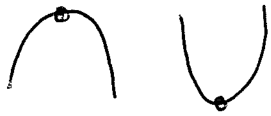
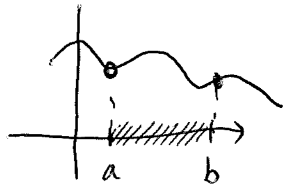
Existence of global min/max

<u>Case 1.</u> The derivative f' changes sign only once



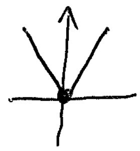
<u>Case 2.</u> The function f is continuous on $[a,b]$.
Then f attains <u>both</u> a global min <u>and</u> a global max on the interval $[a,b]$. Moreover, these are attained at



ⓐ <u>the endpoints a,b</u> OR

ⓑ <u>an interior point $a < x_0 < b$ with $f'(x_0) = 0$</u> OR

ⓒ <u>an interior point $a < x_0 < b$ with $f'(x_0)$ not defined.</u>

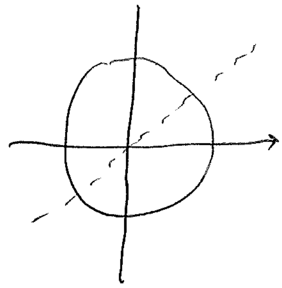


<u>Example 1.</u> Consider $f(x) = \sin x + \cos x$, for instance.
This is a continuous function on the interval $[0, 2\pi]$.
Thus it has a min/max value. We have

$f'(x) = \cos x - \sin x$ defined at all points

with $f'(x) = 0 \iff \sin x = \cos x \iff \tan x = 1$

$\iff x = \pi/4,\ 5\pi/4.$



The possible candidates are

$x = 0$ (endpoint) with $f(0) = \sin 0 + \cos 0 = 1$

$x = 2\pi$ (endpoint) with $f(2\pi) = \sin(2\pi) + \cos(2\pi) = 1$

$x = \pi/4$ $(f' = 0)$ with $f(\pi/4) = \sin \pi/4 + \cos \pi/4 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$

$x = 5\pi/4$ $(f' = 0)$ with $f(5\pi/4) = -\sqrt{2}$.

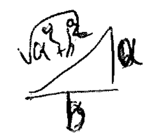
The min is $-\sqrt{2}$ attained at $x = 5\pi/4$, the max is $\sqrt{2}$ when $x = \pi/4$.

<u>Note:</u> Physicists usually write

$$f(x) = \sin x + \cos x = \sqrt{2}\left(\frac{\sqrt{2}}{2}\sin x + \frac{\sqrt{2}}{2}\cos x\right)$$

$$f(x) = \sqrt{2}\left(\cos \pi/4 \cos x + \sin \pi/4 \sin x\right) = \sqrt{2}\cos(\pi/4 - x)$$

and similarly



$$f(x) = a\sin x + b\cos x = \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x\right).$$

Example 2. Consider $f(x) = x^4 - 2x^2 - 1$ on $[0,2]$.

In this case, $f'(x) = 4x^3 - 4x = 4x(x^2-1) = 4x(x+1)(x-1)$.

The min/max must occur at one of

$x=0$ with $f(0) = -1$

$x=2$ with $f(2) = 16-8-1 = 7$

$\cancel{x=-1}$

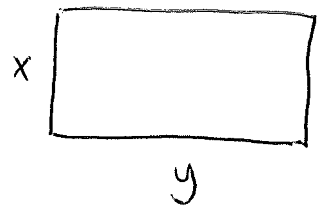
$x=1$ with $f(1) = 1-2-1 = -2$.

We get min $= -2$ when $x=1$, max $= 7$ when $x=2$.

We showed that $-2 \le x^4 - 2x^2 - 1 \le 7$ for all $0 \le x \le 2$.

$\boxed{\text{Optimisation problems}}$

① Out of all rectangles with perimeter 40, which one has the largest possible area?



Let x be the width, y the length.

We know $2x + 2y = 40$ so $x + y = 20$ so $y = 20 - x$.

Maximise Area $= xy = x(20-x) = 20x - x^2$.

Restrictions on x. Since $x \geq 0$ and $y \geq 0$

$x \geq 0$ and $20 - x \geq 0$, we get $0 \le x \le 20$.

We maximise $f(x) = 20x - x^2$ on $[0,20]$.

Since $f'(x) = 20 - 2x = 2(10-x)$, we check

$x = 10$ ---- $f(10) = 20 \times 10 - 10^2 = 100$

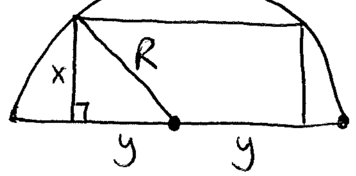
$x = 0$ ---- $f(0) = 0$

$x = 20$ --- $f(20) = 0$.

Thus, the largest possible value is $f(10) = 100$ when $x = 10 = y$.

② Consider a rectangle inscribed in a semicircle of radius R with one of its sides along the diameter. How large can the area of the rectangle be?

Let x be the width, y be half of the length.



Those are related by $x^2 + y^2 = R^2$

so $y = \sqrt{R^2 - x^2}$.

We maximise area $= 2xy = 2x\sqrt{R^2-x^2}$ when $0 \le x \le R$.

That's maximising $f(x) = 2x\sqrt{R^2-x^2}$ on $[0,R]$.

Direct approach $f'(x) = 2\sqrt{R^2-x^2} + 2x\left(\sqrt{R^2-x^2}\right)'$

$$= 2\sqrt{R^2-x^2} + \cancel{2}x \frac{1}{\cancel{2}\sqrt{R^2-x^2}} \cdot (-2x)$$

$$= \frac{2R^2 - 2x^2 - 2x^2}{\sqrt{R^2-x^2}} = \frac{2R^2 - 4x^2}{\sqrt{R^2-x^2}} = \frac{2(R^2-2x^2)}{\sqrt{R^2-x^2}}.$$

We check .... $x = 0$ with $f(0) = 0$

$x = R$ with $f(R) = 0$

$R^2 = 2x^2$ with $x = \frac{R}{\sqrt{2}}$ and $f\left(\frac{R}{\sqrt{2}}\right) = 2\frac{R}{\sqrt{2}}\sqrt{R^2 - \frac{R^2}{2}} = R^2$.

The largest area arises when $x = \frac{R}{\sqrt{2}}$ and $y = \frac{R}{\sqrt{2}}$.

Indirect approach. Maximising $2x\sqrt{R^2-x^2}$ amounts to maximising $(2x)^2(R^2-x^2)$

so we can look at $g(x) = 4x^2(R^2-x^2) = 4R^2x^2 - 4x^4$

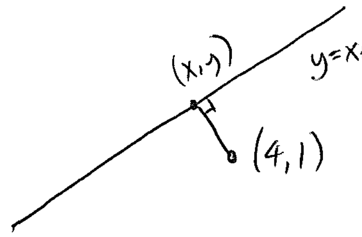
which gives $g'(x) = 8R^2x - 16x^3 = 8x(R^2 - 2x^2)$

and $2x^2 = R^2$ as before.

(3) We find the point on the line $y = x+1$ that lies closest to $(4,1)$.

We minimise distance between $(x,y) = (x, x+1)$

and $(4,1)$.



Distance $= \sqrt{(x-4)^2 + (y-1)^2} = \sqrt{(x-4)^2 + x^2}$

Let us minimise $f(x) = (x-4)^2 + x^2$ then, where x is arbitrary.

We get $f'(x) = 2(x-4) + 2x = 2x - 8 + 2x = 4x - 8 = 4(x-2)$.

This is positive when $x > 2$ and negative when $x < 2$,

so $f(2)$ is the global min. We get this for $(2,3)$.

		2	
f'	$-$		$+$
f	$\searrow$		$\nearrow$

(4) An application in economics. A company produces sweets with cost function $C(x) = 200 + 40x$ when $0 \le x \le 50$ kg of sweets are produced. The price for these sweets is $P(x) = 100 - 2x$, decreasing in x. How much should the company produce to maximise profits?

We have to maximise

$$\begin{aligned} \text{Profits} &= \text{Revenue} - \text{Cost} \\ &= x \cdot P(x) - C(x) \\ &= 100x - 2x^2 - 200 - 40x = \\ &= -2x^2 + 60x - 200. \end{aligned}$$

We get $\Pi(x) = -2x^2 + 60x - 200$ with $0 \le x \le 50$.

Candidates are ... $x = 0$

$x = 50$

$$0 = \Pi'(x) = -4x + 60 = -4(x-15).$$

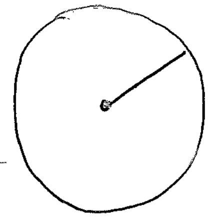
We have $\Pi(0) = -200$ and $\Pi(50) = -2 \times 2500 + 3000 - 200 = -2200$

and $\Pi(15) = -2 \times 225 + 900 - 200 = 250$.

Thus, max profit is $\Pi(15) = 250$.

$\boxed{\text{Related rates}}$ When two quantities are related to one another, their rates of change/derivatives are also related. The independent variable is usually time.

Example 1. A circular pond has a radius r increasing at 2m/sec. How fast is its area increasing when $r = 4$m?



Let $r(t)$ = radius and $A(t)$ = area = $\pi r(t)^2$.

Then $A'(t) = 2\pi r(t) \cdot r'(t)$ by the chain rule.

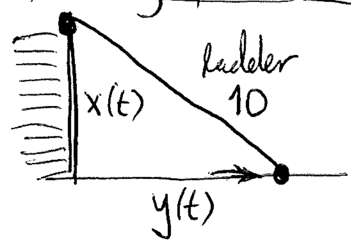
We know $r' = 2$ and we are given $r = 4$, so we substitute:

$$A'(t) = 2\pi \times 4 \times 2 = 16\pi$$

at that particular moment.

Example 2. A ladder is resting against a wall, it is 10 ft long. The base is sliding along the floor at 1 ft/sec. How fast is the top of the ladder sliding down the wall when the base is 6 ft away from the wall?

Let x, y be the sides of the triangle.



Since $\boxed{x(t)^2 + y(t)^2 = 10^2}$

$$2x(t)\,x'(t) + 2y(t)\,y'(t) = 0$$

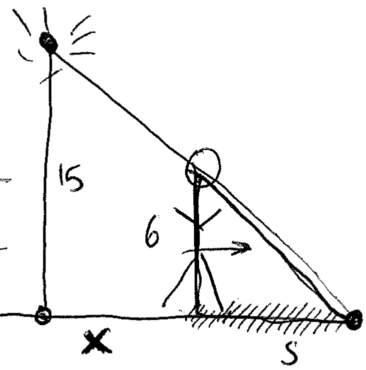
$$x(t)\,x'(t) + y(t)\,y'(t) = 0.$$

We know $y' = 1$, $y = 6$ and we need x'. In fact,

$$xx' + yy' = 0 \Rightarrow x' = -\frac{yy'}{x} = -\frac{6}{x} = -\frac{6}{\sqrt{100-y^2}} = -6/8.$$

Namely, y, y' are known and $x = \sqrt{10-y^2}$ can be determined.

Example 3. A street light is on top of a 15-ft pole. A 6ft-tall person is walking away from the pole at the rate of 3ft/sec. How fast is the tip of his/her shadow moving?



Let x be the distance between person & pole

s be the length of the shadow.

We need to somehow relate x with s. We have

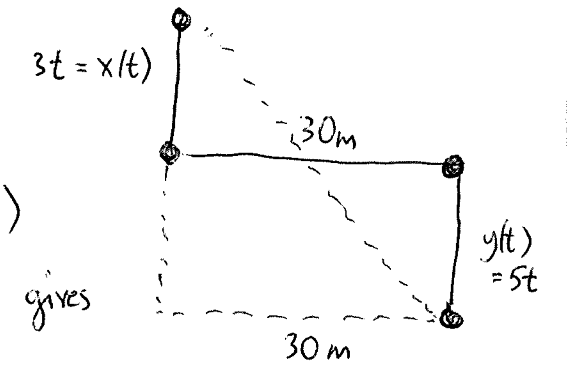
$$\frac{6}{s} = \frac{15}{s+x} \quad \text{by similar triangles}$$

So $6s + 6x = 15s$ so $6x = 9s$.

Differentiating gives $6x' = 9s'$ and we know $x' = 3$

So $s' = \frac{6x'}{9} = \frac{18}{9} = 2$ ft/sec.

Example 4. A man and a woman start walking at the same time. The man walks north at 3m/sec ~~but star~~ and the woman south at 5m/sec, but starting from a point 30m to the east. How fast is the distance between them changing 5 seconds later?



Let $x(t)$ be distance covered by the man
$y(t)$ be distance covered by the woman.

We need to relate x, y with the distance $d(t)$ between the two. In fact, Pythagoras' theorem gives

$$d(t)^2 = 30^2 + (x(t) + y(t))^2$$

$t=5$

$x=15$

$y=25$

$$\Rightarrow \cancel{2} d(t)\, d'(t) = \cancel{2}(x(t)+y(t)) \cdot (x'(t) + y'(t))$$

$$\Rightarrow d'(t) = \frac{(x+y)(x'+y')}{d} = \frac{40 \times 8}{\sqrt{30^2 + (x+y)^2}}$$

$$\Rightarrow d'(t) = \frac{40 \times 8}{\sqrt{30^2+40^2}} = \frac{4\cancel{0} \times 8}{5\cancel{0}} = \frac{32}{5} = 6.4 \text{ m/sec.}$$

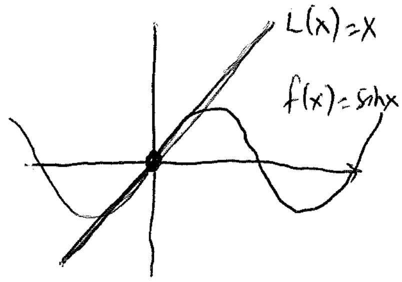
$\boxed{\text{Linear / Tangent Approximation}}$ If f is differentiable at x_0, then $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ and so $f'(x_0) \simeq \frac{f(x)-f(x_0)}{x-x_0}$ at points near x_0. This gives the approximation $f(x) \simeq f'(x_0)(x-x_0) + f(x_0)$ at points near x_0. The right hand side is the line through $(x_0, f(x_0))$ with slope $f'(x_0)$. We call $L(x) = f'(x_0)(x-x_0) + f(x_0)$ the linear / tangent approximation.

Example 1. Consider $f(x) = \sin x$ at the point $x_0 = 0$.

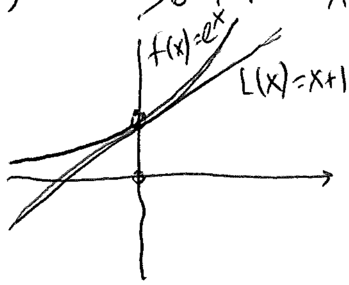
In this case $f(x_0) = \sin x_0 = \sin 0 = 0$

$f'(x_0) = \cos x_0 = \cos 0 = 1$

So the linear approximation is $L(x) = f'(x_0)(x-x_0) + f(x_0) = x$.
We thus have $\sin x \approx x$ for points near $x=0$.



__Example 2.__ Consider $f(x) = e^x$ at the point $x_0 = 0$.
Then $f(x_0) = e^0 = 1$
and $f'(x_0) = e^0 = 1$ so $L(x) = \cancel{f'(x_0)}^{1}(x-x_0) + \cancel{f(x_0)}^{1} = x - \cancel{x_0} + 1 = x+1$.
Thus $f(x) \approx x+1$ at points near $x=0$.



__Example 3.__ Consider $f(x) = \dfrac{x^2+3}{x^3+1}$ at $x_0 = 1$.

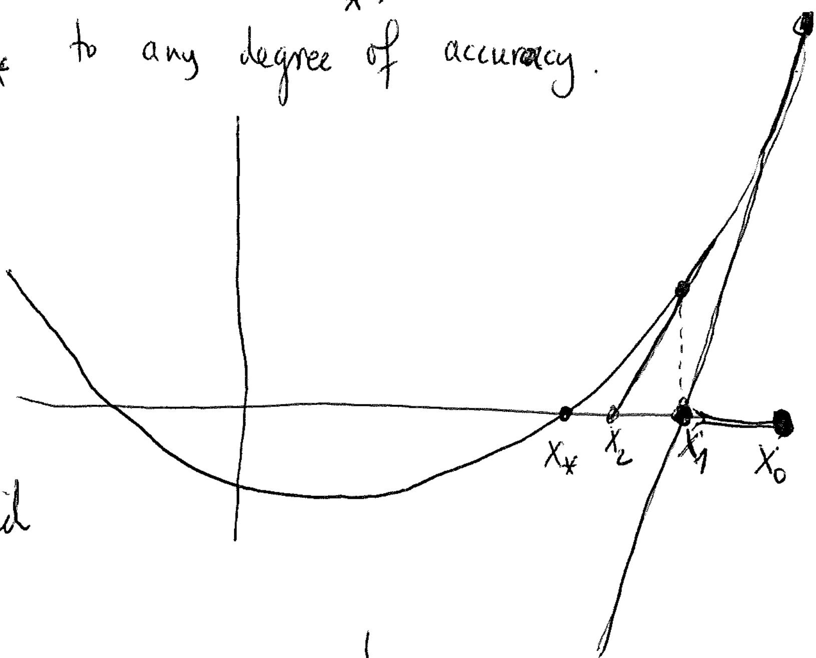
Then $$f'(x) = \frac{2x(x^3+1) - 3x^2(x^2+3)}{(x^3+1)^2} \Rightarrow f'(1) = \frac{2\times 2 - 3\times 4}{2^2} = -2$$

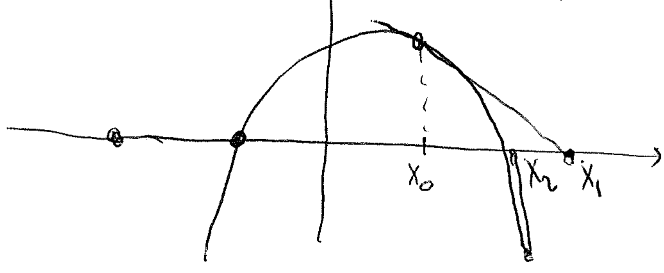
and $f(1) = \dfrac{1+3}{1+1} = 2$ so the linear approximation is

$$L(x) = f'(x_0)(x-x_0) + f(x_0) = -2(x-1) + 2 = -2x+4.$$

$\boxed{\text{Newton's method} \mid \text{for solving } f(x)=0.}$

Suppose that f is known to have a root x_*.
We are hoping to approximate x_* to any degree of accuracy.
We start with a guess x_0.
We find the lin. app. at x_0.
This lin. app. intersects the x-axis at the point x_1, say.
We now use that point x_1 to find the lin. app. at x_1 and proceed in that way.





