellipsoid. Assuming that a is positive, we get $-a \leq x \leq a$ and one may rearrange terms to find that

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \quad \Longrightarrow \quad y^2 = b^2 \left(1 - \frac{x^2}{a^2}\right).$$

The volume of the ellipsoid is the integral of $\pi f(x)^2 = \pi y^2$ and this is given by

$$\pi \int_{-a}^{a} y^2\, dx = b^2 \pi \int_{-a}^{a} \left(1 - \frac{x^2}{a^2}\right) dx = b^2 \pi \left[x - \frac{x^3}{3a^2}\right]_{-a}^{a} = \frac{4ab^2\pi}{3}. \qquad \square$$

5.6 Mass, centre of mass and work

Theorem 5.22 – Mass and centre of mass

Consider a thin rod which extends between the points $a \le x \le b$ and let $\delta(x)$ denote its density at the point x. The overall mass M of the rod is then

$$M = \int_a^b \delta(x)\,dx,$$

The centre of mass $\overline{x}$ is given by a similar formula and one has

$$\overline{x} = \frac{1}{M}\int_a^b x\delta(x)\,dx. \tag{5.4}$$

Definition 5.23 – Work

In physics, the amount of work that is required to move an object by d units using a constant force F in the direction of motion is defined as the product

$$\text{Work} = \text{Force} \cdot \text{Displacement} = F \cdot d.$$

Example 5.24 We compute the mass and the centre of mass for a thin rod with density

$$\delta(x) = 2x^2 + 3x + 4, \qquad 0 \le x \le 1.$$

The mass of the rod is given by the integral of its density function, namely

$$M = \int_0^1 (2x^2 + 3x + 4)\,dx = \left[\frac{2x^3}{3} + \frac{3x^2}{2} + 4x\right]_0^1 = \frac{2}{3} + \frac{3}{2} + 4 = \frac{37}{6}.$$

The centre of mass is given by equation (5.4) and one easily finds that

$$\overline{x} = \frac{1}{M}\int_0^1 x\delta(x)\,dx = \frac{6}{37}\int_0^1 (2x^3 + 3x^2 + 4x)\,dx = \frac{6}{37}\left[\frac{x^4}{2} + x^3 + 2x^2\right]_0^1 = \frac{21}{37}. \qquad \square$$

Example 5.25 A rectangular tank has length 4m, width 3m and height 2m. Suppose it is full with water of density 1000 kg/m^3. How much work does it take to pump out the water through a hole in the top? To find the answer, we consider a typical cross section of the tank. Assume that it has arbitrarily small height dx and lies x metres from the top. Then its volume is $V = 4 \cdot 3 \cdot dx$ and its mass is $m = (12\,dx) \cdot 1000$. The force needed to pump out this part is mass times acceleration, so $F = mg$ and the overall amount of work is

$$\text{Work} = \int F \cdot x = \int_0^2 12{,}000g \cdot x\,dx = 12{,}000g\left[\frac{x^2}{2}\right]_0^2 = 24{,}000g. \qquad \square$$

5.7 Improper integrals

- An improper integral is one that is not defined in the usual way because either the values of x or the values of $f(x)$ become infinite within the interval of integration.
- In these cases, one may integrate the given function over a generic interval $[a, b]$ and then compute the limit as the endpoints a, b approach the endpoints of interest.

Example 5.26 We use limits to compute the improper integral

$$I_1 = \int_0^1 \frac{dx}{\sqrt{x}}.$$

In this case, the integrand becomes infinite at $x = 0$, so we avoid this point and compute

$$\int_a^1 \frac{dx}{\sqrt{x}} = \int_a^1 x^{-1/2}\, dx = \left[\frac{x^{1/2}}{1/2}\right]_a^1 = 2 - 2\sqrt{a}.$$

Taking the limit as $a \to 0^+$, we conclude that the original integral is equal to $I_1 = 2$. □

Example 5.27 Let $c > 0$ be a given constant and consider the improper integral

$$I_2 = \int_0^\infty e^{-cx}\, dx.$$

Since the interval of integration is infinite, we start by computing the finite analogue

$$\int_0^L e^{-cx}\, dx = \left[-\frac{e^{-cx}}{c}\right]_0^L = -\frac{e^{-cL}}{c} + \frac{1}{c}.$$

