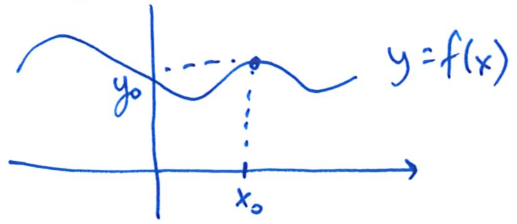
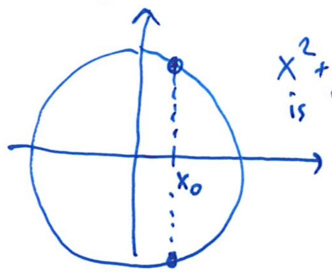
Function. A rule or a formula that assigns a unique value $f(x)$ to each admissible value of x, say $f(x)=x^2+1$.

Domain. The admissible values of x. One has to avoid zero denominators, square roots of negative numbers etc.

Range/Image. The possible values for $f(x)$. One typically writes $y=f(x)$ and plots the points (x,y) in the xy-plane.





$x^2+y^2=1$ is not the graph of a function

Example. We find the domain and range of $f(x)=\dfrac{2x-1}{x-2}$.

Domain ..... we cannot have $x=2$. Domain consists of $x \neq 2$.

Range ...... possible values for $f(x)$. This is easy to determine, if we can solve for x in terms of y.

Now,
$$y=\frac{2x-1}{x-2} \iff y(x-2)=2x-1 \iff yx-2y=2x-1$$
$$\iff yx-2x=2y-1 \iff x(y-2)=2y-1$$
$$\iff x=\frac{2y-1}{y-2}.$$

Based on this formula, we have the restriction $y \neq 2$. The range consists of all $y \neq 2$.

Example. We find the domain and range of $f(x)=\sqrt{\dfrac{1-x}{x}}$.

- For the domain ..... we need $x \neq 0$ and $\dfrac{1-x}{x} \geq 0$.

This gives $x>0$ and $1-x \geq 0$ $\rightsquigarrow$ $0<x\leq 1$

$\boxed{\text{OR}}$ $x<0$ and $1-x\le 0$ $\rightsquigarrow$ $1\le x<0$ which is absurd.

The domain consists of all $0<x\le 1$. This is the interval

$$(0,1] = \{\text{values of } x \text{ with } 0<x\le 1\}.$$

We similarly denote

$$(0,5) = \{x \text{ with } 0<x<5\}$$
$$[a,b) = \{x \text{ with } a\le x<b\}.$$

- For the range, we solve for x in terms of y. We get

$$y=\sqrt{\frac{1-x}{x}} \underset{\text{implies}}{\Longrightarrow} y^2=\frac{1-x}{x} \iff xy^2=1-x$$
$$\iff xy^2+x=1$$
$$\iff x(y^2+1)=1 \iff x=\frac{1}{y^2+1}.$$

In this case, we note that $y\ge 0$ because $y=\sqrt{\frac{1-x}{x}}$. That is the only restriction on y. The range is $[0,+\infty)$.

$\boxed{\text{Injective functions}}$ We say that f is injective or 1-1, if it maps distinct values of x to distinct values of $f(x)$. In other words, $\boxed{f(x_1)=f(x_2) \Longrightarrow x_1=x_2}$ ---- for injective f.

<u>Example.</u> We check $f(x)=\frac{3x-2}{5x-8}$ is injective.

Indeed, $f(x_1)=f(x_2) \Longrightarrow \dfrac{3x_1-2}{5x_1-8}=\dfrac{3x_2-2}{5x_2-8}$

$$\Longrightarrow (3x_1-2)(5x_2-8) = (3x_2-2)(5x_1-8)$$
$$\Longrightarrow \cancel{15x_1x_2}-24x_1-10x_2+\cancel{16} = \cancel{15x_1x_2}-24x_2-10x_1+\cancel{16}$$
$$\Longrightarrow 24x_1+10x_2 = 24x_2+10x_1$$
$$\Longrightarrow 14x_1 = 14x_2 \Longrightarrow x_1=x_2.$$

This proves f is injective.