Precise formula: if we start at the point x_n, the lin. appr. at that point is $L(x) = f'(x_n)(x-x_n) + f(x_n)$ and $L(x)=0$ means $f'(x_n)(x-x_n) + f(x_n) = 0$,

namely $x - x_n = -\dfrac{f(x_n)}{f'(x_n)}$.

This gives $x = x_n - \dfrac{f(x_n)}{f'(x_n)}$. We can thus define

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \text{as long as } f'(x_n) \neq 0.}$$

Example. We use Newton's method to approximate $\sqrt{2}$ ... or $\sqrt{a}$. We need a root of $f(x) = x^2 - 2$. Let $x_0 = 1$ be our initial guess. The formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 2}{2x_n}$$

$$\Rightarrow \quad x_{n+1} = x_n - \frac{x_n}{2} + \frac{1}{x_n} = \frac{x_n}{2} + \frac{1}{x_n}.$$

We get .....

$x_0 = 1$

$x_1 = \frac{1}{2} + \frac{1}{1} = \frac{3}{2} = 1.5$

$x_2 = \frac{x_1}{2} + \frac{1}{x_1} = \frac{3}{4} + \frac{2}{3} = \frac{17}{12} = 1.4166\ldots$

$x_3 = \frac{x_2}{2} + \frac{1}{x_2} = \frac{17}{24} + \frac{12}{17} = 1.4142156$

$x_4 = 1.414213562$ (correct to 9 decimals).

We could have also started with $x_0 = 2$, for instance. In that case $x_1 = \frac{x_0}{2} + \frac{1}{x_0} = 1 + \frac{1}{2} = 3/2$ and the other terms are identical.

Example Consider $f(x) = x^3 - 4x^2 - 3x + 1$ on $(0,2)$.

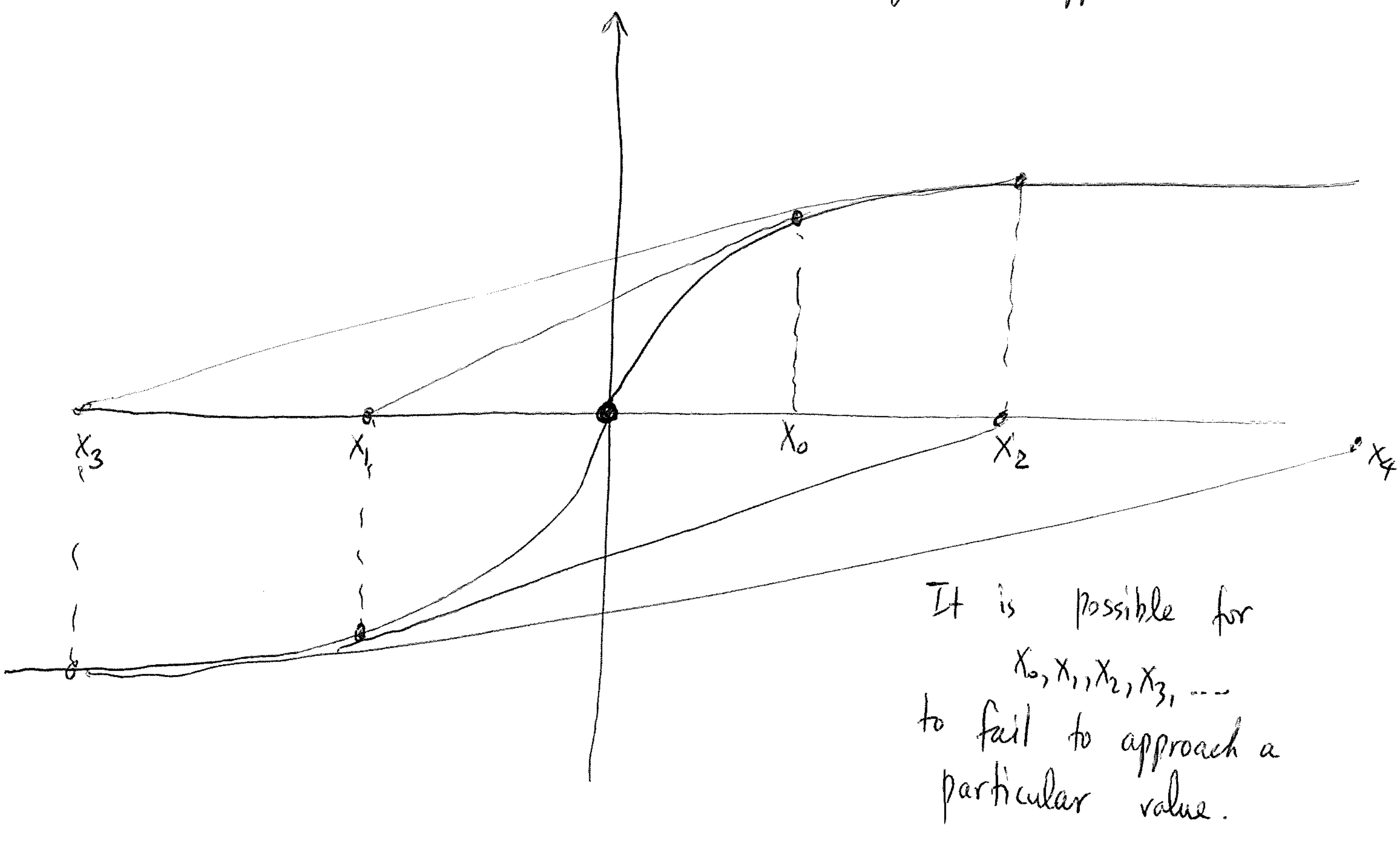
Since f is continuous with $f(0) = 1$ and $f(2) = -13$, there is a root in $(0,2)$ by Bolzano's theorem. We can show it is unique. To approximate it, we use Newton's method with $x_0 = 2$.

We have
$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 4x_n^2 - 3x_n + 1}{3x_n^2 - 8x_n - 3}}$$
and use this formula repeatedly to get

$$x_1 = 0.1428$$
$$x_2 = 0.2635$$
$$x_3 = 0.253309$$
$$x_4 = 0.253239.$$

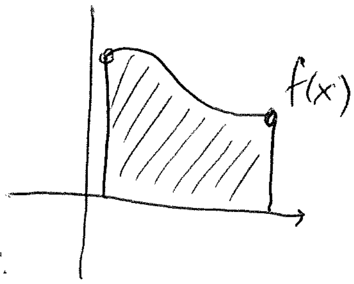
This makes the root $x = 0.253$ within 3 decimal places.

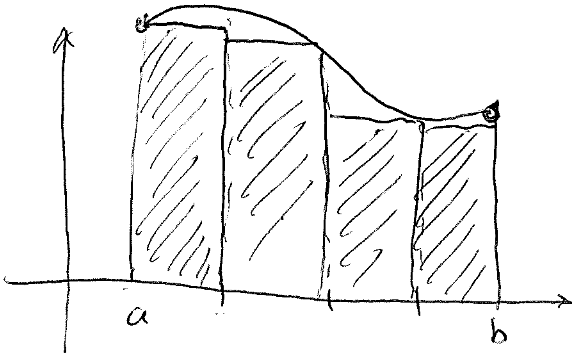
Note: This method does not work in all cases, but it works very often and the numbers x_n get to approach a value/root.



It is possible for $x_0, x_1, x_2, x_3, \ldots$ to fail to approach a particular value.

Integrals We hope to compute the area that lies under the graph of a positive function f. A natural starting point is to obtain approximations.





We can divide $[a,b]$ into smaller subintervals and approximate the area by the area of a rectangle over each subinterval. If we consider a large number of subintervals, we get a better approximation of the area.

Sigma notation We use $\sum$ to denote sums such as

$\sum_{k=1}^{5} a_k = a_1 + a_2 + a_3 + a_4 + a_5$, we call k the index of summation and have $\sum_{k=1}^{5} a_k = \sum_{i=1}^{5} a_i$. One can shift the index of summation to get

$$\sum_{k=1}^{5} a_k = \sum_{k=0}^{4} a_{k+1}.$$

Lower sums Consider $f(x)$ on the interval $[a,b]$. We divide this into n subintervals $[x_k, x_{k+1}]$ and let m_k be the min value attained by f .... $m_k = \min_{[x_k,x_{k+1}]} f(x)$. The lower sum is then

$$L(f) = \sum_{k=0}^{n-1} \underbrace{m_k}_{\text{height}} \cdot \underbrace{(x_{k+1}-x_k)}_{\text{base}}.$$

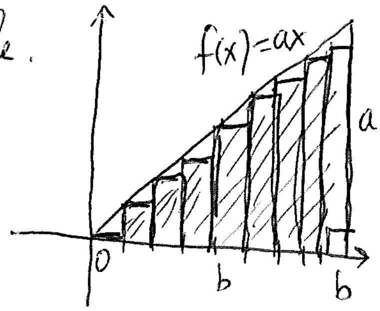
This is the sum of areas of rectangles (below the graph).

Upper sums are defined similarly using the max values,

$$\mathcal{U}(f) = \sum_{k=0}^{n-1} M_k \cdot (x_{k+1} - x_k) = \sum_{k=0}^{n-1} M_k \, \Delta x_k$$

with $M_k = \max f$ on $[x_k x_{k+1}]$ and $\Delta x_k = x_{k+1} - x_k = \text{base}$.

Example We work out the lower/upper sums for $f(x) = ax$ on $[0,b]$ with $a,b > 0$. That approximates the area of a triangle. We expect the area to be $A = \frac{1}{2}ab^2$.



Lower sums Consider n subintervals $[x_k, x_{k+1}]$.

The min value of f is $m_k = f(x_k) = ax_k$ for each k.

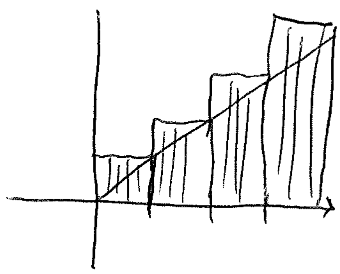
We have n subintervals of length $\frac{b}{n}$ each so $x_k = \frac{kb}{n}$ for each k.

We get
$$L(f) = \sum_{k=0}^{n-1} m_k \cdot \Delta x_k = \sum_{k=0}^{n-1} ax_k \cdot \frac{b}{n}$$
$$= \sum_{k=0}^{n-1} \frac{kba}{n} \cdot \frac{b}{n} = \frac{ab^2}{n^2} \sum_{k=0}^{n-1} k.$$

We need the formula
$$\sum_{k=0}^{n-1} k = 0 + 1 + 2 + 3 + \dots + (n-1) = \frac{(n-1)n}{2}$$

which gives
$$L(f) = \text{lower sum} = \frac{ab^2}{n^{\cancel{2}}} \cdot \frac{\cancel{n}(n-1)}{2}$$
$$L(f) = \frac{ab^2(n-1)}{2n}.$$

Upper sums Consider n subintervals $[x_k, x_{k+1}]$



The max value of f is $M_k = f(x_{k+1}) = ax_{k+1}$

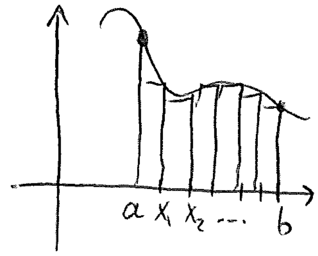
and
$$\mathcal{U}(f) = \sum_{k=0}^{n-1} M_k \cdot \Delta x_k = \sum_{k=0}^{n-1} ax_{k+1} \cdot \frac{b}{n}$$
$$= \sum_{k=0}^{n-1} a\frac{(k+1)b}{n} \cdot \frac{b}{n} = \frac{ab^2}{n^2} \sum_{k=0}^{n-1} (k+1) = \frac{ab^2 \cdot \cancel{n}(n+1)}{n^{\cancel{2}} \cdot 2}$$

In short, $L(f) = \frac{ab^2}{2} \cdot \frac{n-1}{n}$ and $\mathcal{U}(f) = \frac{ab^2}{2} \cdot \frac{n+1}{n}$.

If we now let $n \to \infty$, we see that lower/upper sums approach $\frac{ab^2}{2}$.

We can define the area as the limit of these sums.

Integrability Suppose f is defined on $[a,b]$.



Let $x_0 < x_1 < x_2 < \dots < x_n$ be the points that divide the interval into n subintervals of length $\frac{b-a}{n}$.

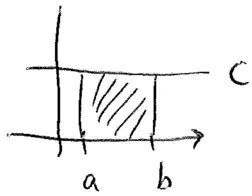
We say that f is <u>integrable</u> on $[a,b]$, if the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*) \cdot \Delta x$$

exists when x_k^* is an arbitrary point of $[x_{k-1}, x_k]$ and its value is independent of the points chosen. In that case, we write

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x \,. \quad (*)$$

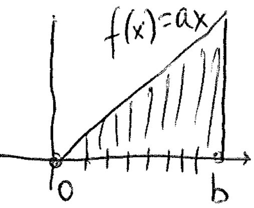
<u>Example 1.</u> Consider $f(x) = c$, a constant function.



In that case, $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} f(x_k^*)\,\Delta x = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} c\,\frac{b-a}{n}$

So $$\int_a^b c\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x = \lim_{n\to\infty} \cancel{n} \cdot c\,\frac{(b-a)}{\cancel{n}} = c(b-a).$$

<u>Example 2.</u> Consider $f(x) = ax$ on $[0,b]$ with $a, b > 0$.



We looked at the <u>lower sums</u> corresponding to min value

with $x_{k-1} \le x_k^* \le x_k$

$$\sum_{k=1}^{n} f(x_k^*)\,\Delta x = \sum_{k=1}^{n} f(x_{k-1})\,\Delta x$$

$$= \sum_{k=1}^{n} a x_{k-1}\,\Delta x = \sum_{k=1}^{n} a\,\frac{(k-1)b}{n} \cdot \frac{b}{n}$$

$$= \frac{ab^2}{n^2} \sum_{k=1}^{n} (k-1) = \frac{ab^2}{n^2}\,\frac{(n-1)n}{2}.$$

We also looked at <u>upper sums</u> corresponding to max value

$$\sum_{k=1}^{n} f(x_k^*)\,\Delta x = \sum_{k=1}^{n} f(x_k)\,\Delta x = \sum_{k=1}^{n} a x_k\,\Delta x = \sum_{k=1}^{n} a\,k\frac{b}{n} \cdot \frac{b}{n}$$

$$= \frac{ab^2}{n^2} \cdot \frac{n(n+1)}{2}.$$

We can now look at arbitrary sums (Riemann sums):

$$m_k \leq f(x_k^*) \leq M_k \Rightarrow m_k \Delta x \leq f(x_k^*)\Delta x \leq M_k \Delta x$$

$$\Rightarrow \underbrace{\underset{\text{lower sums}}{\sum m_k \Delta x} \leq \underset{\text{Riemann sums}}{\sum f(x_k^*)\Delta x} \leq \underset{\text{upper sums}}{\sum M_k \Delta x}}$$

If we can show lower/upper sums have the same limit, then Riemann Sums approach the same limit as well. Now,

$$\lim_{n\to\infty} \sum_{k=1}^{n} m_k \Delta x = \lim_{n\to\infty} \frac{ab^2}{n^2}\frac{(n-1)n}{2} = \lim_{n\to\infty} \frac{ab^2}{\cancel{n^2}}\frac{\cancel{n^2}}{2} = \frac{ab^2}{2}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} M_k \Delta x = \lim_{n\to\infty} \frac{ab^2}{n^2}\frac{n(n+1)}{2} = \lim_{n\to\infty} \frac{ab^2}{\cancel{n^2}}\frac{\cancel{n^2}}{2} = \frac{ab^2}{2}.$$

By the Squeeze Law, this proves $\int_0^b f(x)\,dx = \frac{ab^2}{2}$.

Example 3. Let $f(x) = \begin{Bmatrix} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{Bmatrix}$. In that case, the lower sums give $\sum_{k=1}^{n} m_k \Delta x = \sum_{k=1}^{n} 0\,\Delta x = 0$ and the upper sums give $\sum_{k=1}^{n} M_k \Delta x = \sum_{k=1}^{n} 1\Delta x = \sum_{k=1}^{n} \frac{b-a}{n} = b-a$.

This means $\lim_{n\to\infty} \sum m_k \Delta x = 0$ is not $\lim_{n\to\infty} \sum M_k \Delta x = b-a$, unless $a=b$.

Such a function is not integrable on $[a,b]$ for any $a<b$.

Theorem (Continuous implies integrable) Every continuous function on a closed interval $[a,b]$ is necessarily integrable.

Proof. This will be covered in MAU11204. $\square$

Properties of integrals

① Constant multiples If f is integrable on $[a,b]$, and c is a constant, then cf is also integrable and

$$\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx.$$

② **Linearity of integral** If f and g are integrable on $[a,b]$ then $f+g$ is also integrable on $[a,b]$ and

$$\int_a^b f(x)+g(x)\,dx = \int f(x)\,dx + \int_a^b g(x)\,dx.$$

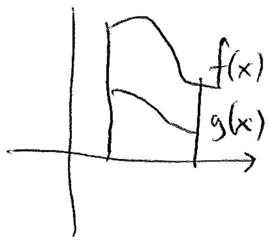
Proof of ①. We know $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^n f(x_k^*)\,\Delta x$ exists.

This implies
$$\lim_{n\to\infty} \sum_{k=1}^n c f(x_k^*)\,\Delta x = \lim_{n\to\infty} c \sum_{k=1}^n f(x_k^*)\,\Delta x$$
$$= c \lim_{n\to\infty} \sum_{k=1}^n f(x_k^*)\,\Delta x = c\int_a^b f(x)\,dx.$$

We get this for any points x_k^*, so $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$. ▨

Proof of ②. If f, g are integrable, then
$$\int_a^b (f(x)+g(x))\,dx = \lim_{n\to\infty} \sum_{k=1}^n [f(x_k^*)+g(x_k^*)]\,\Delta x = \lim_{n\to\infty} \sum_{k=1}^n f(x_k^*)\Delta x + g(x_k^*)\Delta x$$
$$= \lim_{n\to\infty} \sum f(x_k^*)\,\Delta x + \lim_{n\to\infty} \sum g(x_k^*)\,\Delta x$$
$$= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$ ▨

③ **Inequalities & integrals**. Suppose f, g integrable on $[a,b]$.



Suppose $f(x) \geq g(x)$ for all $a \leq x \leq b$.