Letting $L \to \infty$, we see that $cL \to \infty$ as well, so the original integral is equal to

$$I_2 = \lim_{L\to\infty} \int_0^L e^{-cx}\, dx = \lim_{L\to\infty} \frac{1 - e^{-cL}}{c} = \frac{1}{c}.$$ □

Example 5.28 Let $p > 1$ be a given constant and consider the improper integral

$$I_3 = \int_1^\infty x^{-p}\, dx.$$

Once again, one may compute this integral by considering its finite analogue

$$\int_1^L x^{-p}\, dx = \left[\frac{x^{1-p}}{1-p}\right]_1^L = \frac{L^{1-p}}{1-p} - \frac{1}{1-p}.$$

Since the exponent $1 - p$ is negative, one has $L^{1-p} \to 0$ as $L \to \infty$ and this implies that

$$I_3 = \lim_{L\to\infty} \int_1^L x^{-p}\, dx = -\frac{1}{1-p} = \frac{1}{p-1}.$$ □

Chapter 6

Techniques of integration

6.1 Integration by parts

Theorem 6.1 – Integration by parts

If f, g are differentiable functions and their derivatives f', g' are continuous, then

$$\int f(x) \cdot g'(x)\, dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x)\, dx. \tag{6.1}$$

- It is quite common, and also convenient, to express the last equation in the form

  $$\int u\, dv = uv - \int v\, du. \tag{6.2}$$

  This alternative version arises by letting $u = f(x)$ and $v = g(x)$ for simplicity.

- Some typical integrals that may be computed using integration by parts are

  $$\int p(x) \cdot e^{ax}\, dx, \qquad \int p(x) \cdot \sin(ax)\, dx, \qquad \int p(x) \cdot \cos(ax)\, dx,$$

  where $p(x)$ is a polynomial and $a \neq 0$ is a given constant. In each of these cases, one needs to integrate by parts n times, where n is the degree of the polynomial.

- Some other integrals for which integration by parts might be needed are those that involve $\sin^{-1}$, $\tan^{-1}$ or ln. These are functions whose derivatives are much simpler, so they are all natural choices for the variable u that appears in equation (6.2).

Example 6.2 We use integration by parts to compute the integral

$$\int xe^x\, dx.$$

Letting $u = x$ and $dv = e^x\, dx$, we find that $du = dx$ and $v = e^x$. It easily follows that

$$\int xe^x\, dx = uv - \int v\, du = xe^x - \int e^x\, dx = xe^x - e^x + C. \qquad \square$$

Example 6.3 We use integration by parts to compute the integral

$$\int x\cos(2x)\,dx.$$

In this case, we take $u = x$ and $dv = \cos(2x)\,dx$. Since $du = dx$ and $v = \frac{1}{2}\sin(2x)$, we get

$$\int x\cos(2x)\,dx = \frac{x}{2}\,\sin(2x) - \frac{1}{2}\int \sin(2x)\,dx = \frac{x}{2}\,\sin(2x) + \frac{1}{4}\,\cos(2x) + C. \qquad \square$$

Example 6.4 We use integration by parts to compute the integral

$$\int x^2 \ln x\,dx.$$

Letting $u = \ln x$ and $dv = x^2\,dx$, we find that $du = \frac{1}{x}\,dx$ and $v = \frac{x^3}{3}$. This implies that

$$\int x^2 \ln x\,dx = \frac{x^3}{3}\,\ln x - \int \frac{x^3}{3}\cdot\frac{1}{x}\,dx = \frac{x^3}{3}\ln x - \int \frac{x^2}{3}\,dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C. \qquad \square$$

Example 6.5 We use a double integration by parts to compute the integral

$$\int e^{ax}\sin(bx)\,dx, \qquad b \neq 0.$$

If we let $u = e^{ax}$ and $dv = \sin(bx)\,dx$, then $du = ae^{ax}\,dx$ and $v = -\frac{1}{b}\cos(bx)$, so

$$\int e^{ax}\sin(bx)\,dx = -\frac{1}{b}\,e^{ax}\cos(bx) + \frac{a}{b}\int e^{ax}\cos(bx)\,dx.$$