Example. We check $f(x)=x^2$ is not injective.

In fact $f(2)=4$ and $f(-2)=4$, even though $2,-2$ are distinct.

Alternatively, one may argue that

$$f(x_1)=f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1^2 - x_2^2 = 0$$

$$\Rightarrow (x_1+x_2)(x_1-x_2) = 0$$

$$\Rightarrow x_1+x_2=0 \text{ or } x_1=x_2$$

$$\Rightarrow x_2=-x_1 \text{ or } x_2=+x_1.$$

This proves f is not injective.

$\boxed{\text{Surjective functions}}$ One says $f: X \to Y$ is surjective, if the range is all of Y. We typically write $f(x)=\frac{x+1}{x^2+1}$ as a function $f:\mathbb{R}\to\mathbb{R}$ without checking that the range is all of $\mathbb{R}$.

Example 1. Consider $f(x)=\frac{x}{x+1}$ as a function $f:[0,1]\to\mathbb{R}$.

Is that surjective? Is any value attained? What is the range?

As before, $y=\frac{x}{x+1} \Leftrightarrow y(x+1)=x \Leftrightarrow yx+y=x$

$$\Leftrightarrow y = x-yx \Leftrightarrow y=x(1-y) \Leftrightarrow x=\frac{y}{1-y}.$$

Because of the denominator, $y\neq 1$ so the range is not $\mathbb{R}$.

Example 2. We show $f(x)=\frac{x}{x+1}$ is surjective as a function $f:[0,1]\to[0,\frac{1}{2}]$

In other words, the values $0\le x\le 1$ are mapped to $0\le f(x)\le \frac{1}{2}$.

⊙ Suppose $0\le x\le 1$. Then $f(x)=\frac{x}{x+1}\geq 0$ and $f(x)=\frac{x}{x+1}\le \frac{1}{2}$.

The first part is clear and the second part follows

since $\frac{x}{x+1}\le \frac{1}{2} \Leftrightarrow 2x\le x+1 \Leftrightarrow x\le 1$

⊙ Suppose $0 \leq y \leq 1/2$. We need to check $0 \leq x \leq 1$.
To do this, we solve for x in terms of y as before.
We have $y = \frac{x}{x+1} \iff x = \frac{y}{1-y}$ .... as above (Example 1).
We are assuming $0 \leq y \leq 1/2$ and we need $0 \leq x \leq 1$.
In fact, $x = \frac{y}{1-y} \geq 0$ because $y \geq 0$ and $1-y > 0$ ✓

and $x = \frac{y}{1-y} \leq 1 \iff y \leq 1-y$
$\iff 2y \leq 1 \iff y \leq 1/2$. ✓

$\boxed{\text{Bijective functions}}$ We say $f: X \to Y$ is bijective, if f is both injective & surjective. In this case, every value in Y is attained and attained exactly once.

Theorem 1. Quadratic functions

Consider $f(x) = ax^2 + bx + c$ with $a \neq 0$.

Case 1. If the discriminant $\Delta = b^2 - 4ac$ is non-negative, then $f(x)$ has real roots $x_1 = \frac{-b-\sqrt{\Delta}}{2a}$ and $x_2 = \frac{-b+\sqrt{\Delta}}{2a}$. In addition, one may factor $f(x) = a(x-x_1)(x-x_2)$.

Case 2. If the discriminant Δ is negative, then $f(x)$ has no real roots and it cannot be factored.