Then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

Proof of ③ When $f(x) \geq 0$, $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^n f(x_k^*)\,\Delta x \geq 0$.

When $f(x) \geq g(x)$, $f(x) - g(x) \geq 0$ so $\int_a^b (f(x)-g(x))\,dx \geq 0$

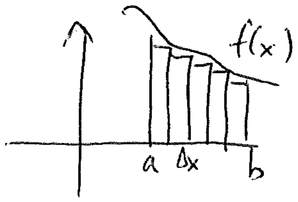
by above, so $\int_a^b f(x)\,dx + \int (-g(x))\,dx \geq 0$

so $\int_a^b f(x)\,dx - \int g(x)\,dx \geq 0$

so $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$. ▨

Recall that integrals are defined using Riemann sums

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x ,$$



where x_k^* is an arbitrary point from $[x_{k-1}, x_k]$ and the points $x_0 < x_1 < \dots < x_n$ subdivide $[a,b]$ into n equal parts.

Fundamental theorem of calculus, part 1

Suppose F is differentiable and $F' = f$ is continuous. Then $\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = F(b) - F(a)$.

Proof. We compute $\int_a^b F'(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} F'(x_k^*)\,\Delta x$ using the MVT which gives $F'(x_k^*) = \dfrac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}} = \dfrac{F(x_k) - F(x_{k-1})}{\Delta x}$.

This gives

$$\int_a^b F'(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \left[F(x_k) - F(x_{k-1})\right] \quad \text{.... one term, minus previous term (telescoping)}$$

$$= \lim_{n\to\infty} \Big[F(x_n) \cancel{- F(x_{n-1})} \cancel{+ F(x_{n-1})} \cancel{- F(x_{n-2})} \cancel{+ F(x_{n-2})} - F(x_{n-3}) + \dots + F(x_2) - F(x_1) + F(x_1) - F(x_0) \Big]$$

All terms get to cancel except for $F(x_n) - F(x_0) = F(b) - F(a)$. ◼

Example 1. To compute $\int_0^1 x^2\,dx$, we express x^2 as the derivative of a function, namely $x^2 = \left(\frac{x^3}{3}\right)'$, to get

$$\int_0^1 x^2\,dx = \int_0^1 \left(\frac{x^3}{3}\right)' dx = \frac{1^3}{3} - \frac{0^3}{3} = 1/3 .$$

Notation We sometimes write $\int_a^b F'(x)\,dx = F(b) - F(a) = \left[F(x)\right]_a^b$.

We call an antiderivative of f a function whose derivative is f.

Thus F = antiderivative of f means $F' = f$ and then

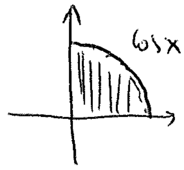
$$\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = \Big[F(x)\Big]_a^b.$$

We could also write $\int f(x)\,dx = F(x)$ .... indefinite integral

$\int_a^b f(x)\,dx$ = definite integral = number.

Example 2. To compute $\int_0^{\pi/2} \cos x\,dx$, we write

$$\int_0^{\pi/2} (\sin x)'\,dx = \Big[\sin x\Big]_0^{\pi/2} = \sin \pi/2 - \sin 0 = 1.$$

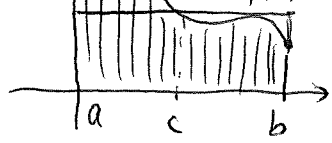


Antiderivatives of standard functions

- $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$, if $n \neq -1$
- $\int \frac{1}{x}\,dx = \ln|x| + C$
- $\int \cos x\,dx = \sin x + C$
- $\int \sin x\,dx = -\cos x + C$
- $\int \sec^2 x\,dx = \tan x + C$
- $\int \csc^2 x\,dx = -\cot x + C$
- $\int \sec x \tan x\,dx = \sec x + C$
- $\int \csc x \cot x\,dx = -\csc x + C$
- $\int \dfrac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$
- $\int \dfrac{1}{1+x^2}\,dx = \tan^{-1} x + C.$
- $\int e^{ax}\,dx = \frac{1}{a} e^{ax} + C$, if $a \neq 0$.

Mean value theorem for integrals

Suppose $f(x)$ is continuous on $[a,b]$. The mean value of f is defined as $\dfrac{1}{b-a}\int_a^b f(x)\,dx$. That's the height of the rectangle that gives the same area. We have



$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx \quad \text{at some point } a<c<b.$$

Proof. Let m be the min value, M be the max value.

Then $m \le f(x) \le M$ for all $a \le x \le b$.

So $\int_a^b m \, dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx$

$\Rightarrow \quad m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

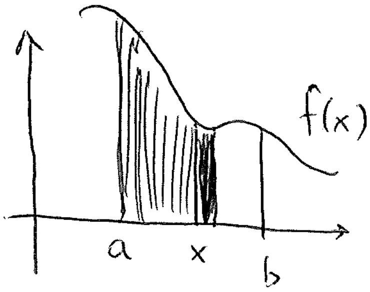
$\Rightarrow \quad m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M$.

We get the result using the intermediate value theorem.

Namely, m & M are attained $\Rightarrow$ any value $m \le \alpha \le M$ is attained. ▨

Fundamental theorem of calculus, part 2

Suppose f is continuous on $[a,b]$.
Then $\int_a^x f(t)\,dt$ is a function of x
which satisfies $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.



For instance, $\int_0^x \cos t\,dt = [\sin t]_0^x = \sin x - \cancel{\sin 0}$

So $\frac{d}{dx}\int_0^x \cos t\,dt = \cos x$.

Proof. We need to compute $G'(x_0)$ when $G(x) = \int_a^x f(t)\,dt$.

Using the limit definition of derivatives, we get

$$G'(x_0) = \lim_{x\to x_0} \frac{G(x)-G(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{\int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt}{x-x_0}$$

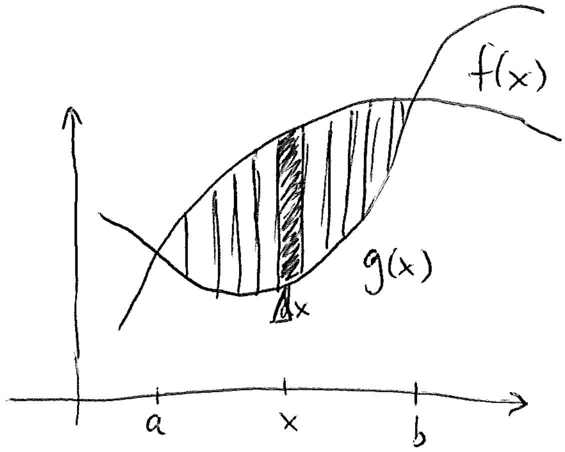
$$= \lim_{x\to x_0} \frac{\cancel{\int_a^{x_0} f(t)\,dt} + \int_{x_0}^x f(t)\,dt - \cancel{\int_a^{x_0} f(t)\,dt}}{x-x_0}$$

$$= \lim_{x\to x_0} \text{Average value of } f \text{ on } [x_0, x].$$

Since $x \to x_0$, the average value approaches $f(x_0)$. ▨

Applications of integrals

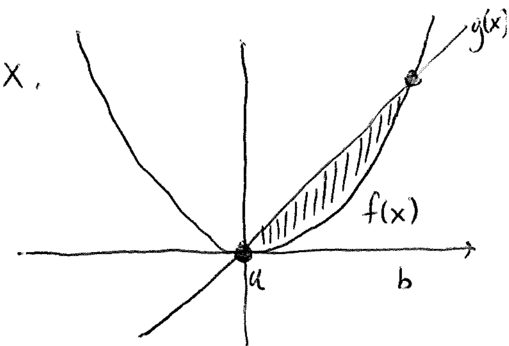
① Area between two graphs



Suppose $f(x) \geq g(x)$ for all $a \leq x \leq b$.
We can compute the area between the two graphs as

$$\lim_{n \to \infty} \sum_{k=1}^{n} \underbrace{[f(x)-g(x)]}_{\text{height}} \underbrace{\Delta x}_{\text{base}}, \quad \text{so}$$

$$\boxed{\text{Area} = \int_a^b [f(x)-g(x)]\,dx.}$$



__Example__. Consider $f(x) = x^2$ and $g(x) = 3x$.
The graphs meet when $x^2 = 3x$

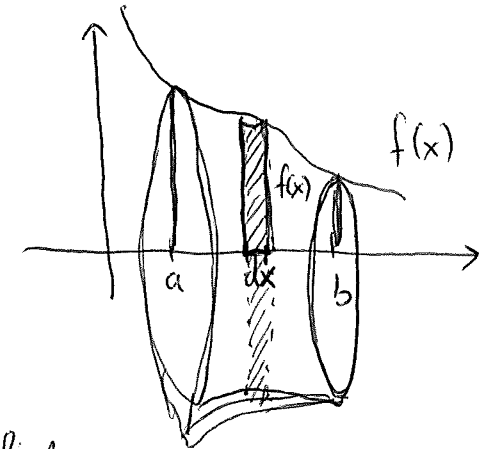
$$x^2 - 3x = 0$$
$$x(x-3) = 0 \quad \text{or} \quad x = 0, 3.$$

The area is $\int_0^3 [g(x)-f(x)]\,dx$ since line $g(x)$ lies on top.

Thus, $\text{Area} = \int_0^3 (3x - x^2)\,dx = \int_0^3 \left(\frac{3x^2}{2} - \frac{x^3}{3}\right)' dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3$

$\Rightarrow \text{Area} = \frac{27}{2} - \frac{27}{3} = \frac{27}{6} = \frac{9}{2}.$

② Volume of a solid of revolution

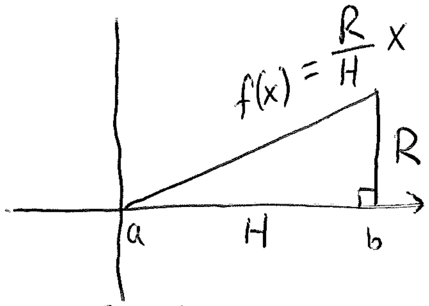


Consider the graph of $f(x)$ on $[a,b]$.
Rotation around the x-axis gives rise to a solid. To compute its volume, we can argue that

volume = sum of volumes of cylinders
= sum of $\pi \cdot \text{radius}^2 \cdot \text{height}$.

This gives $\boxed{\text{volume} = \int_a^b \pi \cdot f(x)^2 \cdot dx}$

Example. We compute the volume of a cone of height H and radius R. We can obtain this cone by rotating a triangle. In this case, $f(x) = \frac{R}{H}x$

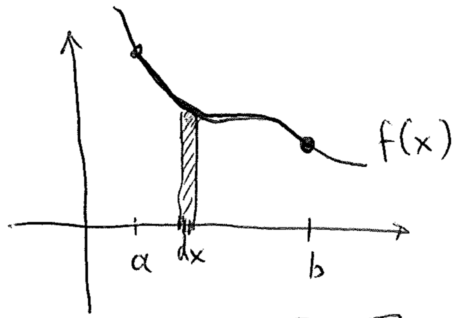


so volume $= \displaystyle\int_a^b \pi f(x)^2\, dx$

$$= \int_0^H \pi \left(\frac{Rx}{H}\right)^2 dx = \int_0^H \frac{\pi R^2}{H^2} x^2\, dx$$

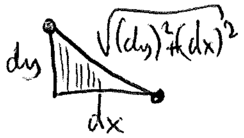
$$= \int_0^H \left(\frac{\pi R^2}{H^2}\frac{x^3}{3}\right)' dx = \frac{\pi R^2}{H^2}\cdot\frac{H^3}{3} = \frac{1}{3}\pi R^2 H.$$

3 Arc length of a graph



Consider the graph of $f(x)$ when $a \leq x \leq b$.

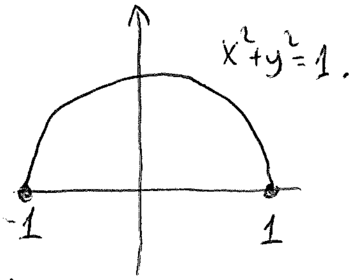
To compute its length, we look at a small subinterval and compute the hypotenuse



$$\sqrt{(dx)^2+(dy)^2} = \sqrt{(dx)^2 + f'(x)^2(dx)^2} = \sqrt{1+f'(x)^2}\, dx.$$

$dy = f'(x)\,dx$

This gives $\boxed{\text{length} = \int_a^b \sqrt{1+f'(x)^2}\, dx}$



Example. We compute the circumference of a circle. Consider the unit circle (of radius 1). We have

$$x^2+y^2=1 \Rightarrow y^2 = 1-x^2 \Rightarrow y = \sqrt{1-x^2}$$ for the upper semicircle.

The length of this upper part is: $\displaystyle\int_{-1}^1 \sqrt{1+f'(x)^2}\, dx$ with $f(x) = y = \sqrt{1-x^2}$.

We get $f(x) = \sqrt{1-x^2} \Rightarrow f'(x) = \dfrac{1}{\cancel{2}\sqrt{1-x^2}}\cdot(-\cancel{2}x) = -\dfrac{x}{\sqrt{1-x^2}}$

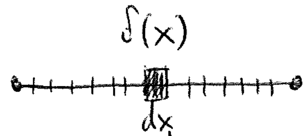
so $f'(x)^2 = \dfrac{x^2}{1-x^2}$ so $1+f'(x)^2 = 1+\dfrac{x^2}{1-x^2} = \dfrac{\cancel{x^2}\,1}{1-x^2}$.

Arc length $= \int_{-1}^{1} \sqrt{1+f'(x)^2}\,dx = \int_{-1}^{1} \sqrt{\frac{1}{1-x^2}}\,dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx$.

We know $(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$ so we get

$$\int_{-1}^{1} (\sin^{-1} x)'\,dx = \left[\sin^{-1} x\right]_{-1}^{1} = \sin^{-1} 1 - \sin^{-1}(-1) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$$

(4) Mass of one-dimensional objects (rods)



We define density $= \frac{\text{mass}}{\text{volume}} = \frac{\text{mass}}{\text{length}}$ for one-dimensional objects.

If a rod has density $\delta(x)$ at the point x, then its mass is

mass = sum of masses of small parts

= sum of density × length so $\boxed{M = \int_a^b \delta(x)\,dx.}$

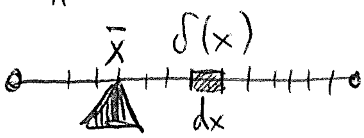
__Example.__ Suppose $\delta(x) = \sin(\pi x)$ with $0 \leq x \leq 1$.

Then mass $= \int_0^1 \delta(x)\,dx = \int_0^1 \sin(\pi x)\,dx$

$$= \int_0^1 \left(-\frac{\cos(\pi x)}{\pi}\right)'\,dx = \left[-\frac{\cos(\pi x)}{\pi}\right]_0^1$$

$$= \frac{-\cos\pi}{\pi} + \frac{\cos 0}{\pi} = 2/\pi.$$

(5) Centre of mass for a rod.



We wish to compute the centre of mass $\bar{x}$.

If the density is $\delta(x)$ and we have a small piece at the point x with length dx, the corresponding torque is

Torque = force × displacement

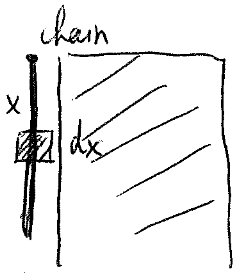
= mass × acceleration × displacement

$$= \underbrace{\delta(x)\,dx}\cdot g\ (x-\bar{x})$$

We need the sum to be zero, so $\int (x-\bar{x})\,\delta(x)\,dx = 0$.

This gives $\int x\,\delta(x)\,dx = \bar{x}\int \delta(x)\,dx$, or $\boxed{\bar{x} = \frac{\int x\,\delta(x)\,dx}{\int \delta(x)\,dx}}$.

$\boxed{\text{Work}}$ In physics, the work produced by a force F while displacing an object by d units (in the direction of F) is defined as $W = F \cdot d$ when F, d are constant. In general, when $F(x)$, $d(x)$ are depending on x, the work is $W = \int F(x) \cdot d(x) \, dx$.



Example. A chain is hanging from the top of a building. Assume it is 5m long and weighs 4kg/m. How much work is needed to pull it up to the top?

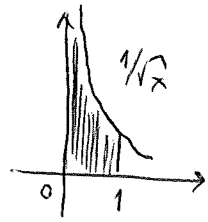
Consider a small part of the chain x m from the top of length dx. Its displacement is x, while the force needed to pull it up is its weight mg = density $\times$ length $\times$ g = $4g\,dx$ and

$$\text{work} = \int mg\,x = \int_0^5 4gx\,dx = \left[\frac{4gx^2}{2}\right]_0^5 = 50g.$$

$\boxed{\text{Improper integrals}}$ Those are integrals involving infinite values of x or y. They are defined as limits. For instance, one has

$$\int_0^\infty f(x)\,dx = \lim_{N\to\infty} \int_0^N f(x)\,dx.$$

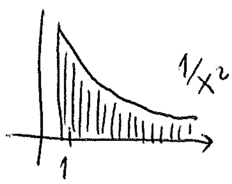
<u>Example 1.</u> $\int_0^1 \frac{1}{\sqrt{x}}\,dx$. This is improper due to $x=0$.



We have $$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0} \int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0} \int_a^1 x^{-1/2}\,dx$$

$$= \lim_{a\to 0} \left[\frac{x^{1/2}}{1/2}\right]_a^1 = \lim_{a\to 0} \frac{\sqrt{1}-\sqrt{a}}{1/2} = 2.$$

<u>Example 2.</u> $\int_1^\infty \frac{1}{x^2}\,dx$. This is



$$\lim_{N\to\infty} \int_1^N x^{-2}\,dx = \lim_{N\to\infty} \left[\frac{x^{-1}}{-1}\right]_1^N = \lim_{N\to\infty} \left(-\frac{1}{N}+1\right) = 1.$$

<u>Example 3.</u> $$\int_0^\infty e^{-3x}\,dx = \lim_{N\to\infty} \int_0^N e^{-3x}\,dx = \lim_{N\to\infty} \left[\frac{e^{-3x}}{-3}\right]_0^N$$

$$= \lim_{N\to\infty} \left(-\frac{e^{-3N}}{3} + \frac{1}{3}\right) = \frac{1}{3}.$$

$\boxed{\text{Integration by parts}}$ According to the product rule,

$$[f(x)\cdot g(x)]' = f'(x)\,g(x) + f(x)\,g'(x)$$

so $$f(x)\,g(x) = \int f'(x)\,g(x)\,dx + \int f(x)\,g'(x)$$

so $$\int \underline{f(x)\,g'(x)\,dx} = f(x)\,g(x) - \int f'(x)\,g(x)\,dx.$$

We sometimes write this as $u = f(x)$, $\frac{du}{dx} = f'(x)$, $du = f'(x)\,dx$ and $v = g(x)$, $\frac{dv}{dx} = g'(x)$, $dv = g'(x)\,dx$

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

- This formula relates two integrals and we hope the latter is simpler. Some typical choices of u are polynomials (whose derivative du is simpler) or logarithmic (for similar reasons).
- Typical examples ... $\int p_n(x)\cdot \sin(ax)\,dx$, $\int p_n(x)\cdot\cos(ax)\,dx$, $\int p_n(x)\cdot e^{ax}\,dx$ can be computed using $u = p_n(x)$.