Next, we take $u = e^{ax}$ and $dv = \cos(bx)\,dx$. Since $du = ae^{ax}\,dx$ and $v = \frac{1}{b}\sin(bx)$, one has

$$\begin{aligned}\int e^{ax}\sin(bx)\,dx &= -\frac{1}{b}\,e^{ax}\cos(bx) + \frac{a}{b}\left[\frac{1}{b}\,e^{ax}\sin(bx) - \frac{a}{b}\int e^{ax}\sin(bx)\,dx\right]\\ &= -\frac{1}{b}\,e^{ax}\cos(bx) + \frac{a}{b^2}\,e^{ax}\sin(bx) - \frac{a^2}{b^2}\int e^{ax}\sin(bx)\,dx.\end{aligned}$$

Here, the rightmost integral is the same as the leftmost integral, so we actually have

$$\left(1 + \frac{a^2}{b^2}\right)\int e^{ax}\sin(bx)\,dx = -\frac{1}{b}\,e^{ax}\cos(bx) + \frac{a}{b^2}\,e^{ax}\sin(bx).$$

Once we now multiply the last equation by $b^2/(a^2+b^2)$, we may finally conclude that

$$\int e^{ax}\sin(bx)\,dx = -\frac{be^{ax}\cos(bx)}{a^2+b^2} + \frac{ae^{ax}\sin(bx)}{a^2+b^2} + C. \qquad \square$$

6.2 Integration by substitution

- A very useful tool for simplifying integrals is provided by the formula
$$\int g(f(x)) \cdot f'(x)\, dx = \int g(u)\, du. \tag{6.3}$$
- Here, the idea is to introduce a variable $u = f(x)$ to replace the leftmost integral by another integral which is much simpler and also expressible in terms of u alone.

Example 6.6 We use an appropriate substitution to compute the integral
$$\int \frac{(\ln x)^3}{x}\, dx.$$
If we take $u = \ln x$, then we have $du = \frac{1}{x}\, dx$ and this is easily seen to imply that
$$\int \frac{(\ln x)^3}{x}\, dx = \int u^3\, du = \frac{1}{4}\, u^4 + C = \frac{1}{4}\, (\ln x)^4 + C. \qquad \square$$

Example 6.7 We use integration by substitution to compute the integral
$$\int x^2(x^3+6)^4\, dx.$$
In this case, the choice $u = x^3 + 6$ is suitable because $du = 3x^2\, dx$ and this gives
$$\int x^2(x^3+6)^4\, dx = \frac{1}{3} \int u^4\, du = \frac{1}{15}\, u^5 + C = \frac{1}{15}\, (x^3+6)^5 + C. \qquad \square$$

Example 6.8 We use an appropriate substitution to compute the integral
$$\int \frac{2x+7}{(x+2)^2}\, dx.$$
In this case, we take $u = x + 2$ to merely simplify the denominator. Since $du = dx$, we get
$$\begin{aligned}\int \frac{2x+7}{(x+2)^2}\, dx &= \int \frac{2(u-2)+7}{u^2}\, du = \int 2u^{-1}\, du + \int 3u^{-2}\, du \\ &= 2\ln|u| - 3u^{-1} + C = 2\ln|x+2| - \frac{3}{x+2} + C. \qquad \square\end{aligned}$$

Example 6.9 We use integration by substitution to compute the integral
$$\int \cos\sqrt{x}\, dx.$$
If we let $u = \sqrt{x}$ to simplify the square root, then $x = u^2$ and $dx = 2u\, du$, so
$$\int \cos\sqrt{x}\, dx = \int \cos u \cdot 2u\, du = 2\int u \cos u\, du.$$
Next, we need to integrate by parts. Letting $dv = \cos u\, du$ and $v = \sin u$, we find that
$$\begin{aligned}\int \cos\sqrt{x}\, dx &= 2u\sin u - 2\int \sin u\, du = 2u\sin u + 2\cos u + C \\ &= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C. \qquad \square\end{aligned}$$