Proof. One can write $\frac{f(x)}{a} = x^2 + \frac{b}{a}x + \frac{c}{a}$ and complete the square to get

$$\frac{f(x)}{a} = \underbrace{x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2}_{\text{complete square}} + \frac{c}{a} - \left(\frac{b}{2a}\right)^2.$$

This gives

$$\frac{f(x)}{a} = \left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2} = \left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}.$$

Case 1 ...... $\Delta \geq 0$. Then

$$\frac{f(x)}{a} = \left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{\Delta}}{2a}\right)^2 = \left(x + \frac{b}{2a} + \frac{\sqrt{\Delta}}{2a}\right)\left(x + \frac{b}{2a} - \frac{\sqrt{\Delta}}{2a}\right)$$
$$= (x - x_1)(x - x_2).$$

Case 2 .... $\Delta < 0$. Then $\frac{f(x)}{a} \geq -\frac{\Delta}{4a^2} > 0$ and f has no roots. $\blacksquare$

Example 1. (Factorisation and range of quadratics) Let $f(x) = 2x^2 - 7x + 3$.

Then $\Delta = b^2 - 4ac = 49 - 24 = 25 \geq 0$

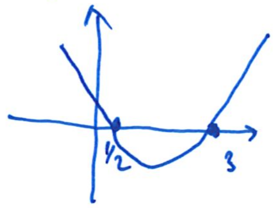
so we get roots $x_1 = \frac{-b-\sqrt{\Delta}}{2a} = \frac{7-5}{4} = 1/2$, $x_2 = \frac{7+5}{4} = 3$

and $f(x) = a(x-x_1)(x-x_2) = 2(x - 1/2)(x-3)$.

- To find the range, we solve $y = 2x^2 - 7x + 3$ for x. That's quadratic, $2x^2 - 7x + 3 - y = 0$ and its discriminant is $\Delta' = b^2 - 4ac = 49 - 8(3-y) = 25 + 8y$.

If $25+8y \geq 0$ or $8y \geq -25$ or $y \geq -25/8$, the equation has roots. In particular, the value $y = 2x^2-7x+3$ is attained by some x.
If $25+8y<0$ or $y<-25/8$, then $y = 2x^2-7x+3$ has no solutions and the value y is not attained.

The range of f is $[-25/8, +\infty)$.



Example 2. Let $f(x) = -2x^2+3x-4$.
To find the range, we start with $y = -2x^2+3x-4$ and get $2x^2-3x+4+y=0 \Rightarrow \Delta = b^2-4ac = 9-8(4+y) = -23-8y$.
We have solutions only when $-23-8y \geq 0$, namely $8y \leq -23$ or $y \leq -23/8$. The range is $(-\infty, -23/8]$.

Factorisation of polynomials

Theorem 2. (Factor theorem) Suppose $f(x)$ is a polynomial that has $x=\alpha$ as a root. In that case, $x-\alpha$ is a factor of $f(x)$.
In other words, $f(x) = (x-\alpha)\,g(x)$ for some polynomial $g(x)$.

Theorem 3. (Rational root theorem) Suppose $f(x)$ is an INTEGER polynomial $f(x) = a_n x^n + \dots + a_2 x^2 + a_1 x + a_0$. If $f(x)$ has a rational root, then the root has the form p/q with p, q relatively prime (no common factor other than 1)
p a divisor of a_0
and q a divisor of a_n.

Example (Factorisation of cubic polynomials) Let $f(x) = x^3+x^2-3x+1$.
We look for roots and factors.

- Possible rational roots: p/q with p dividing 1, q dividing 1
  so $p = \pm 1$, $q = \pm 1$
  possibilities are ± 1

⊙ Let's check: $f(1) = 1+1-3+1 = 0$ so $x=1$ is a root

$f(-1) = -1+1+3+1 = 4$ so $x=-1$ is not a root.

Since $x=1$ is a root of $f(x)$, $x-1$ is a factor of $f(x)$.

⊙ Thus $f(x) = (x-1)\,g(x)$ for some polynomial $g(x)$.

We need to determine $g(x) = \dfrac{f(x)}{x-1}$ and proceed.