Example 1. Consider $\int \underline{x}\,\underline{e^x\,dx}$. Let $u = x$, $dv = e^x\,dx$, $du = dx$, $v = e^x$

We get $\int xe^x\,dx = uv - \int v\,du = xe^x - \int e^x\,dx = xe^x - e^x + C.$

Example 2. Consider $\int \underline{x^2}\,\underline{\cos x\,dx}$. Let $u = x^2$, $dv = \cos x\,dx$, $du = 2x\,dx$, $v = \sin x$

We get $\int x^2\cos x\,dx = x^2\sin x - \int \underline{2x}\,\underline{\sin x\,dx}$. Now take $u = 2x$, $dv = \sin x\,dx$, $du = 2dx$, $v = -\cos x$

Thus $\int 2x\sin x\,dx = -2x\cos x + \int 2\cos x\,dx = -2x\cos x + 2\sin x$

and $\int x^2\cos x\,dx = x^2\sin x + 2x\cos x - 2\sin x + C.$

Example 3. Consider $\int x^n \ln x \, dx$ with $n \neq -1$.

We let $u = \ln x$, $dv = x^n dx$

$du = \frac{1}{x} dx$, $v = \frac{x^{n+1}}{n+1}$. This gives

$$\int x^n \ln x \, dx = \frac{x^{n+1}}{n+1} \ln x - \int \frac{x^{n+1}}{n+1} \frac{1}{x} dx$$

$$= \frac{x^{n+1} \ln x}{n+1} - \frac{1}{n+1} \int x^n dx$$

$$= \frac{x^{n+1} \ln x}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C.$$

Example 4. Consider $\int e^{ax} \cdot \cos(bx) \, dx$ with $b \neq 0$.

One can take $u = e^{ax}$ or $u = \cos(bx)$. Let's take

$u = e^{ax}$, $dv = \cos(bx) \, dx$

$du = a e^{ax} dx$, $v = \frac{1}{b} \sin(bx)$.

We get $\int e^{ax} \cos(bx) \, dx = \frac{e^{ax} \sin(bx)}{b} - \frac{a}{b} \int e^{ax} \sin(bx) \, dx$. (*)

We integrate by parts once again... $u = e^{ax}$, $dv = \sin(bx) \, dx$

$du = a e^{ax} dx$, $v = -\frac{1}{b} \cos(bx)$

We get $\int e^{ax} \sin(bx) \, dx = -\frac{e^{ax} \cos(bx)}{b} + \frac{a}{b} \int e^{ax} \cos(bx) \, dx$. (**)

Combining (*) and (**) gives

$$\int e^{ax} \cos(bx) \, dx = \frac{e^{ax} \sin(bx)}{b} - \frac{a}{b} \left(- \frac{e^{ax} \cos(bx)}{b} + \frac{a}{b} \int e^{ax} \cos(bx) \, dx \right)$$

$$\Rightarrow \int e^{ax} \cos(bx) \, dx = \frac{e^{ax} \sin(bx)}{b} + \frac{a}{b^2} e^{ax} \cos(bx) - \frac{a^2}{b^2} \int e^{ax} \cos(bx) \, dx$$

$$\Rightarrow \left(1 + \frac{a^2}{b^2}\right) \int e^{ax} \cos(bx) \, dx = \frac{b e^{ax} \sin(bx) + a e^{ax} \cos(bx)}{b^2}$$

$$\Rightarrow \int e^{ax} \cos(bx) \, dx = \frac{b e^{ax} \sin(bx) + a e^{ax} \cos(bx)}{a^2 + b^2}.$$

$\boxed{\text{Integration by substitution}}$ This is based on the formula

$$\int g(f(x)) \cdot \underline{f'(x)\, dx} = \int g(u)\, \underline{du}$$

where $u = f(x)$ and $du = f'(x)\,dx$. It is an intermediate step for computing the x-integral in terms of the u-integral. Note that everything needs to be expressed in terms of u.

$u = 3x+1$
$du = 3\,dx$
$\frac{1}{3} du = dx$

(1) $\displaystyle\int \underline{\cos(3x+1)}\, dx = \int \underline{\cos u}\, \frac{1}{3}\, du$

$\displaystyle = \frac{1}{3} \sin u + C = \frac{1}{3} \sin(3x+1) + C.$

$u = x^2+1$
$du = 2x\, dx$

(2) $\displaystyle\int \frac{2x}{x^2+1}\, dx = \int \frac{du}{u}$

$= \ln|u| + C = \ln(x^2+1) + C.$

$u = e^x$
$du = e^x dx$

(3) $\displaystyle\int \frac{e^x}{1+e^{2x}}\, dx = \int \frac{du}{1+u^2} = \tan^{-1} u + C$

$= \tan^{-1} e^x + C$

Another approach would be ... $u = 1+e^{2x}$ ... $\displaystyle\int \frac{du}{2u\sqrt{u-1}}$

$du = 2e^{2x}\, dx$

$du = \underline{2e^x}\, \underline{e^x\, dx}$

$\displaystyle e^x dx = \frac{du}{2e^x} = \frac{du}{2\sqrt{u-1}}$

This seems more complicated.

(4) $\displaystyle\int \frac{x^2}{(x+1)^3}\, dx$, Take $u = x+1$, $du = dx$ so that

$$\int \frac{(u-1)^2}{u^3}\, du = \int \frac{u^2-2u+1}{u^3}\, du = \int\left(\frac{1}{u} - \frac{2}{u^2} + \frac{1}{u^3}\right) du$$

$$= \ln|u| + \frac{2}{u} - \frac{1}{2u^2} + C$$

$$= \ln|x+1| + \frac{2}{x+1} - \frac{1}{2(x+1)^2} + C.$$

⑤ $\displaystyle\int \frac{\ln x}{x}\,dx = \int u\,du = \frac{1}{2}u^2 + C$

$\displaystyle = \frac{1}{2}(\ln x)^2 + C.$

$u = \ln x$
$du = \frac{1}{x}dx$

⑥ $\displaystyle\int \sin(\sqrt{x})\,dx = \int \sin u \cdot 2u\,du$

$\displaystyle = \int 2u \sin u\,du.$

$u = \sqrt{x}$
$u^2 = x$
$2u\,du = dx$

We can now use integration by parts. Take $u = u$, $dv = 2\sin u\,du$
$du = du$, $v = -2\cos u$

Then $\displaystyle\int 2u\sin u\,du = -2u\cos u + 2\int \cos u\,du$

$= -2u\cos u + 2\sin u + C$

$= -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C.$

⑦ $\displaystyle\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{du}{u}$

$= -\ln|u| + C = -\ln|\cos x| + C$

$u = \cos x$
$du = -\sin x\,dx$

⑧ $\int \tan^{-1}x\,dx$ or $\int \sin^{-1}x\,dx$ or $\int \ln x\,dx$.

Those can be computed using integration by parts.

~~We let $u = \tan^{-1}$~~

$\displaystyle\int \tan^{-1}x\,dx = x\tan^{-1}x - \int \frac{x\,dx}{1+x^2}$ Take $u = \tan^{-1}x$, $dv = dx$

$du = \frac{1}{1+x^2}dx$, $v = x$

$\displaystyle = x\tan^{-1}x - \frac{1}{2}\int \frac{dw}{w}$ Take $w = 1+x^2$

$\frac{1}{2}dw = \cancel{2}x\,dx$

$\displaystyle = x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C.$

Similarly, $\displaystyle\int \sin^{-1}x\,dx = x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx$

$z = 1-x^2$
$dz = -2x\,dx$

$\displaystyle = x\sin^{-1}x + \frac{1}{2}\int \frac{dz}{\sqrt{z}}$

$= x\sin^{-1}x + \sqrt{z} + C = x\sin^{-1}x + \sqrt{1-x^2} + C.$

$\boxed{\text{Integrating } \sin^m x \cdot \cos^n x \text{ for any integers } m, n \geq 0}$

• The idea is to use the formulas $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $\sin^2 x + \cos^2 x = 1$.

Case 1. The substitution $u = \sin x$ helps when n is odd

This is because powers of sine become powers of u

and even powers of cosine become $\cos^{2k} x = (\cos^2 x)^k = (1-\sin^2 x)^k$
$= (1-u^2)^k$.

However, we also need one extra copy of cosine because of $du = \cos x \, dx$. This works for odd number of cosines.

Example 1. $\int \sin^4 x \cdot \cos x \, dx = \int u^4 \, du$

$u = \sin x$
$du = \cos x \, dx$

$= \frac{u^5}{5} + C = \frac{\sin^5 x}{5} + C.$

Example 2. $\int \sin^4 x \cdot \cos^3 x \, dx = \int \sin^4 x \cdot \cos^2 x \; \cos x \, dx$

$= \int \sin^4 x \cdot (1-\sin^2 x) \cdot \cos x \, dx$

$= \int u^4 (1-u^2) \, du = \int (u^4 - u^6) \, du$

$= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C.$

Example 3. $\int \cos^5 x \, dx = \int \cos^4 x \cdot \cos x \, dx$

$= \int (\cos^2 x)^2 \cdot \cos x \, dx = \int (1-\sin^2 x)^2 \cos x \, dx$

$= \int (1-u^2)^2 \, du = \int (1 - 2u^2 + u^4) \, du$

$= \sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + C.$

Case 2. The substitution $u = \cos x$ helps when m is odd.

This is because $du = -\sin x \, dx$ and even powers of sine are fine.

Example 4. $\int \sin^3 x \cdot \cos^2 x \, dx = \int \sin^2 x \cos^2 x \; \sin x \, dx$

$u = \cos x$
$du = -\sin x \, dx$

$$= \int (1-\cos^2 x) \cdot \cos^2 x \cdot \underline{\sin x \, dx}$$

$$= \int (1-u^2)\, u^2 \cdot (-du)$$

$$= \int (-u^2 + u^4)\, du = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C.$$

Case 3. When m,n are both even, one needs to use half-angle formulas

More precisely, $\cos(2x) = \cos(x+x) = \cos^2 x - \sin^2 x$

So $\cos(2x) = 1 - \sin^2 x - \sin^2 x = 1 - 2\sin^2 x$

So $2\sin^2 x = 1 - \cos(2x)$ ....

So $\cos^2 x = 1 - \sin^2 x = 1 - \frac{1-\cos(2x)}{2}$

$$\boxed{\sin^2 x = \frac{1-\cos(2x)}{2}}$$

$$\boxed{\cos^2 x = \frac{1+\cos(2x)}{2}}$$

These formulas can be used to halve the powers of sine / cosine.

Example 5. $\int \sin^2 x \, dx = \int \frac{1-\cos(2x)}{2} dx = \int \left(\frac{1}{2} - \frac{\cos(2x)}{2}\right) dx$

$$= \frac{x}{2} - \frac{\sin(2x)}{4} + C.$$

Example 6. $\int \sin^2 x \cdot \cos^2 x \, dx = \int \frac{1-\cos(2x)}{2} \cdot \frac{1+\cos(2x)}{2} dx$

$$= \frac{1}{4} \int (1 - \cos^2(2x))\, dx = \frac{x}{4} - \frac{1}{4}\int \cos^2(2x)\, dx$$

$$= \frac{x}{4} - \frac{1}{4}\int \frac{1+\cos(4x)}{2} dx = \frac{x}{4} - \frac{1}{8}\int (1+\cos(4x))\, dx$$

$$= \frac{x}{4} - \frac{x}{8} - \frac{1}{8}\,\frac{\sin(4x)}{4} = \frac{x}{8} - \frac{\sin(4x)}{32} + C.$$

$$\boxed{\text{Integrating } \sec^m x \cdot \tan^n x \text{ for any integers } m,n \geq 0}$$

The idea is to use $(\sec x)' = \sec x \tan x$,

$(\tan x)' = \sec^2 x$ and also

$\sin^2 x + \cos^2 x = 1 \Rightarrow \boxed{\tan^2 x + 1 = \sec^2 x}$

Thus, even powers of $\sec x$ are expressible in terms of $\tan x$.

Case 1. The substitution $u=\sec x$ helps when n is odd

Example 7. $\int \sec^2 x \cdot \tan^3 x \, dx = \int \sec x \tan^2 x \;\; \sec x \tan x \, dx$

$$= \int u\,(u^2-1)\,du = \int (u^3-u)\,du$$

$$= \frac{\sec^4 x}{4} - \frac{\sec^2 x}{2} + C.$$

$u=\sec x$
$du=\sec x \tan x \, dx$

Case 2. The substitution $u=\tan x$ helps when m is even

Namely, $du = \sec^2 x \, dx$ makes use of 2 copies of secant.

Example 8. $\int \sec^4 x \tan^3 x \, dx = \int \sec^2 x \cdot \tan^3 x \cdot \sec^2 x \, dx$

$u=\tan x$
$du=\sec^2 x \, dx$

$$= \int (1+\tan^2 x) \tan^3 x \cdot \sec^2 x \, dx$$

$$= \int (1+u^2)\,u^3 \, du = \int (u^3+u^5)\,du$$

$$= \frac{\tan^4 x}{4} + \frac{\tan^6 x}{6} + C.$$

Case 3. When n is even and m is odd, we can express as $\sec^k x$.

Namely, $\sec^m x \cdot \tan^n x = \sec^{2l+1} x \cdot \tan^{2k} x$

$$= \sec^{2l+1} x \cdot (\tan^2 x)^k$$

$$= \sec^{2l+1} x \cdot (\sec^2 x - 1)^k$$ and this involves powers of secant alone.

It remains to integrate powers of secant. We'll do that using reduction formulas that relate ~~powers~~ integrals of $\sec^n x$ to integrals of smaller powers of $\sec x$.

Example Consider $I_n = \int x^n e^x \, dx$, an integration by parts.

We have $I_n = \int u\,dv = uv - \int v\,du$

$u=x^n$, $dv=e^x dx$
$du=nx^{n-1}dx$, $v=e^x$

$$= x^n e^x - n\int x^{n-1} e^x \, dx$$

which gives $I_n = x^n e^x - n \cdot I_{n-1}$ for any integer n.

Reduction formulas

Those are formulas that relate an integral I_n depending on some integer n to a similar integral I_m for some smaller $m<n$.
There are lots of such formulas for $\int (\ln x)^n\,dx$, $\int \sin^n x\,dx$, $\int \sec^n x\,dx$, $\int \tan^n x\,dx$, $\int x^n e^{ax}\,dx$, $\int \frac{dx}{(x^2+a^2)^n}$.

Example 1. Consider $I_n = \int \underbrace{(\ln x)^n}\,\underbrace{dx}$, for instance.

We integrate by parts ... $u=(\ln x)^n$, $dv=dx$
$du = n(\ln x)^{n-1}\cdot\frac{1}{x}\,dx$, $v=x$.

We get $I_n = \int u\,dv = uv - \int v\,du = x(\ln x)^n - \int n(\ln x)^{n-1}\cancel{\tfrac{1}{x}}\cancel{x}\,dx$

$$\therefore \quad \boxed{I_n = x(\ln x)^n - nI_{n-1}}.$$

We can use this to compute ... $I_0 = \int \cancel{(\ln x)^0}\,dx = x$

$$I_1 = x(\ln x)^1 - I_0 = x\ln x - x + C$$
$$I_2 = x(\ln x)^2 - 2I_1 = x(\ln x)^2 - 2x\ln x + 2x + C \quad \text{etc.}$$

Example 2. Consider $I_n = \int \sin^n x\,dx$.

We integrate by parts ..... $u = \sin^{n-1}x$ $dv = \sin x\,dx$
$du = (n-1)\sin^{n-2}x\cdot\cos x\,dx$, $v = -\cos x$

We get $I_n = \int u\,dv = uv - \int v\,du = -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,\underline{\cos^2 x}\,dx$

$$\Rightarrow \quad I_n = -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,(1-\sin^2 x)\,dx$$
$$\Rightarrow \quad I_n = -\sin^{n-1}x\cos x + (n-1)I_{n-2} + (1-n)\,\cancel{\int \sin^n x\,dx}\,I_n$$
$$\Rightarrow \quad \boxed{I_n = -\frac{\sin^{n-1}x\cos x}{n} + \frac{(n-1)}{n}I_{n-2}}$$

Thus, $I_0 = \int \cancel{(\sin^0 x)}^{1}\,dx = x$, $I_1 = \int \sin x\,dx = -\cos x$

and then $I_2 = -\frac{\sin x\cos x}{2} + \frac{1}{2}I_0 = -\frac{\sin x\cos x}{2} + \frac{x}{2} + C$

$$I_3 = -\frac{\sin^2 x\cos x}{3} + \frac{2}{3}I_1 = -\frac{\sin^2 x\cos x}{3} - \frac{2}{3}\cos x + C \quad \text{etc.}$$

Example 3. Consider $I_n = \int \sec^n x \, dx = \int \sec^{n-2} x \cdot \underbrace{\sec^2 x \, dx}$

We get $u = \sec^{n-2} x$, $dv = \sec^2 x \, dx$

$du = (n-2)\sec^{n-3} x \cdot \sec x \tan x \, dx$, $v = \tan x$ and so

$$I_n = \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \cdot \tan^2 x \, dx$$

$$\Rightarrow I_n = \sec^{n-2} x \tan x + (2-n)\int \sec^{n-2} x \cdot (\sec^2 x - 1)\, dx$$

$$\Rightarrow I_n = \sec^{n-2} x \tan x + (2-n) I_n + (n-2) I_{n-2}$$

$$\Rightarrow \boxed{I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}.}$$

We need $I_0 = \int dx = x$ and $I_1 = \int \sec x \, dx$. The latter is

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$ because

$$\left[\ln|\sec x + \tan x|\right]' = \frac{1}{\sec x + \tan x} \cdot (\sec x + \tan x)' = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x.$$

We'll derive this formula in a different way later.