Polynomial division is similar to division of integers.

$$
\begin{array}{r}
77 \\
3 \,\overline{)\, 231} \\
\underline{21} \\
21 \\
\underline{21} \\
0
\end{array}
\qquad\qquad
\begin{array}{r}
x^2+2x-1 \\
x-1 \,\overline{)\, x^3+x^2-3x+1} \\
\underline{x^3-x^2} \\
2x^2-3x+1 \\
\underline{2x^2-2x} \\
-x+1 \\
\underline{-x+1} \\
0
\end{array}
$$

In this case, the remainder is zero and we have $f(x) = (x-1)(x^2+2x-1)$.

⊙ Let's worry about the quadratic now. Roots of x^2+2x-1.

Discriminant $\Delta = b^2-4ac = 4+4 = 8 \geq 0$

We get roots $x_1 = \dfrac{-b-\sqrt{\Delta}}{2a} = \dfrac{-2-\sqrt{8}}{2} = -1-\sqrt{2}$

and $x_2 = \dfrac{-2+\sqrt{8}}{2} = -1+\sqrt{2}$.

These roots are irrational! In any case, we get

$$f(x) = (x-1)(x-x_1)(x-x_2).$$

<u>Example 2.</u> We factor $g(x) = \frac{1}{5}x^3 - \frac{1}{2}x^2 + \frac{2}{5}x - \frac{1}{10}$.

Let's clear denominators: $10\,g(x) = 2x^3-5x^2+4x-1$.

This is now an INTEGER polynomial.

⊙ Possible roots: p/q with p dividing -1, q dividing 2

$p/q = \pm 1, \pm 1/2$

We check $10\,g(1) = 2-5+4-1 = 0$ so $x=1$ is a root

$10\,g(-1) = -2-5-4-1 < 0$ so $x=-1$ not a root

$10\,g(1/2) = 2/8 - 5/4 + 2 - 1 = 0$ so $x = 1/2$ is a root

$10\,g(-1/2) = -2/8 - 5/4 - 2 - 1 < 0$ so $x=-1/2$ not a root.

⊙ This means $x-1$, $x-1/2$ are factors.

We get $10\,g(x) = (x-1)\cdot h(x)$ for some polynomial $h(x)$.

Thus $10\,g(x) = (x-1)\cdot\underline{(2x^2-3x+1)}$

$= (x-1)(2x-1)(x-1)$

$= (x-1)^2(2x-1)$.

$$
\begin{array}{r}
2x^2 - 3x + 1 \\
x-1 \;\big)\; 2x^3 - 5x^2 + 4x - 1 \\
2x^3 - 2x^2 \\
\hline
-3x^2 + 4x - 1 \\
-3x^2 + 3x \\
\hline
x - 1 \\
x - 1 \\
\hline
0
\end{array}
$$

One needs to divide the polynomials to find that $x=1$ is a double root.

<u>Example 3.</u> Let $f(x) = 2x^3 - 9x^2 + 10x - 3$.

Possible roots: p/q with p dividing 3, q dividing 2

$p/q = \pm 1, \pm 1/2, \pm 3, \pm 3/2$.

We check: $x=1$, $x=3$, $x=1/2$ are the only roots.

This already implies $x-1$, $x-3$, $x-1/2$ are factors

so $f(x) = 2(x-1)(x-3)(x-1/2)$.

$\boxed{\text{Proof of rational root theorem}}$ Write $f(x) = a_n x^n + \dots + a_1 x + a_0$.

We look for rational roots $x = p/q$ with p, q relatively prime.

We need $0 = f(p/q) = a_n \frac{p^n}{q^n} + a_{n-1}\frac{p^{n-1}}{q^{n-1}} + \dots + a_1 \frac{p}{q} + a_0$

$\Longrightarrow \quad 0 = a_n p^n + a_{n-1} p^{n-1} q + \dots + a_1 p q^{n-1} + a_0 q^n$.

All the terms have a p factor except for $a_0 q^n$.

Now p divides $a_n p^n + a_{n-1} p^{n-1} q + \dots + a_1 p q^{n-1} = -a_0 q^n$

so p divides $a_0 q^n$ so p divides a_0.

Similarly, q divides $a_{n-1} p^{n-1} q + \dots + a_1 p q^{n-1} + a_0 q^n = -a_n p^n$

so q divides $a_n p^n$ so q divides a_n. $\blacksquare$

$\boxed{\text{Proof of factor theorem.}}$ Suppose $f(x)$ is a polynomial and $x = \alpha$ is a root. We need to check $x - \alpha$ is a factor.

Divide $f(x)$ by $x - \alpha$.

The division proceeds until the remainder is a constant (has no x in it).

$$\begin{array}{r|l} & a_n x^{n-1} + \dots \\ \hline x - \alpha & f(x) \\ & a_n x^n + \dots + a_1 x + a_0 \\ & x^{n-1} \\ & \vdots \\ & r \end{array}$$

We get $f(x) = Q(x)(x - \alpha) + r$.

Since $f(\alpha) = 0$, this gives

$$\begin{array}{r|l} & Q(x) \\ \hline x - \alpha & f(x) \\ & \vdots \\ & r \end{array}$$

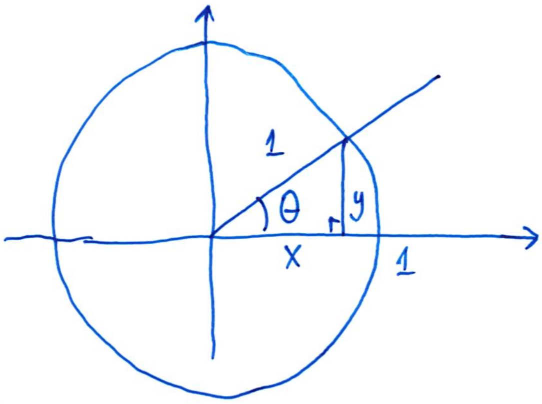
$$0 = f(\alpha) = \cancel{Q(\alpha)(\alpha - \alpha)} + r$$

so $r = 0$, the remainder is zero. $\blacksquare$

Trigonometric functions

Those are defined in terms of an angle θ measured in radians. Here, 2π rad $= 360°$ so π rad $= 180°$.

Consider the triangle that we get for this angle θ. We define



$$\sin\theta = \frac{y}{1}, \quad \cos\theta = \frac{x}{1}, \quad \tan\theta = \frac{y}{x}$$

$$\csc\theta = \frac{1}{y}, \quad \sec\theta = \frac{1}{x}, \quad \cot\theta = \frac{x}{y}.$$

The main functions are $y = \sin\theta$ and $x = \cos\theta$.

Then $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$, $\csc\theta = \dfrac{1}{\sin\theta}$.

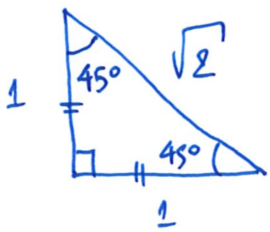
The most important identities are:

$x^2 + y^2 = 1$ ... by Pythagoras' theorem ... so $\boxed{\sin^2\theta + \cos^2\theta = 1}$.

Dividing by $\cos^2\theta$ gives .... $\boxed{\tan^2\theta + 1 = \sec^2\theta}$

Dividing by $\sin^2\theta$ gives .... $\boxed{1 + \cot^2\theta = \csc^2\theta}$

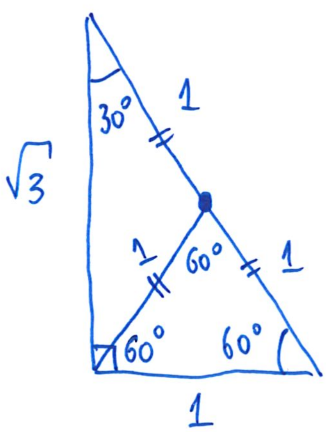
Some special values of trigonometric functions



$$\sin 45° = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\cos 45° = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\tan 45° = \tan \pi/4 = 1$$