$\boxed{\text{Trigonometric substitutions}}$ Those are used to simplify square roots $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$. To simplify the first, we write

$$\sqrt{a^2-x^2} = \sqrt{a^2\left(1-\frac{x^2}{a^2}\right)} = a\sqrt{1-\left(\frac{x}{a}\right)^2}$$ and take $\frac{x}{a} = \sin\theta$

so that $1-\left(\frac{x}{a}\right)^2 = 1-\sin^2\theta = \cos^2\theta$.

Namely, take $x = a\sin\theta$ so that $\theta = \sin^{-1}\frac{x}{a}$ and

$$\sqrt{a^2-x^2} = a\sqrt{1-\sin^2\theta} = a\cos\theta,$$ as long as $a>0$

and $-\pi/2 \le \theta \le \pi/2$.

Example 1. We compute $\int \frac{dx}{\sqrt{1-x^2}}$. If we take $x = \sin\theta$,

then $1-x^2 = 1-\sin^2\theta = \cos^2\theta$, so

$$\int \frac{\cancel{\cos\theta}\, d\theta}{\cancel{\cos\theta}} = \int d\theta = \theta + C = \sin^{-1} x + C.$$

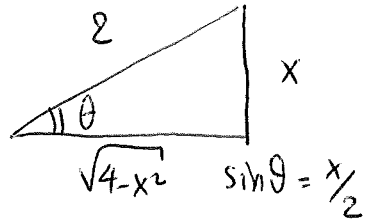
Example 2. We compute $\int \frac{dx}{x^2\sqrt{4-x^2}}$.

If we take $x = 2\sin\theta$, then $x^2 = 4\sin^2\theta$ and $\sqrt{4-x^2} = \sqrt{4\cos^2\theta} = 2\cos\theta$

so we get $$\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{\cancel{2\cos\theta}\, d\theta}{4\sin^2\theta \cdot \cancel{2\cos\theta}} = \frac{1}{4}\int \csc^2\theta \, d\theta .$$

Recall $\int \sec^2\theta \, d\theta = \tan\theta$ and $\int \csc^2\theta \, d\theta = -\cot\theta + C$, so we get
$$\frac{1}{4}\int \csc^2\theta \, d\theta = -\frac{1}{4}\cot\theta + C = -\frac{1}{4}\cot\left(\sin^{-1}\left(\frac{x}{2}\right)\right) + C.$$

To simplify this formula, we note that
$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{+\sqrt{4-x^2}}{x}$$ which gives



$$\int \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{1}{4}\cot\theta + C = -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C \quad \text{--- when } x \geq 0.$$

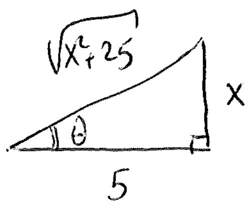
When $x \leq 0$, we still have $\cot(\sin^{-1}\frac{x}{2}) = \frac{\sqrt{4-x^2}}{x}$ because the two sides change by a sign when x becomes $-x$.

$\boxed{\text{Integrals involving } \sqrt{a^2+x^2}}$ In this case, we write

$$\sqrt{a^2+x^2} = \sqrt{a^2\left(1+\frac{x^2}{a^2}\right)} = a\sqrt{1+\left(\frac{x}{a}\right)^2}$$ and take $\frac{x}{a} = \tan\theta$ ($1+\tan^2\theta = \sec^2\theta$)

to get $\sqrt{a^2+x^2} = a\sqrt{1+\tan^2\theta} = a\sec\theta$, as long as $a > 0$ and $-\pi/2 < \theta \leq \pi/2$.

Example 3. We compute $\int \frac{dx}{\sqrt{x^2+25}}$.



If we take $\tan\theta = \frac{x}{5}$ or $x = 5\tan\theta$, then
$dx = 5\sec^2\theta \, d\theta$ and $\sqrt{x^2+5^2} = 5\sec\theta$.

We get $$\int \frac{dx}{\sqrt{x^2+25}} = \int \frac{\cancel{5}\sec^2\theta \, d\theta}{\cancel{5}\sec\theta} = \int \sec\theta \, d\theta$$
$$= \ln|\sec\theta + \tan\theta| + C$$

and we need to express this in terms of x.

When $x \geq 0$ ..... we get $\ln\left|\frac{\sqrt{x^2+25}}{5} + \frac{x}{5}\right| + C$.

This gives $\int \frac{dx}{\sqrt{x^2+25}} = \ln\left|\frac{\sqrt{x^2+25}}{5} + \frac{x}{5}\right| + C$ for any x.
(We may assume $x \geq 0$).

$\boxed{\text{Integrals involving } \sqrt{x^2-a^2}}$ In this case, we write

$\sqrt{x^2-a^2} = a\sqrt{\left(\frac{x}{a}\right)^2 - 1}$ and $\frac{x}{a} = \sec\theta$ to simplify. $\left(\begin{array}{c} 1+\tan^2\theta = \sec^2\theta \\ \tan^2\theta = \sec^2\theta - 1 \end{array}\right)$

We get $\sqrt{x^2-a^2} = a\sqrt{\sec^2\theta - 1} = a|\tan\theta|$ as long as $a > 0$
and $-\pi/2 < \theta < \pi/2$.

In short .... $\sqrt{a^2-x^2}$ .... $x = a\sin\theta$ .... get $\sqrt{a^2-x^2} = a\cos\theta$
.... $\sqrt{a^2+x^2}$ .... $x = a\tan\theta$ .... get $\sqrt{a^2+x^2} = a\sec\theta$
.... $\sqrt{x^2-a^2}$ .... $x = a\sec\theta$ .... get $\sqrt{x^2-a^2} = a|\tan\theta|$.

Integration of rational functions

There is a standard approach for integrating $\frac{P(x)}{Q(x)}$ with P, Q polynomials.
It depends on the factorisation of the denominator $Q(x)$.

Consider $\frac{1}{x-3} + \frac{2}{x-2} = \frac{x-2+2x-6}{(x-3)(x-2)} = \frac{3x-8}{(x-3)(x-2)}$.

To integrate the fraction on the right, we decompose it ~~to~~ as a sum of two fractions that are easier to integrate. Here

$$\int \frac{3x-8}{(x-3)(x-2)}\,dx = \int \frac{dx}{x-3} + \int \frac{2\,dx}{x-2}$$

$$= \ln|x-3| + 2\ln|x-2| + C.$$

We have to decompose general fractions into simpler fractions as above.

Partial fractions decomposition Consider a rational function $\frac{P(x)}{Q(x)}$ with $\deg P(x) < \deg Q(x)$. If $Q(x) = f_1(x) f_2(x) \cdots f_n(x)$ is the product of polynomials $f_1, \ldots, f_n$ (that have no factor in common), then one may decompose

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{f_1(x) \cdots f_n(x)} = \sum_{i=1}^{n} \frac{g_i(x)}{f_i(x)}$$ with $\deg g_i < \deg f_i$ for each i.

Example. Consider $\frac{3x-5}{x^2-4x+3} = \frac{3x-5}{(x-1)(x-3)}$.

In this case, $\frac{3x-5}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3}$ for some A, B.

We clear denominators ... $3x-5 = A(x-3) + B(x-1)$.

This is an identity that holds for all x.

- Taking $x=3$ gives $9-5 = 2B \Rightarrow B = 2$

  Taking $x=1$ gives $3-5 = A(-2) \Rightarrow A = 1$.

  Thus $\frac{3x-5}{(x-1)(x-3)} = \frac{1}{x-1} + \frac{2}{x-3}$ and then

$$\int \frac{3x-5}{(x-1)(x-3)}\,dx = \ln|x-1| + 2\ln|x-3| + C.$$

- Alternatively, one could compare coefficients $\begin{cases} A+B=3 & \text{--- coeff of } x \\ -3A-B=-5 & \text{--- const. coeff} \end{cases}$

  Solving this system gives $A=1$, $B=2$ as before.

Example. We compute $\int \frac{x^2+4}{x^3-x}\,dx = \int \frac{x^2+4}{x(x^2-1)}\,dx = \int \frac{x^2+4}{x(x-1)(x+1)}\,dx$.

Partial fractions decomposition ... $\frac{x^2+4}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$

for some constants A, B, C we need to determine. Then

$$x^2+4 = A(x+1)(x-1) + Bx(x+1) + Cx(x-1) \quad \circledast$$

for all x. We pick some values for x ...

$\boxed{x=0}$ ... $4 = -A + 0 + 0 \Rightarrow A = -4$

$\boxed{x=1}$ ... $5 = 2B \Rightarrow B = 5/2$

$\boxed{x=-1}$ ... $5 = 2C \Rightarrow C = 5/2$

This gives
$$\frac{x^2+4}{x(x-1)(x+1)} = \frac{-4}{x} + \frac{5/2}{x-1} + \frac{5/2}{x+1}$$

$$\Rightarrow \int \frac{x^2+4}{x(x-1)(x+1)}\,dx = -4\ln|x| + \frac{5}{2}\ln|x-1| + \frac{5}{2}\ln|x+1| + C.$$

Example. Consider $\int \frac{3x-5}{x^3+x^2} = \int \frac{3x-5}{x^2(x+1)}$

We get
$$\frac{3x-5}{x^2(x+1)} = \frac{Ax+B}{x^2} + \frac{C}{x+1}$$
using partial fractions.

Thus $3x-5 = (Ax+B)(x+1) + Cx^2$ for some A, B, C.

We take

$\boxed{x=0}$ ... $-5 = B$

$\boxed{x=-1}$ ... $-8 = C$

and $\boxed{x=1}$ ... $-2 = 2(A+B) + C = 2A - 10 - 8$

$2A = 16 \Rightarrow A = 8$.

Then
$$\int \frac{3x-5}{x^2(x+1)}\,dx = \int \frac{8x-5}{x^2}\,dx + \int \frac{-8}{x+1}\,dx$$
$$= \int \frac{8}{x}\,dx - \int \frac{5}{x^2}\,dx - \int \frac{8}{x+1}\,dx$$
$$= 8\ln|x| + \frac{5}{x} - 8\ln|x+1| + C.$$

Partial fractions decomposition

Suppose $\frac{P(x)}{Q(x)}$ is a proper rational function, namely one with $\deg P < \deg Q$. Suppose also $Q(x) = f_1(x) \cdots f_k(x)$ with the polynomials f_i having no (non-constant) factor in common. Then

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{f_1(x)\cdots f_k(x)} = \sum_{i=1}^{k} \frac{g_i(x)}{f_i(x)}$$

with $g_i(x)$ polynomials and $\deg g_i < \deg f_i$.

$\boxed{\text{Note \#1.}}$ When $\frac{P(x)}{Q(x)}$ is not proper, one can use division to write

$$\frac{P(x)}{Q(x)} = \text{polynomial} + \text{proper rational function}.$$

Example. Consider $\int \frac{x^3 - x^2 + 2}{x^2 - 1}\,dx$. This is improper! We use division with remainder to get

$$\begin{array}{r|l} & x-1 \\ \hline x^2-1 & x^3-x^2+2 \\ & \underline{x^3-x} \\ & -x^2+x+2 \\ & \underline{-x^2+1} \\ & x+1 \end{array}$$

$$x^3 - x^2 + 2 = (x-1)(x^2-1) + \underbrace{x+1}_{\text{remainder}}$$

$$\Rightarrow \frac{x^3 - x^2 + 2}{x^2 - 1} = \underbrace{x-1}_{\text{quotient}} + \frac{x+1}{x^2-1} \leftarrow \text{proper rational function.}$$

$$\Rightarrow \int \frac{x^3-x^2+2}{x^2-1}\,dx = \frac{x^2}{2} - x + \int \frac{x+1}{x^2-1}\,dx$$

$$= \frac{x^2}{2} - x + \int \frac{dx}{x-1} = \frac{x^2}{2} - x + \ln|x-1| + C.$$

$\boxed{\text{Note \#2.}}$ The assumption about non-constant factors means we cannot separate like terms. Those terms are treated together as in

$$\frac{x+1}{x^2(x-1)} = \frac{Ax+B}{x^2} + \frac{C}{x-1}$$

or

$$\frac{3x^2 - 4x + 5}{(x-1)^2(x+2)^2} = \frac{Ax+B}{(x-1)^2} + \frac{Cx+D}{(x+2)^2}.$$

Namely, we cannot expect to have $\frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x} + \frac{C}{x-1}$ because this simplifies to $\frac{A+B}{x} + \frac{C}{x-1} = \frac{(A+B)(x-1) + Cx}{x(x-1)}$ and the repeated factor is lost.

Example 1. Consider $\displaystyle\int \frac{x^2+4}{x^3-x}\,dx = \int \frac{x^2+4}{x(x-1)(x+1)}\,dx$

We get $\displaystyle\frac{x^2+4}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$ for some A, B, C

$$\Rightarrow \quad x^2+4 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

Take $x=0$ ... $4 = -A$ ... $A = -4$

$x=-1$ ... $5 = +2C$ ... $C = +5/2$

$x=1$ ... $5 = 2B$ ... $B = 5/2$

We get $\displaystyle\int \frac{x^2+4}{x(x-1)(x+1)}\,dx = \int\left(\frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}\right)dx$

$$= -4\ln|x| + \frac{5}{2}\ln|x-1| + \frac{5}{2}\ln|x+1| + K.$$

Example 2. Consider $\displaystyle\int \frac{3x+1}{(x-1)^2(x+2)}\,dx$. Partial fractions give

$$\frac{3x+1}{(x-1)^2(x+2)} = \frac{Ax+B}{(x-1)^2} + \frac{C}{x+2} \quad \text{for some } A, B, C$$

$$\Rightarrow \quad 3x+1 = (Ax+B)(x+2) + C(x-1)^2$$

Take $x=-2$ ..... $-6+1 = 9C$ .... $C = -5/9$

$x=0$ .... $1 = 2B + C$ ... $B = \frac{1-C}{2} = 7/9$

$x=1$ .... $4 = 3(A+B) \Rightarrow A = \frac{4}{3} - B = \frac{4}{3} - 7/9 = 5/9$.

This gives

$$\int \frac{3x+1}{(x-1)^2(x+2)}\,dx = \int \frac{\frac{5}{9}x + \frac{7}{9}}{(x-1)^2}\,dx + \int \frac{-5/9}{x+2}\,dx$$

$u = x-1$

$$= \int \frac{\frac{5}{9}(u+1) + \frac{7}{9}}{u^2}\,du - \frac{5}{9}\ln|x+2| + K$$

$$= \frac{5}{9}\int \frac{du}{u} + \frac{4}{3}\int \frac{du}{u^2} - \frac{5}{9}\ln|x+2| + K$$

$$= \frac{5}{9}\ln|x-1| - \frac{4}{3(x-1)} - \frac{5}{9}\ln|x+2| + K.$$

More generally, one can integrate $\frac{P(x)}{(x-x_0)^k}$ using the substitution $u = x - x_0$ to get

$$\int \frac{P(x)}{(x-x_0)^k}\,dx = \int \frac{P(u+x_0)}{u^k}\,du$$

and then expand the polynomial to obtain powers of u alone. This settles <u>any</u> partial fraction that corresponds to a linear factor $(x-x_0)^k$.

<u>Example 3.</u> Consider $\int \frac{x^4-2x+3}{x^3+x}\,dx$, an improper fraction.

We cannot use partial fractions directly!!!

$$\begin{array}{r|l} & x \\ \hline x^3+x & x^4-2x+3 \\ & x^4+x^2 \\ \hline & -x^2-2x+3 \end{array}$$

Division gives $x^4-2x+3 = x(x^3+x) \underbrace{-x^2-2x+3}_{\text{remainder}}$

$$\Rightarrow \frac{x^4-2x+3}{x^3+x} = x - \frac{x^2+2x-3}{x^3+x}.$$

We have a <u>proper</u> rational function now. We get

$$\frac{x^2+2x-3}{x^3+x} = \frac{x^2+2x-3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} \quad \text{for some } A,B,C$$

$$\Rightarrow \quad x^2+2x-3 = A(x^2+1) + x(Bx+C)$$

Take $x=0$ ..... $-3 = A$

$x=1$ ..... $0 = 2A+B+C$ ..... $B+C=6$

$x=-1$ ..... $-4 = 2A+B-C$ ..... $B-C=2$

Thus $2B=8$, $B=4$, $C=2$ and we conclude that

$$\int \frac{x^2+2x-3}{x^3+x}\,dx = -\int \frac{3}{x}\,dx + \int \frac{4x+2}{x^2+1}\,dx$$

$u=x^2+1$

$$= -3\ln|x| + \int \frac{4x}{x^2+1}\,dx + \int \frac{2\,dx}{x^2+1}$$

$$= -3\ln|x| + 2\ln(x^2+1) + 2\tan^{-1}x + K$$

So $$\int \frac{x^4-2x+3}{x^3+x}\,dx = \frac{x^2}{2} + 3\ln|x| - 2\ln(x^2+1) - 2\tan^{-1}x + K.$$

Partial fractions (a digression)

- The greatest common divisor of two integers can be expressed as a linear combination of the two integers, namely
$\gcd(5,3) = 1$ and thus $5x + 3y = 1$ for some integers x, y.
- We have a similar statement for polynomials, namely
$\gcd(f,g) = 1$ implies $f \cdot p + g \cdot q = 1$ for some polynomials p, q.

This is the identity that leads to partial fractions:

$$fp + gq = 1 \implies \frac{p}{g} + \frac{q}{f} = \frac{1}{fg}.$$

We can decompose $\dfrac{A}{fg} = \dfrac{Aq}{f} + \dfrac{Ap}{g}$ for some polynomials p, q.

▨ To prove this statement, we use division with remainder.
For instance, consider 8 and 5. We compute $\gcd(8,5)$ by

$$\begin{aligned}
8 &= 5 \times 1 + 3 \quad \leftarrow \text{remainders}\\
5 &= 3 \times 1 + 2\\
3 &= 2 \times 1 + 1\\
2 &= 1 \times 2 + 0
\end{aligned}$$

Remainders decrease at every step and eventually become 0.
<u>Claim</u>: gcd = last nonzero remainder
If x divides 8 and 5, then x divides <u>all</u> remainders.

Moreover, gcd is a linear combination of 8 and 5, namely

$$\begin{aligned}
1 &= 3 - 2 \times 1 && \text{... lin. comb. of } 2, 3\\
&= 3 - (5 - 3) \times 1 && \text{... eliminate } 2\\
&= 2 \times 3 - 5 && \text{... eliminate } 3\\
&= 2 \times (8 - 5) - 5\\
&= 2 \times 8 - 3 \times 5.
\end{aligned}$$

We need this argument for polynomials f, g.
Suppose that $\gcd(f,g) = 1$. Then division of polynomials gives

$$f = g \cdot Q_1 + R_1 \qquad \text{with} \qquad \deg R_1 < \deg g$$

$$g = R_1 \cdot Q_2 + R_2 \qquad \text{with} \qquad \deg R_2 < \deg R_1 < \deg g.$$

The degrees decrease at every step, so we eventually get the greatest common divisor which is constant c and then we can express that constant $c = P \cdot f + Q \cdot g$ as a lin. combination.
to get $1 = \frac{1}{c} Pf + \frac{1}{c} Qg$ as a lin. combination.