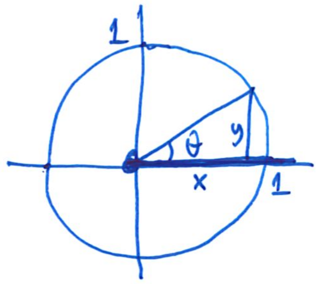
$\sin 30° = \sin\frac{\pi}{6} = 1/2$, $\sin 60° = \sqrt{3}/2$

$\cos 30° = \cos \pi/6 = \sqrt{3}/2$, $\cos 60° = 1/2$

$\tan 30° = \tan \pi/6 = 1/\sqrt{3}$, $\tan 60° = \sqrt{3}$

⊙ Graphs of sine and cosine

Recall that $x = \cos\theta$ and $y = \sin\theta$.



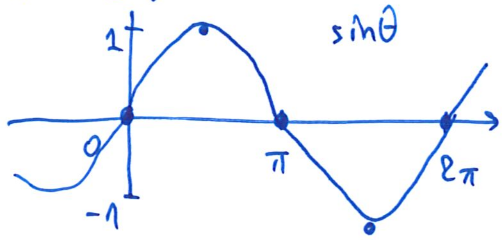
Focusing on the y-coordinate sine,
we look at various angles $0 \le \theta \le 2\pi$.

When $\theta = 0$, the y-coordinate is $\sin\theta = 0$

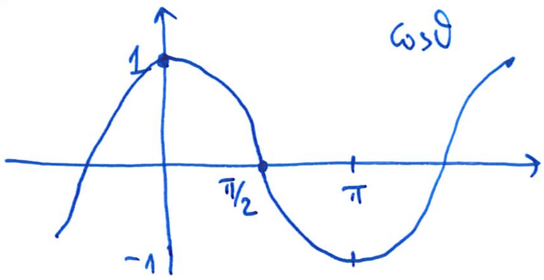
When $\theta = \pi/2$, the y-coordinate is $\sin \pi/2 = 1$

When $\theta = \pi$, the y-coordinate is $\sin\pi = 0$.

The graph looks roughly like



⊙ For cosine, the same approach works,
but we need to keep track of the x-coordinate.



Addition formulas for sine and cosine

Given two angles x and y, one has

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

Example 1. Consider two angles θ_1, θ_2 with sum $\theta_1 + \theta_2 = \pi$.

Then $\sin\theta_2 = \sin(\pi - \theta_1) = \overset{0}{\cancel{\sin\pi}} \cos\theta_1 - \overset{-1}{\cancel{\cos\pi}} \sin\theta_1 = \sin\theta_1$

and $\cos\theta_2 = \cos(\pi - \theta_1) = \overset{-1}{\cancel{\cos\pi}} \cos\theta_1 + \overset{0}{\cancel{\sin\pi}} \sin\theta_1 = -\cos\theta_1$

Example 2. Consider two angles θ_1, θ_2 with sum $\theta_1 + \theta_2 = \pi/2$.

Then $\sin\theta_2 = \sin(\pi/2 - \theta_1) = \overset{1}{\cancel{\sin\pi/2}} \cos\theta_1 - \overset{0}{\cancel{\cos\pi/2}} \sin\theta_1 = \cos\theta_1$

$\cos\theta_2 = \cos(\pi/2 - \theta_1) = \overset{0}{\cancel{\cos\pi/2}} \cos\theta_1 + \overset{1}{\cancel{\sin\pi/2}} \sin\theta_1 = \sin\theta_1$.

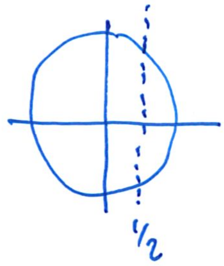
Example 3. We find the angles $0 \le \theta \le 2\pi$ with $2\cos^2\theta + 5\cos\theta = 3$.

This is $2x^2 + 5x - 3 = 0$ with $x = \cos\theta$.

We have $$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-5 \pm \sqrt{25+24}}{4} = \frac{-5 \pm 7}{4} \begin{cases} \to x_1 = 1/2 \\ \to x_2 = -3. \end{cases}$$

For $\cos\theta = -3$ .... we get no solutions.

For $\cos\theta = 1/2$ .... we get two solutions.



One of them is $\theta = \pi/3 = 60°$

The other one is $2\pi - \pi/3 = 5\pi/3$.

Exponential functions

Those have the form $f(x) = a^x$ for some $a > 0$.

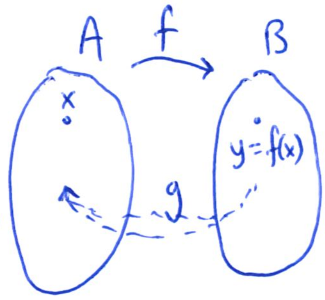
We define powers a^x in the usual way.

- When x is a positive integer, we have $a^1 = a$, $a^2 = a \cdot a$ etc.
  Namely, a^x is just a product of x copies of a.
- When x is a rational number $x = m/n$ with m, n positive,
  $a^x = a^{m/n} = (a^{1/n})^m$ with $a^{1/n}$ being the nth root of a.
  For instance, $a^{1/2} = \sqrt{a}$ because $(a^{1/2})^2 = a^{2/2} = a^1 = a$.
  This is only defined when $a \geq 0$.
- When x is an arbitrary number, we define a^x by approximations.
  For instance, $a^{\sqrt{2}}$ is computed by noting that $\sqrt{2} = 1.4142135\ldots$
  and then looking at a^1, $a^{1.4} = a^{14/10}$, $a^{1.41} = a^{141/100}$ etc.
  This gives approximations of $a^{\sqrt{2}}$ to any degree of accuracy.
- Negative powers are defined, as usual, by $a^{-x} = \dfrac{1}{a^x}$.

We then have the following properties ....

$$\boxed{a^x \cdot a^y = a^{x+y}, \quad a^0 = 1 \text{ by definition}, \quad \frac{a^x}{a^y} = a^{x-y} \quad \text{and} \quad (a^x)^y = a^{xy}.}$$

Inverse functions



Suppose $f : A \to B$ is <u>bijective</u>.

Then the equation $y = f(x)$ has a unique solution x for each y in B. We denote this value by $x = f^{-1}(y)$ and we call $g = f^{-1}$ the inverse function $g : B \to A$.

Note that g has two properties:

$$g(f(x)) = g(y) = f^{-1}(y) = x \quad \text{for all } x \text{ in } A$$

$$f(g(y)) = f(f^{-1}(y)) = f(x) = y \quad \text{for all } y \text{ in } B.$$

Logarithmic functions Consider the exponential function $f(x) = a^x$ with $a > 0$. This is <u>injective</u> whenever $a \neq 1$ because

$$f(x_1) = f(x_2) \Rightarrow a^{x_1} = a^{x_2} \Rightarrow a^{x_1 - x_2} = 1 \Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2 .$$

It is also <u>surjective</u> as a function $f : \mathbb{R} \to (0, \infty)$.
We can then define $\log_a$ as the inverse function.
It has the following properties:

① $\log_a a^x = x$	② $a^{\log_a x} = x$
③ $\log_a (x \cdot y) = \log_a x + \log_a y$	④ $\log_a x^r = r \log_a x$
⑤ $\log_a \dfrac{x}{y} = \log_a x - \log_a y$	⑥ $\log_a 1 = 0$.

Note that ① and ② follow by definition ... $\cancel{\log_a a}^{x} = x$.

To prove ③, write $x \cdot y = a^{\log_a x} \cdot a^{\log_a y} = a^{\log_a x + \log_a y}$

so $\log_a (x \cdot y) = \cancel{\log_a a}^{\log_a x + \log_a y}$ proving ③.