<u>Complex analysis</u> Any real polynomial $f(x)$ factors over the complex numbers as $f(x) = c(x-x_1)(x-x_2)\cdots(x-x_n)$ with linear factors only. Some roots may be real, the others are conjugate pairs $a \pm bi$ which give factors $(x-a-bi)(x-a+bi)$

$$= (x-a)^2 - (bi)^2 = (x-a)^2 + b^2.$$

Those are quadratic factors that cannot be factored over $\mathbb{R}$.
We only have to deal with <u>linear</u> factors $(x-x_i)^k$
and <u>quadratic</u> factors $((x-a)^2+b^2)^k$.

<u>Example</u> Consider $\displaystyle\int \frac{3x+5}{x^2+4x+8}\,dx$.

We check the denominator for roots .... $\Delta = b^2 - 4ac = 16 - 4 \times 8 = -16$ is negative $\Rightarrow$ no real roots $\Rightarrow$ we cannot factor over $\mathbb{R}$.
In that case, we can complete the square

$$x^2 + 4x + 8 = \underbrace{x^2 + 4x + 4} + 4 = (x+2)^2 + 2^2$$

We get $\displaystyle\int \frac{3x+5}{x^2+4x+8}\,dx = \int \frac{3x+5}{(x+2)^2+2^2}\,dx.$ $u = x+2$

(ii) To integrate denominators $(x+a)^2+b^2$, we can let $u=x+a$ to make the denominator $u^2+b^2 = b^2\left(\left(\frac{u}{b}\right)^2+1\right)$ and then $w=\frac{u}{b}$ will make this $b^2(w^2+1)$, a denominator that corresponds to $\frac{w}{w^2+1}$ or $\frac{1}{w^2+1}$, for instance.

In our case, $u=x+2$ gives

$$\int \frac{3x+5}{(x+2)^2+4}\,dx = \int \frac{3(u-2)+5}{u^2+4}\,du = \int \frac{3u-1}{u^2+4}\,du \quad \text{and } w=\frac{u}{2} \text{ gives } u=2w$$

$$= \int \frac{6w-1}{4w^2+4}\,2\,dw$$

$$= \frac{1}{2}\int \frac{6w-1}{w^2+1}\,dw$$

So

$$\int \frac{3x+5}{(x+2)^2+4}\,dx = \frac{1}{2}\int \frac{6w\,dw}{w^2+1} - \frac{1}{2}\int \frac{dw}{w^2+1}$$

$$= \frac{3}{2}\ln(w^2+1) - \frac{1}{2}\tan^{-1} w + C$$

$$= \frac{3}{2}\ln\left(\frac{u^2}{4}+1\right) - \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + C$$

$$= \frac{3}{2}\ln\left(\left(\frac{x+2}{2}\right)^2+1\right) - \frac{1}{2}\tan^{-1}\frac{x+2}{2} + C.$$

(iii) More generally, linear factors $(x-x_0)^k$ can be handled by $u=x-x_0$ and quadratic factors $(x-a)^2+b^2$ can be handled by $w=\frac{x-a}{b}$ which reduces the factors to $b^2w^2+b^2 = b^2(w^2+1)$, a familiar denominator. One can then integrate any rational function.

Various integration examples

① $\int \frac{e^x\,dx}{1+e^x}$ ..... we take $u = e^x \Rightarrow du = e^x\,dx$. This gives

$$\int \frac{du}{1+u} = \ln(1+u) + C = \ln(1+e^x) + C.$$

② $\int \frac{dx}{1+e^x}$ .... we take $u = e^x$ once again. Then

$$\int \frac{e^x\,dx}{e^x(1+e^x)} = \int \frac{du}{u(1+u)}.$$

Partial fractions $\frac{1}{u(1+u)} = \frac{A}{u} + \frac{B}{1+u}$ for some A, B. Thus $1 = A(1+u) + Bu$.

When $u=0$, we get $A=1$. When $u=-1$, we get $B=-1$.

$$\therefore \int \frac{du}{u(1+u)} = \int \left(\frac{1}{u} - \frac{1}{1+u}\right) du = \ln|u| - \ln|1+u| + C$$
$$= \ln e^x - \ln(1+e^x) + C$$
$$= x - \ln(1+e^x) + C.$$

③ $\int \frac{\sin^3 x}{\cos^4 x}\,dx$ .... we could try $u = \cos x$, $du = -\sin x\,dx$ to get

$$\int \frac{\sin^2 x \cdot \underbrace{(\sin x\,dx)}}{\cos^4 x} = -\int \frac{(1-u^2)\,du}{u^4} = \int \left(\frac{u^2-1}{u^4}\right) du$$
$$= \int (u^{-2} - u^{-4})\,du = -\frac{1}{u} + \frac{1}{3u^3} + C$$
$$= -\sec x + \frac{\sec^3 x}{3} + C.$$

Alternatively, one could write

$$\int \frac{\sin^3 x}{\cos^3 x} \cdot \frac{1}{\cos x}\,dx = \int \tan^3 x \cdot \sec x\,dx$$

and let $w = \sec x$, $dw = \sec x \tan x$

$$= \int \tan^2 x \cdot \underbrace{\sec x \tan x\,dx}$$
$$= \int (w^2-1)\,dw = \frac{\sec^3 x}{3} - \sec x + C.$$

④ $\int 2x e^{x^2}\,dx$ = easy --- take $u = x^2$, $du = 2x\,dx$

$= \int e^u\,du = e^u + C = e^{x^2} + C.$

⑤ $\int 2x^3 e^{x^2}\,dx = \int x^2 \cdot e^{x^2} \cdot \underbrace{2x\,dx} = \int u e^u\,du$

$= ue^u - \int e^u\,du$

$u = u \quad dv = e^u du$
$du = du \quad v = e^u$

$= ue^u - e^u + C = x^2 e^{x^2} - e^{x^2} + C.$

The corresponding integrals $\int 2e^{x^2}\,dx$, $\int 2x^2 e^{x^2}\,dx$ are <u>not</u> expressible in terms of standard functions (polynomials, exp, ln, trig). Some other examples are $\int \frac{\sin x}{x}\,dx$, $\int \frac{e^x}{x}\,dx$, $\int \frac{e^x}{x^2}\,dx$, $\int x^x\,dx$ etc.

⑥ $\int \frac{2x\,dx}{(1+x^2)^2}$ = easy ---- take $u = 1+x^2$, $du = 2x\,dx$

$\int \frac{du}{u^2} = \int u^{-2}\,du = -\frac{1}{u} + C = -\frac{1}{1+x^2} + C.$

⑦ $\int \frac{2\,dx}{(1+x^2)^2}$ = harder ---- the substitution $u = 1+x^2$ does not help!

Let's try $x = \tan\theta$, $dx = \tan\theta \sec\theta\,d\theta$. We get

$\int \frac{2\tan\theta\sec\theta\,d\theta}{(1+\tan^2\theta)^2} = \int \frac{2\tan\theta\sec\theta\,d\theta}{\sec^4\theta} = 2\int \frac{\tan\theta}{\sec^3\theta}\,d\theta$

$= 2\int \frac{\sin\theta}{\cancel{\cos\theta}} \cdot \cancel{\cos^3\theta}^{\cos^2\theta}\,d\theta$

$= 2\int \cos^2\theta \sin\theta\,d\theta$ .... now $u = \cos\theta$, $du = -\sin\theta\,d\theta$

$= -2\int u^2\,du$

$= -\frac{2u^3}{3} + C = -\frac{2}{3}\cos^3\theta + C.$

It remains to relate $x = \tan\theta$ to $\cos\theta$.

Now, $\sec^2\theta = 1+\tan^2\theta \Rightarrow \frac{1}{\cos^2\theta} = 1+x^2 \Rightarrow \cos\theta = \frac{1}{\sqrt{1+x^2}}$.

The answer is then $-\frac{2}{3}\frac{1}{(1+x^2)^{3/2}} + C$.

(8*) $$\int \sec x\, dx = \int \frac{1}{\cos x}\, dx = \int \frac{\cos x}{1-\sin^2 x}\, dx$$

$$= \int \frac{\cos x\, dx}{(1-\sin x)(1+\sin x)} = \int \frac{\frac{1}{2}\cos x\, dx}{1-\sin x} + \int \frac{\frac{1}{2}\cos x}{1+\sin x}\, dx$$

$$= -\frac{1}{2}\ln|1-\sin x| + \frac{1}{2}\ln|1+\sin x| + C$$

$$= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| \cdot \left|\frac{1+\sin x}{1+\sin x}\right| = \frac{1}{2}\ln\frac{(1+\sin x)^2}{\cos^2 x}$$

$$= \cancel{\frac{1}{2}}\ln\left(\frac{1+\sin x}{\cos x}\right)^{\cancel{2}} = \ln|\sec x + \tan x| + C.$$

(9) $\int x \cdot \underline{\underline{\tan^{-1} x}}\, dx$ .... $u = \tan^{-1} x$, $dv = x\, dx$

$du = \frac{1}{1+x^2}\, dx$, $v = \frac{1}{2}x^2$

Integrate by parts to get

$$\int x\tan^{-1}x\, dx = \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}\, dx$$

$$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int \frac{x^2+1-1}{x^2+1}\, dx$$

$$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int\left(1-\frac{1}{x^2+1}\right)dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{1}{2}\tan^{-1}x + C.$$

Another integration example

We compute $\int \frac{5x^2-4x+3}{x^3-1}\,dx$.

The denominator factors as $x^3-1 = (x-1)(x^2+x+1)$ with the quadratic having $\Delta = 1-4=-3$ and no roots. We can write

$$\frac{5x^2-4x+3}{x^3-1} = \frac{5x^2-4x+3}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}.$$

This gives $5x^2-4x+3 = A(x^2+x+1) + (Bx+C)(x-1)$

$\underline{\underline{x=0}}$ $\quad 3 = A - C$

$\underline{\underline{x=1}}$ $\quad 4 = 3A$ ---- so $A = 4/3$ and $C = A-3 = -5/3$

$\underline{\underline{x=-1}}$ $\quad 12 = A-2(C-B) = A-2C+2B$ and $B = \frac{12-A+2C}{2} = \frac{12-4/3-10/3}{2}$,

namely $B = \frac{22}{6} = 11/3$.

We get $\int \frac{5x^2-4x+3}{x^3-1}\,dx = \int \frac{4/3}{x-1}\,dx + \frac{1}{3}\int \frac{11x-5}{x^2+x+1}\,dx$. $(*)$

The first integral is just $\frac{4}{3}\ln|x-1|$. To compute the second, we complete the square: $x^2+x+1 = \underbrace{x^2+2x\left(\tfrac{1}{2}\right)+\left(\tfrac{1}{2}\right)^2}-\left(\tfrac{1}{2}\right)^2+1 = \left(x+\tfrac{1}{2}\right)^2+\frac{3}{4}$.

Substitute $u = x+\frac{1}{2}$ to make this $u^2+3/4$. Then $du = dx$ and

$$\frac{1}{3}\int \frac{11x-5}{x^2+x+1}\,dx = \frac{1}{3}\int \frac{11\left(u-\frac{1}{2}\right)-5}{u^2+\frac{3}{4}}\,du = \frac{1}{3}\int \frac{11u-\frac{21}{2}}{u^2+3/4}\,du.$$

$\odot$ $\int \frac{11u}{u^2+3/4}\,du = \int \frac{\frac{11}{2}\,dw}{w} = \frac{11}{2}\int \frac{dw}{w} = \frac{11}{2}\ln|w|$ $\quad$ $w = u^2+3/4$, $dw = 2u\,du$

$= \frac{11}{2}\ln(u^2+3/4) = \frac{11}{2}\ln(x^2+x+1)$.

$\odot$ $\int \frac{-\frac{21}{2}\,du}{u^2+3/4}$ can be simplified by writing $u^2+\frac{3}{4} = \frac{3}{4}\left(\frac{u^2}{3/4}+1\right)$

$= \frac{3}{4}\left(z^2+1\right)$ with $z = \frac{2u}{\sqrt{3}}$.

We get $\int \frac{-\frac{21}{2}}{\frac{3}{4}(z^2+1)}\left(\frac{\sqrt{3}}{2}\,dz\right) = -7\sqrt{3}\int \frac{dz}{z^2+1}$

$= -7\sqrt{3}\tan^{-1} z = -7\sqrt{3}\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right) = -7\sqrt{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$

and thus $\int \frac{5x^2-4x+3}{x^3-1}\,dx = \frac{4}{3}\ln|x-1| + \frac{11}{6}\ln(x^2+x+1) - \frac{7\sqrt{3}}{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + K.$

Sequences and series A sequence is a function $f: \mathbb{N} \to \mathbb{R}$, where $\mathbb{N}$ = positive integers. It is determined by $f(1), f(2), f(3), \dots$ and usually denoted by $\{a_n\}$, where $n \geq 1$.

<u>A convergent sequence</u> is one that has a limit as $n \to \infty$, namely one for which $\lim_{n\to\infty} a_n = L$ is finite.

<u>A monotonic sequence</u> is one that is increasing with $a_{n+1} \geq a_n$ for all n <u>or</u> decreasing with $a_{n+1} \leq a_n$ for all n.

<u>A bounded sequence</u> is a sequence $\{a_n\}$ such that $|a_n| \leq M$ for some finite M and all integers $n \geq 1$.

<u>Example.</u> Consider $a_n = \frac{n}{n^2+1}$. Then $a_1 = 1/2$, $a_2 = 2/5$, $a_3 = \frac{3}{10}$, etc.
This seems to be decreasing to 0. We can check

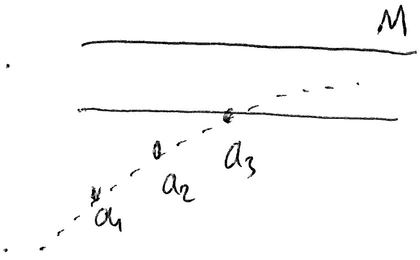
1. convergence ---- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n^2\cancel{+1}} = \lim_{n\to\infty} \frac{n}{n^2} = 0$.
2. monotonicity ---- define $f(x) = \frac{x}{x^2+1}$ for <u>any</u> x, not just integers.
   Then $f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} \leq 0$ <u>for any $x \geq 1$</u>,
   so the sequence is decreasing!
3. boundedness ---- we know $a_1 \geq a_2 \geq a_3 \geq \dots \geq 0$
   and thus $1/2 \geq a_n \geq 0$ for any n.

<u>Example.</u> Consider $a_n = \frac{2^n}{n!}$. To check monotonicity, we check
$a_1 = \frac{2}{1!} = 2$, $a_2 = \frac{4}{2} = 2$, $a_3 = \frac{8}{6} = 4/3$. This seems decreasing.
We claim ---- $a_{n+1} \leq a_n$ for all n.
To prove this --- $\frac{2^{n+1}}{(n+1)!} \leq \frac{2^n}{n!} \iff \frac{2^{n+1}}{2^n} \leq \frac{(n+1)!}{n!}$

$$\iff 2 \leq n+1 \iff 1 \leq n.$$

Since $n \geq 1$, we get $a_{n+1} \leq a_n$ for all n and thus $\{a_n\}$ is decreasing.

Theorem. Suppose $\{a_n\}$ is monotonic and bounded.
Then $\{a_n\}$ is also convergent, namely
$\lim_{n\to\infty} a_n$ exists (and is finite).



Proof. We know that $|a_n| \leq M$ because $\{a_n\}$ is bounded.
Suppose $\{a_n\}$ is increasing and let M be the "smallest number" satisfying $a_n \leq M$ for all n. Then ~~$a_n \leq |a_n| \leq M$~~
~~and~~ we claim that $\boxed{\lim_{n\to\infty} a_n = M}$. We need: given any $\varepsilon > 0$ there exists N such that $|a_n - M| < \varepsilon$ for all $n \geq N$.
Since $a_n \leq M$ ..... this reads $M - a_n < \varepsilon$ for all $n \geq N$.

Now, M = smallest number bigger than all terms
$\Rightarrow$ $M - \varepsilon$ is not bigger than all terms
$\Rightarrow$ there exists $a_N > M - \varepsilon$
$\Rightarrow$ $a_n > M - \varepsilon$ for any term $n \geq N$
$\Rightarrow$ $\varepsilon > M - a_n$ for any $n \geq N$, as needed. ▨

$\boxed{\text{Example}}$ Consider a sequence $\{a_n\}$ defined by $a_1 = 1$, $a_2 = \sqrt{3}$, $a_3 = \sqrt{2+\sqrt{3}}$ etc so that $a_{n+1} = \sqrt{2+a_n}$ for each $n \geq 1$.
This is a recursive definition of $\{a_n\}$. We compute $\lim_{n\to\infty} a_n$.

Step 1. We show the limit exists.
We claim that $1 \leq a_n \leq a_{n+1} \leq 3$ for all n.
Use induction. When $n=1$, we get $1 \leq a_1 \leq a_2 \leq 3$
or $1 \leq 1 \leq \sqrt{3} \leq 3$ ... that is true.

Assume for some n, say $1 \leq a_k \leq a_{k+1} \leq 3$.
To prove it for the next, $\sqrt{3} \leq \sqrt{a_k + 2} \leq \sqrt{a_{k+1}+2} \leq \sqrt{5}$
and so $\sqrt{3} \leq a_{k+1} \leq a_{k+2} \leq \sqrt{5}$
and so $1 \leq \sqrt{3} \leq a_{k+1} \leq a_{k+2} \leq \sqrt{5} \leq 3$.
We get the same statement for $k+1$ as well.

Step 2. Once we know the limit exists, we recall

$a_{n+1} = \sqrt{2+a_n}$. We get $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2+a_n}$.

We know $\lim_{n\to\infty} a_n = L$ exists, $L = \sqrt{2+L}$

so $L^2 = 2+L$, $L^2 - L - 2 = 0$

so $(L-2)(L+1) = 0$

and $L=2$, $L=-1$.

Since $1 \le a_n \le a_{n+1} \le 3$, we must have $1 \le L \le 3$ and thus $L=2$.

Convergence of sequences: $\{a_n\}$ converges means $\lim_{n\to\infty} a_n$ exists.

Convergence of series: $\sum_{n=1}^{\infty} a_n$ converges means $\lim_{n\to\infty} (a_1+a_2+a_3+\dots+a_n)$ exists.

More formally, given a sequence $\{a_n\}$, we define the partial sums

$$S_N = a_1 + a_2 + \dots + a_N$$

and check if those have a limit as $N\to\infty$. If they do, we define $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} S_N$.

In practice, sums are usually difficult to compute. We will mostly be interested in convergence, namely if $a_1 + a_2 + \dots + a_n$ has a limit.

Example. Let $a_n = (-1)^n$ so that $a_1 = -1$, $a_2 = +1$ etc.

This sequence does not converge. We say it diverges.

The partial sums are: $S_1 = a_1 = -1$

$S_2 = a_1 + a_2 = -1 + 1 = 0$

$S_3 = a_1 + a_2 + a_3 = -1 + 1 - 1 = -1$

They are oscillating between -1 and 0, so there is no limit.

Example. Let $a_n = 1/2^n$ so that $a_1 = 1/2$, $a_2 = 1/4$ etc.

Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2^n} = 0$ so $\{a_n\}$ converges.

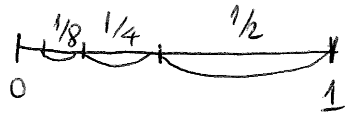
Looking at partial sums, $s_1 = 1/2$

$$s_2 = 1/2 + 1/4 = 3/4$$

$$s_3 = 1/2 + 1/4 + 1/8 = 7/8 \quad \text{and so on.}$$

We can check $s_n = \dfrac{2^n - 1}{2^n}$ by induction and this gives

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2^n-1}{2^n} = \lim_{n\to\infty}\left(1 - \frac{1}{2^n}\right) = 1.$$



In most cases, we do not have a formula for s_n.

<u>Theorem (nth term test)</u> If $\sum_{n=1}^{\infty} a_n$ converges, then one must have $a_n \to 0$ as $n \to \infty$. In other words, if $\lim_{n\to\infty} a_n$ is not zero, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. Consider the partial sums $S_N = a_1 + a_2 + \dots + a_N$.
If the series converges, then S_N converges, so $\lim_{N\to\infty} S_N$ exists.
Call it L. Then

$$S_N = a_1 + \dots + a_N$$

$$S_{N+1} = a_1 + \dots + a_N + a_{N+1}$$

So $$S_{N+1} - S_N = a_{N+1}$$

So $$\lim_{N\to\infty} S_{N+1} - \lim_{N\to\infty} S_N = \lim_{N\to\infty} a_{N+1}$$

So $$0 = L - L = \lim_{N\to\infty} a_{N+1}. \quad \blacksquare$$

$\boxed{\text{Geometric series}}$ Consider the series formed by adding powers of x, namely $a_1 = x$, $a_2 = x^2$ and $a_n = x^n$ more generally. We look at the partial sums

$$S_n = x + x^2 + x^3 + \dots + x^n.$$

Since $x S_n = x^2 + x^3 + \dots + x^n + x^{n+1}$, subtraction gives

$$xS_n - S_n = x^{n+1} - x \quad \text{or} \quad (x-1)S_n = x^{n+1} - x$$

$$\text{or} \quad S_n = \frac{x^{n+1} - x}{x-1}.$$

Theorem. Consider the geometric series $\sum_{n=1}^{\infty} x^n$.

ⓐ If $|x| \geq 1$, then this series diverges.

ⓑ If $|x| < 1$, then it converges and one has $\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$.

Proof. If $x=1$, then the sum is $1+1+1+\dots$ hence infinite. We can deduce this from the n^{th} term test. The same is true when $x=-1$ because $a_n = (-1)^n$ does not go to 0. The same is true when $|x|>1$.

When $|x|<1$, we look at partial sums $S_n = x + x^2 + \dots + x^n$.

This is $S_n = \frac{x^{n+1}-x}{x-1}$ and we can look at

$$x + x^2 + x^3 + \dots = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{x^{n+1}-x}{x-1} = \frac{0-x}{x-1} = \frac{x}{1-x}. \quad ▨$$

$\boxed{\text{Explicit sums}}$ One has $\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$ and $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, if $|x|<1$.

This gives $\sum_{n=1}^{\infty} \frac{3^n}{4^n} = \sum_{n=1}^{\infty} (3/4)^n = \frac{3/4}{1-3/4} = 3$

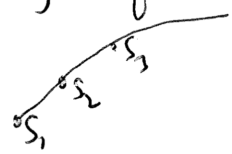
and $\sum_{n=0}^{\infty} \frac{3^{n+2}}{2^{4n+5}} = \sum_{n=0}^{\infty} \frac{3^n \cdot 3^2}{2^{4n}\cdot 2^5} = \frac{9}{32} \sum_{n=0}^{\infty} \left(\frac{3}{16}\right)^n = \frac{9}{32}\frac{1}{1-3/16}$

$= \frac{9}{32}\cdot\frac{16}{13} = 9/26$, for instance.

$\boxed{\text{Convergence of non-negative series}}$

First, we consider $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots$ when $a_n \geq 0$ for all n.

In that case, $S_n = a_1 + a_2 + \dots + a_n$ is an increasing sequence.



Either S_n increases indefinitely (and series diverges)

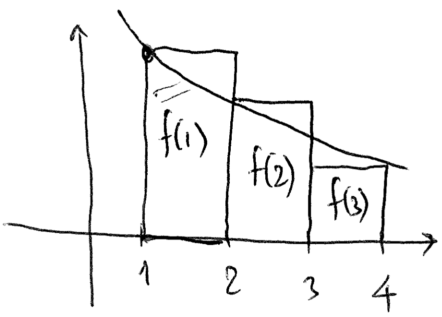
or S_n is bounded $\Rightarrow$ S_n is monotonic & bounded

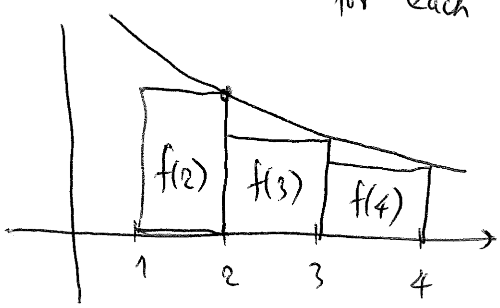
$\Rightarrow$ S_n is convergent

$\Rightarrow$ the series converges.

We just need to know S_n is bounded.

Integral test Suppose $f(x)$ is continuous, positive and decreasing for each $x \geq 1$.





In that case, the series $f(1) + f(2) + f(3) + \dots = \sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_1^n f(x)\, dx$ is bounded for all $n \in \mathbb{N}$.

__Proof.__ Since $f(x)$ is decreasing, one has

$$f(k) \leq f(x) \leq f(k+1) \qquad \text{on} \qquad [k, k+1]$$

so

$$\int_k^{k+1} f(k)\, dx \leq \int_k^{k+1} f(x)\, dx \leq \int_k^{k+1} f(k+1)\, dx \quad \dots \text{for all } k$$

$$f(k) \leq \int_k^{k+1} f(x)\, dx \leq f(k+1) \quad \text{for all } k.$$

We add those inequalities to get

$$f(1) + f(2) + \dots + f(n-1) \leq \int_1^n f(x)\, dx \leq f(2) + f(3) + \dots + f(n)$$

$$\sum_{k=1}^{n-1} f(k) \leq \int_1^n f(x)\, dx \leq \sum_{k=2}^{n} f(k).$$

If the __integral__ is bounded, then $\sum_{k=1}^{n-1} f(k)$ is bounded, so S_{n-1} is bounded and we get convergence.

If the integral is unbounded, then $\sum_{k=2}^{n} f(k)$ is unbounded and we get divergence. ▨

__Theorem 1.__ (Geometric series) $\sum_{n=1}^{\infty} x^n$ converges if and only if $|x| < 1$.

__Theorem 2.__ (p-series) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

__Proof.__ Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$. If $p \leq 0$, then the $n^{\underline{th}}$ term is $a_n = n^{-p} = n$ to a non-negative power and a_n does __not__ go to zero. By the $n^{\underline{th}}$ term test, we get divergence if $p \leq 0$.

Consider the case $p > 0$. Then $f(x) = x^{-p}$ with $p>0$ is continuous on $[1,\infty)$, positive with $f'(x) = -px^{-p-1} < 0$ so we can use the integral test. Thus convergence is equivalent to $\int_1^n f(x)\,dx = \int_1^n x^{-p}\,dx$ being bounded for all n.

When $1-p \neq 0$, we get $\left[\frac{x^{1-p}}{1-p}\right]_1^n = \frac{n^{1-p} - 1^{1-p}}{1-p} = \frac{n^{1-p}-1}{1-p}$.

If $1-p>0$, n^{1-p} becomes unbounded and we get divergence

If $1-p<0$, $n^{1-p} \to 0$ as $n\to\infty$ and we get convergence.

If $p=1$, finally, $\int_1^n x^{-1}\,dx = [\ln x]_1^n = \ln n - \underbrace{\ln 1}_{0}$ is unbounded and we get divergence. ▨

<u>Examples</u>. $\sum_{n=1}^{\infty} \frac{1}{n}$ = harmonic series $= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ is divergent

$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots$ is convergent $(p=2)$

$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is still convergent etc.

<u>Convergent</u> series can be manipulated as usual

$$\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (a_n + b_n)$$

$$c\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} c a_n$$ and so on.

These follow from finite sums by looking at partial/finite sums and taking limits as $n\to\infty$.

Manipulation of divergent series is <u>not</u> justified.

$$S = 1 - 1 + 1 - 1 + 1 - \dots$$
$$S = 1 - (1 - 1 + 1 - 1 + \dots)$$
$$S = 1 - S$$
$$2S = 1$$
$$S = 1/2$$

Convergence tests for non-negative series

- Consider the series $\sum_{n=1}^{\infty} a_n$ with $a_n \geq 0$ for all n.
  Then $S_n = a_1 + a_2 + \dots + a_n$ is increasing and we need only worry about S_n being bounded.

~~①~~ <u>Integral test:</u> $\sum_{n=1}^{\infty} f(n)$ is convergent $\iff$ $\int_1^n f(x)\,dx$ is bounded.

This implies $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent $\iff$ $p > 1$.

② <u>Comparison test</u>: ⓐ If a series is smaller than convergent, it is convergent.
Namely, $\sum b_n$ converges and $a_n \leq b_n$ for large n, then $\sum a_n$ converges.
ⓑ If a series is larger than divergent, it is divergent.
Namely, $\sum b_n$ diverges and $a_n \geq b_n$ for large n, then $\sum a_n$ diverges.

<u>THIS WORKS FOR NON-NEGATIVE SERIES!!</u>

<u>Proof.</u> Suppose $a_n \leq b_n$ for all $n \geq N$.

We look at
$$S_n = a_N + a_{N+1} + \dots + a_n \quad \dots \text{ the partial sum}$$
$$t_n = b_N + b_{N+1} + \dots + b_n \quad \dots \text{ the partial sum.}$$

If $\sum b_n$ converges, then t_n is Convergent, hence bounded and so $S_n \leq t_n$ is also bounded, hence convergent. ▨

<u>Example 1.</u> Consider $\sum_{n=1}^{\infty} \frac{1}{n^2+3n}$, for instance.

We compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$. We have
$$\sum_{n=1}^{\infty} \frac{1}{n^2+3n} \leq \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\substack{\text{convergent} \\ \text{since } p=2>1}}.$$

Then $\sum \frac{1}{n^2+3n}$ is also convergent.

<u>Example 2.</u> Consider $\sum_{n=1}^{\infty} \frac{3}{n+2}$. We compare this with $\sum_{n=1}^{\infty} \frac{3}{n}$.

Unfortunately, $\sum_{n=1}^{\infty} \frac{3}{n}$ is divergent and $\sum \frac{3}{n+2} \leq \underbrace{\sum \frac{3}{n}}_{\text{divergent}}$,
which gives <u>no</u> conclusions.

Let's try to compare with $\frac{1}{n}$. We ~~need~~ claim $\frac{3}{n+2} \geq \frac{1}{n}$.
This is true when $3n \geq n+2$ or $2n \geq 2$ or $n \geq 1$. Then
$\sum \frac{1}{n}$ diverges $(p=1)$ and $\sum \frac{3}{n+2} \geq \sum \frac{1}{n}$, so $\sum \frac{3}{n+2}$ diverges as well.

③ Limit comparison test. Consider the series $\sum a_n$, $\sum b_n$ with $a_n, b_n \geq 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. Then $\sum a_n$ converges $\iff$ $\sum b_n$ converges.

Example 3. Consider $\sum_{n=1}^{\infty} \frac{3n^2+4n+6}{4n^3+3n+1}$. We compare with $\sum \frac{3n^2}{4n^3}$.

Take $a_n = \frac{3n^2+4n+6}{4n^3+3n+1}$, $b_n = \frac{3n^2}{4n^3} = \frac{3}{4n}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3n^2+4n+6}{4n^3+3n+1} \cdot \frac{4n}{3} = \lim_{n\to\infty} \frac{3n^2}{4n^3} \cdot \frac{4n}{3} = \lim_{n\to\infty} \frac{12n^3}{12n^3} = 1.$$

Use Limit Comparison: $\sum b_n = \sum \frac{3}{4n} = \frac{3}{4} \sum \frac{1}{n}$ (divergent $p=1$)

so $\sum a_n$ diverges as well.

// Limit Comparison helps with rational functions!

Example 4. Consider $\sum_{n=1}^{\infty} \frac{\sqrt{n}+3}{n^3+2n+1}$ and $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$.

We know $\sum \frac{1}{n^{5/2}}$ converges (since $p = 5/2 > 1$). Let $a_n = \frac{\sqrt{n}+3}{n^3+2n+1}$, $b_n = \frac{1}{n^{5/2}}$.

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(\sqrt{n}+3)n^{5/2}}{n^3+2n+1} = \lim_{n\to\infty} \frac{n^3+3n^{5/2}}{n^3+2n+1} = \lim_{n\to\infty} \frac{n^3}{n^3} = 1$.

Since $\sum b_n$ converges, we get $\sum a_n$ converges.

④ Ratio test Consider $\sum a_n$ with $a_n \geq 0$ as before. Define

$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

ⓐ If $L < 1$, then $\sum a_n$ converges.

ⓑ If $L > 1$, then $\sum a_n$ diverges.

Note: If $L = 1$, we get no conclusions.

Roughly speaking: $\frac{a_{n+1}}{a_n} \approx L$ for large enough n, say $n \geq N$

so

$$a_{N+1} \approx L a_N$$
$$a_{N+2} \approx L a_{N+1} \approx L^2 a_N$$
$$a_{N+3} \approx L a_{N+2} \approx L^3 a_N$$

and then $a_N + a_{N+1} + a_{N+2} + a_{N+3} + \dots \approx (1 + L + L^2 + L^3 + \dots) a_N$

gives a geometric series which converges for $L < 1$, diverges for $L > 1$.

Example 5. We use the ratio test for $\sum_{n=1}^{\infty} \frac{2^n (n+1)}{n!}$.

We have $a_n = \frac{2^n(n+1)}{n!}$, so $a_{n+1} = \frac{2^{n+1}(n+2)}{(n+1)!}$. Then

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \frac{n+2}{n+1} \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} 2 \cdot \frac{n+2}{n+1} \cdot \frac{1}{n+1} = \lim_{n\to\infty} \frac{2\cancel{n}}{n^{\cancel{2}}} = 0.$$

The ratio test gives: $L < 1 \Rightarrow$ series converges.

In practice, the ratio test helps with powers like 2^n, a^n and factorials $(n+1)!$, $(2n)!$ but not with rational functions.

Example 6. We use the ratio test for $\sum_{n=1}^{\infty} \frac{e^n \sqrt{n+1}}{n^2+4}$.

Let $a_n = \frac{e^n \sqrt{n+1}}{n^2+4}$, $a_{n+1} = \frac{e^{n+1}\sqrt{n+2}}{(n+1)^2+4}$ and compute

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{e^{n+1}}{e^n} \frac{\sqrt{n+2}}{\sqrt{n+1}} \frac{n^2+4}{n^2+2n+5}$$

$$= \lim_{n\to\infty} \frac{e^{n+1}}{e^n} \cdot \underbrace{\lim_{n\to\infty} \sqrt{\frac{n+2}{n+1}}}_{\to 1} \cdot \underbrace{\lim_{n\to\infty} \frac{n^2+4}{n^2+2n+5}}_{\to 1}$$

$$= e.$$

Ratio test says: $L = e > 1$, so the series diverges.

Example 7. For $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2+4}$, the ratio test is useless.

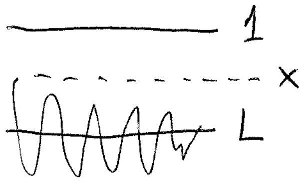
One would use limit comparison: $a_n = \frac{\sqrt{n+1}}{n^2+4}$, $b_n = \frac{\sqrt{n}}{n^2}$.

One has $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ and $\sum b_n = \sum \frac{1}{n^{3/2}}$ ($p = 3/2$ convergent)

Limit comparison gives: $\sum a_n$ converges as well.

Proof of the ratio test. Assume $L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$.

Since $\frac{a_{n+1}}{a_n} \to L$, we can define $x = \frac{1+L}{2}$, the average, to get $\frac{a_{n+1}}{a_n} \le x$ for large n



Assume $\frac{a_{n+1}}{a_n} \le x$ for each $n \geqslant N$, say.

Then $a_{N+1} \le x\, a_N$

$a_{N+2} \le x\, a_{N+1} \le x^2 a_N$

$a_{N+3} \le x\, a_{N+2} \le x^3 a_N$ ..... So $a_{N+1} + a_{N+2} + \dots \le a_N(x + x^2 + x^3 + \dots)$.

Thus $\sum_{n=N+1}^{\infty} a_n$ is smaller than the geometric series $\sum_{n=1}^{\infty} a_N x^n$.

Since $x<1$, the latter converges. Thus $\sum_{n=N+1}^{\infty} a_n$ converges by comparison. ▨

$\boxed{\text{Series involving arbitrary terms, not necessarily non-negative.}}$

- Consider the series $\sum a_n$ for arbitrary terms a_n.
  We look at the series $\sum |a_n|$.

$\boxed{\text{Theorem:}}$ $\boxed{\text{If } \sum |a_n| \text{ converges, then } \sum a_n \text{ converges as well.}}$

In this case, we say $\sum a_n$ converges absolutely. Namely, it converges with/without the absolute values.

Proof. We can express a number x as the difference of non-negative numbers.

Write
$$x = \begin{Bmatrix} x & \text{if } x \geqslant 0 \\ 0 & \text{if } x \le 0 \end{Bmatrix} - \begin{Bmatrix} 0 & \text{if } x \geqslant 0 \\ -x & \text{if } x<0 \end{Bmatrix}.$$

Consider the series $\sum a_n$ and introduce
$$b_n = \begin{Bmatrix} a_n & \text{if } a_n \geqslant 0 \\ 0 & \text{if } a_n < 0 \end{Bmatrix}, \quad c_n = \begin{Bmatrix} 0, & \text{if } a_n \geqslant 0 \\ -a_n, & \text{if } a_n < 0 \end{Bmatrix}.$$

(A) b_n, c_n are non-negative

(B) $b_n - c_n = a_n$

(C) $b_n + c_n = \begin{Bmatrix} a_n & \text{if } a_n \geqslant 0 \\ -a_n & \text{if } a_n < 0 \end{Bmatrix} = |a_n|.$

We are assuming $\sum |a_n|$ converges. Then

$$\sum b_n \leq \sum (b_n + c_n) = \sum |a_n| \Rightarrow \sum b_n \text{ converges}$$

$$\sum c_n \leq \sum (b_n + c_n) = \sum |a_n| \Rightarrow \sum c_n \text{ converges.}$$

The series $\sum a_n$ is just $\sum a_n = \sum (b_n - c_n) = \sum b_n - \sum c_n$. $\blacksquare$

<u>Example 1.</u> Consider $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^3+1}$.

Take absolute values .... look at $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1}$.

Limit comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ gives convergence $(p=2)$

Since $\sum \frac{n}{n^3+1}$ converges, the original series converges.

<u>Example 2.</u> Consider $\displaystyle\sum_{n=1}^{\infty} \frac{n \cdot \sin n}{n^4+1}$.

We look at $\displaystyle\sum_{n=1}^{\infty} \frac{n|\sin n|}{n^4+1} \leq \sum_{n=1}^{\infty} \frac{n}{n^4+1}$.

For the series on the right, we can use limit comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4} = \sum \frac{1}{n^3}$. This converges $(p=3)$, so $\sum \frac{n}{n^4+1}$ converges.

Then $\sum \frac{n|\sin n|}{n^4+1}$ converges by comparison $\Rightarrow \sum \frac{n \sin n}{n^4+1}$ converges as well.

<u>Example 3.</u> Consider $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 2^n n!}{n^n}$. We look at $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$,

Ratio test ... $\displaystyle L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \frac{(n+1)!}{n!} \frac{n^n}{(n+1)^{n+1}}$

$$= \lim_{n\to\infty} 2 \cancel{(n+1)} \frac{n^n}{(n+1)^{\cancel{n+1}}}$$

$$= \lim_{n\to\infty} 2 \left(\frac{n}{n+1}\right)^n.$$

This limit is related to $\displaystyle e = \lim_{n\to\infty} (1+1/n)^n = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$.

We get $L = \frac{2}{e} < 1$ and thus convergence. For the limit,

$\cancel{\lim_{n\to\infty} \frac{\ln(1+1/n)}{n}}$ $\displaystyle M = \lim (1+1/n)^n \Rightarrow \ln M = \lim_{n\to\infty} \frac{\ln(1+1/n)}{1/n} = 1 \Rightarrow M = e.$

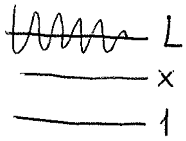
Tests for non-negative series	Tests for arbitrary series
① Comparison test $\quad a_n \le b_n$	① Absolute convergence $\quad \sum \lvert a_n \rvert$
② Limit Comparison test $\quad \lim\limits_{n\to\infty} \frac{a_n}{b_n} = 1$	② Ratio test $\quad L = \lim \left\lvert \frac{a_{n+1}}{a_n} \right\rvert$
③ Ratio test $\quad L = \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$	③ Alternating test $\quad \sum (-1)^{n-1} a_n$

Theorem (Ratio test extended) Consider an arbitrary series $\sum a_n$ and let $L = \lim\limits_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert$. If $L<1$, we get convergence.
If $L>1$, we get divergence.

Proof. Consider the case $L<1$. We look at $\sum |a_n|$, instead. Using the old ratio test, we get $M = \lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim\limits_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = L < 1$.

This gives $\sum |a_n|$ converges $\Rightarrow$ $\sum a_n$ converges (absolute convergence).

For the case $L>1$, pick some $1<x<L$.



Then $\left\lvert \frac{a_{n+1}}{a_n} \right\rvert \ge x$ for large enough n, say for $n \ge N$.

Then
$$|a_{N+1}| \ge x|a_N|$$
$$|a_{N+2}| \ge x|a_{N+1}| \ge x^2|a_N|$$
$$|a_{N+3}| \ge x|a_{N+2}| \ge x^3|a_N|$$
and then $|a_n|$ does not approach 0 as $n\to\infty$ because $x>1$.

We get divergence by the nth term test. ∎

Example 1. Consider $\sum\limits_{n=1}^{\infty} \frac{(-1)^n (n+1)}{2^n(n^2+1)}$. We use the ratio test. We get

$$L = \lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = \lim_{n\to\infty} \left\lvert \frac{(-1)^{n+1}}{(-1)^n} \frac{n+2}{n+1} \frac{2^n}{2^{n+1}} \frac{n^2+1}{(n+1)^2+1} \right\rvert$$
$$= \lim_{n\to\infty} \left\lvert (-1) \cdot \frac{n+2}{n+1} \cdot \frac{1}{2} \frac{n^2+1}{n^2+2n+2} \right\rvert$$
$$= 1 \cdot 1 \cdot \frac{1}{2} \cdot 1 = 1/2 .$$

Since $L<1$, the series converges. The exact same argument gives: $\sum\limits_{n=1}^{\infty} \frac{2^n(n^2+1)}{(-1)^n(n+1)}$ diverges since $L=2$ in that case.

Example 2. Consider $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{e^n (n+1)}$. In this case,

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \frac{\ln(n+1)}{\ln n} \frac{e^n}{e^{n+1}} \frac{n+1}{n+2} \right|$$

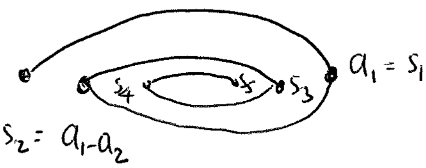
$$= \lim_{n\to\infty} \left| (-1) \cdot \frac{\ln(n+1)}{\ln n} \cdot \frac{1}{e} \cdot \frac{n+1}{n+2} \right|$$

$$= 1 \cdot \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \cdot \frac{1}{e} \cdot 1 .$$

Using L'Hôpital's rule, $\displaystyle\lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} = \lim_{n\to\infty} \frac{1/(n+1)}{1/n} = \lim_{n\to\infty} \frac{n}{n+1} = 1$.

Altogether, $L = 1 \cdot 1 \cdot 1/e \cdot 1 = 1/e < 1$ and the series converges.

$\boxed{\text{Alternating series test}}$ We look at the series $a_1 - a_2 + a_3 - a_4 + \dots$ with each a_n non-negative and <u>decreasing</u> to 0. In that case, the series $\sum (-1)^{n-1} a_n$ converges.



<u>Example</u>. The series $1 - 1/2 + 1/3 - 1/4 + 1/5 - \dots = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ corresponds to taking $a_n = 1/n$ which is non-negative, decreasing to 0. Thus $\sum \frac{(-1)^{n-1}}{n}$ converges. However, it does not converge absolutely since $\sum \frac{1}{n}$ is a p-series with $p=1$.

<u>Example</u>. The series $\sum \frac{(-1)^{n-1}}{n^2+1}$ corresponds to $a_n = \frac{1}{n^2+1}$ which is positive and decreasing to 0. Thus, this series converges. This also converges absolutely since $\sum \frac{1}{n^2+1}$ converges by limit comparison with $\sum \frac{1}{n^2}$.

<u>Proof of test.</u> We look at the partial sums $s_1 = a_1$, $s_2 = a_1 - a_2$, $s_3 = a_1 - a_2 + a_3$ etc. Consider the <u>odd</u> sums first s_1, s_3, s_5 etc.

These are decreasing since $s_{2n+1} = s_{2n-1} - a_{2n} + a_{2n+1} \le s_{2n-1}$ and they are bounded since $s_{2n+1} = a_1 - a_2 + a_3 - \dots + a_{2n+1}$ is bigger than s_2 for all n. Namely, ~~$a_1 - a_2 + (a_3 - a_4) + (a_5 - a_6)$~~ ~~$a_1 - (a_2$~~

$$s_2 - s_1 = -a_2 \le 0 \quad \text{gives} \quad s_2 \le s_1$$

$$s_2 - s_3 = -(s_3 - s_2) = -a_3 \le 0 \quad \text{gives} \quad s_2 \le s_3 \quad \text{and similarly}$$

$$\begin{aligned} s_2 - s_{2n+1} &= -(s_{2n+1} - s_2) \\ &= -(a_3 - a_4 + a_5 - \dots + a_{2n+1}) \\ &= -(a_3 - a_4) - (a_5 - a_6) - \dots - (a_{2n-1} - a_{2n}) - a_{2n+1} \le 0. \end{aligned}$$

Thus s_{2n+1} is decreasing and $s_{2n+1} \geq s_2$ so it is bounded.

We claim s_{2n} is increasing and bounded. Namely,

$$s_{2n+2} = s_{2n} + (a_{2n+1} - a_{2n+2}) \geq s_{2n} \quad \text{for all } n \quad \text{and}$$

$$\begin{aligned} s_{2n} &= a_1 - a_2 + a_3 - a_4 + \dots + a_{2n-1} - a_{2n} \\ &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \dots - (a_{2n-2} - a_{2n-1}) - a_{2n} \\ &\le a_1 - a_{2n} \le a_1 \quad \text{so } s_{2n} \text{ is bounded.} \end{aligned}$$

Thus $\lim_{n\to\infty} s_{2n+1} = L_1$ exists, $\lim_{n\to\infty} s_{2n} = L_2$ exists.

But $L_1 - L_2 = \lim_{n\to\infty} (s_{2n+1} - s_{2n}) = \lim_{n\to\infty} a_{2n+1} = 0$ so $L_1 = L_2$.

This means that <u>all</u> partial sums converge to the same limit. ▨

$\boxed{\text{Power series}}$ We introduce a <u>function</u> $f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$ as an infinite series. This is only defined when the series converges.

We apply the <u>ratio test</u> to determine convergence.

<u>Example</u>. Let $f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$. We have

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \right| = \lim_{n\to\infty} |x| = |x|.$$

If $|x| < 1$ --- convergence. If $|x| > 1$ --- divergence.

We say $R = 1$ is the radius of convergence.

<u>Example</u>. Let $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n+1} x^n = 1 - x + \frac{4}{3}x^2 \dots$.

This is defined for some x. We have

$$\begin{aligned} L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \frac{2^{n+1}}{2^n} \frac{x^{n+1}}{x^n} \frac{n+1}{n+2} \right| \\ &= \lim_{n\to\infty} \left| -1 \cdot 2 \cdot x \cdot \frac{n+1}{n+2} \right| = 2|x| \end{aligned}$$

If $L < 1$, convergence. That's $2|x| < 1$ or $|x| < 1/2$. If $|x| > 1/2$, divergence.

More generally, $\sum c_n x^n$ will converge by the ratio test as long as $\lim\limits_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, namely $\lim\limits_{n\to\infty} \left| \frac{c_{n+1} x^{n+1}}{c_n x^n} \right| < 1$.

This reads $|x| \cdot \lim\limits_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| < 1$ or $|x| < \lim\limits_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|$.

We ~~say~~ call $R = \lim\limits_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|$ the radius of convergence.

<u>Example</u>. Let $f(x) = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$. This power series converges for all x.

In fact,
$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \frac{n!}{(n+1)!} \right| = \lim_{n\to\infty} \frac{|x|}{n+1}$$
$$= |x| \cdot \lim_{n\to\infty} \frac{1}{n+1} = 0$$
for any value of x. Thus $L = 0$ so $L < 1$ and the series converges.

We'll see $f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ is the exponential function.

<u>Example</u>. Let $f(x) = \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$.

In this case,
$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \frac{x^{2n+2}}{x^{2n}} \frac{(2n)!}{(2n+2)!} \right|$$
$$= x^2 \lim_{n\to\infty} \frac{1}{(2n+1)(2n+2)} = x^2 \lim_{n\to\infty} \frac{1}{4n^2} = 0$$
for any value of x. Thus, the series converges for any x.

<u>Introduction of power series</u> Suppose we do have a power series
$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$
that is defined for some x. We wish to determine a_n.

When $x = 0$ ..... we get $f(0) = a_0$ ..... So $\boxed{a_0 = f(0).}$

Suppose we can differentiate both sides to get
$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots$$

Then we get $f'(0) = a_1$ - - - - - so $\boxed{a_1 = f'(0).}$

Once again

$$f''(x) = 2a_2 + 2.3\,a_3 x + 3\cdot 4 a_4 x^2 + \dots$$

so $f''(0) = 2a_2$ - - - - - - - - so $\boxed{a_2 = \frac{1}{2} f''(0).}$

Then $f'''(x) = 2.3.a_3 + 2.3.4a_4 x + \dots$ so $\boxed{a_3 = \frac{1}{3!} f'''(0).}$

Based on that pattern, we expect a_n to be $\frac{1}{n!}$ times $f^{(n)}(0)$, the nth derivative evaluated at $x=0$.

<u>Example</u>. Consider $f(x) = e^x$. We try to write $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

We need $a_n = \frac{f^{(n)}(0)}{n!}$. But $f'(x) = e^x$, $f''(x) = e^x$, $f^{(n)}(x) = e^x$.

This gives $a_n = \frac{e^0}{n!} = \frac{1}{n!}$ and $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

<u>Example</u>. Consider $f(x) = \cos x$. Its derivatives are

$$f'(x) = -\sin x$$
$$f''(x) = -\cos x$$
$$f'''(x) = \sin x$$
$$f''''(x) = \cos x$$

and repeat in cycles of 4.

Thus
$$f(0) = \cos 0 = 1$$
$$f'(0) = -\sin 0 = 0$$
$$f''(0) = -\cos 0 = -1$$
$$f'''(0) = \sin 0 = 0$$
$$f''''(0) = \cos 0 = 1$$
and repeat in cycles of 4.

We get: $\cos x = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

an <u>even</u> function.

Theorem (Differentiation of power series)

Power series $\sum_{n=0}^{\infty} a_n x^n$ can be differentiated term by term.

Namely, if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for all $|x| < R$, then

$g(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$ also converges for $|x| < R$ and $g(x) = f'(x)$.

Warning The derivative of an infinite sum is not necessarily the sum of the derivatives. It only works for power series $\sum a_n x^n$.

Example. Take $f_n(x) = \frac{x^2}{(1+x^2)^n}$ for each $n \geq 0$, and let $f(x) = \sum_{n=0}^{\infty} f_n(x)$.

Then $f_n(x)$ is defined for all x, continuous / differentiable for all x.

We claim $f(x)$ is not continuous / differentiable. In fact,

$f(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n} = x^2 \sum_{n=0}^{\infty} \left(\frac{1}{1+x^2}\right)^n$ is a geometric series which converges when $\frac{1}{1+x^2} < 1$, namely when $1+x^2 > 1$ or $x \neq 0$.

Assuming $x \neq 0$, we get $f(x) = \dfrac{x^2}{1 - \frac{1}{1+x^2}} = \dfrac{x^2}{\frac{\cancel{1}+x^2-\cancel{1}}{1+x^2}} = 1 + x^2$

but $x = 0$ clearly gives $f(0) = 0$. Thus, f is not continuous at $x=0$.

Taylor series Given a function f that is infinitely differentiable at $x=0$, we define its N^{th} Taylor polynomial

$$T_N(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(N)}(0)}{N!}x^N$$

and its Taylor series

$$T(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

Example 1. (Exponential) When $f(x) = e^x$, the Taylor series is

$$T(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n.$$

We claim $f(x) = T(x)$ for all x. . . . . so that $\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$

Convergence: Ratio test gives $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{x^n} \frac{n!}{(n+1)!}\right|$

$$= \lim_{n\to\infty} \frac{|x|}{n+1} = 0$$

This proves convergence for all x. We can differentiate:

$$T(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \implies T'(x) = \sum_{n=0}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

$$\implies T'(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = T(x).$$

<u>$T(x) = e^x$:</u> $$\left(\frac{T(x)}{e^x}\right)' = \frac{T'(x)e^x - T(x)e^x}{e^{2x}} = \frac{T(x)e^x - T(x)e^x}{e^{2x}} = 0$$

so $$\frac{T(x)}{e^x} = \text{constant} = \frac{T(0)}{e^0} = \frac{1}{1} = 1 \quad \text{and} \quad T(x) = e^x.$$

<u>Example 2.</u> (Sine and cosine) Consider the Taylor series for sine/cosine:

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$g(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

We claim $f(x) = \sin x$ and $g(x) = \cos x$ for all x.

Convergence for all x follows by the ratio test. We can differentiate:

$$f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(2n+1)x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = g(x),$$

$$g'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2n)x^{2n-1} = \sum_{n=1}^{\infty} \frac{(-1)^n (2n)}{(2n)!} x^{2n-1}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!} x^{2n-1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} x^{2n+1} = -f(x).$$

This proves $f'(x) = g(x)$, $g'(x) = -f(x)$.

<u>$f(x) = \sin x$, $g(x) = \cos x$:</u> ~~Indeed~~ Indeed, consider

$$H(x) = (f(x) - \sin x)^2 + (g(x) - \cos x)^2.$$

Then

$$H'(x) = 2(f(x) - \sin x)(g(x) - \cos x) + 2(g(x) - \cos x)(-f(x) + \sin x)$$

$$\implies H'(x) = 0$$

$$\implies H(x) = \text{constant} = H(0) = (f(0) - \sin 0)^2 + (g(0) - \cos 0)^2 = 0$$

$$\implies (f(x) - \sin x)^2 + (g(x) - \cos x)^2 = 0 \quad \text{for all } x$$

$$\implies f(x) = \sin x, \quad g(x) = \cos x \quad \text{for all } x.$$

Euler's formula We know $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$.

If we use complex numbers to compute e^{ix} (for real x), $i = \sqrt{-1}$

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} \frac{(ix)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(ix)^{2n+1}}{(2n+1)!} \quad \ldots \text{even/odd}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

namely $$\boxed{e^{ix} = \cos x + i \sin x .}$$