To prove ④, we write

$$x^r = \left(a^{\log_a x}\right)^r = a^{r \log_a x}$$

$$\Rightarrow \log_a x^r = \cancel{\log_a a}^{r \log_a x} = r \log_a x .$$

Finally, ⑤ follows since $\log_a \frac{x}{y} = \log_a (x y^{-1}) = \log_a x + \log_a y^{-1}$ by ③

$= \log_a x - \log_a y$ by ④.

Also, ⑥ follows since $\log_a 1 = \cancel{\log_a a}^{0} = 0$.

⊘ Note that $\log_a x$ is <u>only</u> defined when $x > 0$.

<u>Example.</u> We find the inverse function f^{-1} when $f : (1/3, \infty) \to \mathbb{R}$ is given by $f(x) = 3 - \log_2 (3x - 1)$. We need $3x - 1 > 0$ because of the logarithm. We proceed to solve $y = f(x)$ in terms of x.
In this case

$$y = 3 - \log_2(3x-1) \Rightarrow \log_2(3x-1) = 3 - y$$

$$\Rightarrow 2^{\log_2(3x-1)} = 2^{3-y} \Rightarrow 3x - 1 = 2^{3-y}$$

$$\Rightarrow 3x = 2^{3-y} + 1 \Rightarrow x = \frac{2^{3-y}+1}{3}.$$

The inverse function is then $f^{-1}(y) = \dfrac{2^{3-y}+1}{3}$. Its domain is $\mathbb{R}$.

Example. We find the inverse function f^{-1} when $f : [1,\infty) \to [1,\infty)$ is defined by $f(x) = 2x^2 - 4x + 3$. We solve $y = f(x)$ for x to get

$$y = 2x^2 - 4x + 3 \Rightarrow 2x^2 - 4x + (3-y) = 0$$

$$\Rightarrow x = \frac{4 \pm \sqrt{16 - 8(3-y)}}{2 \cdot 2} = \frac{4 \pm \sqrt{8y-8}}{4}.$$

Note that $y \geq 1$ by assumption, so $\sqrt{8y-8}$ is defined.

Also, we need to have $x \geq 1$ by assumption. This gives

$$x = 1 \pm \frac{\sqrt{8y-8}}{4} \Rightarrow x = 1 + \frac{\sqrt{8y-8}}{4} \Rightarrow f^{-1}(y) = 1 + \frac{\sqrt{8y-8}}{4},$$

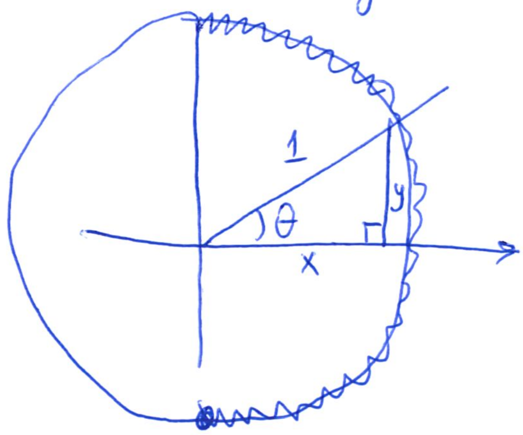
a unique solution.

Inverse trigonometric functions

We define the inverse functions for sine, cosine and tangent.

Recall that $\sin\theta = y$, $\cos\theta = x$, $\tan\theta = \dfrac{y}{x}$.

None of these is bijective, though.



Now, $\sin : [-\pi/2, \pi/2] \to [-1,1]$ is bijective

$\cos : [0, \pi] \to [-1,1]$ is bijective

and $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is bijective.

We ~~have~~ define

$$\sin^{-1} : [-1,1] \to [-\pi/2, \pi/2]$$

$$\cos^{-1} : [-1,1] \to [0, \pi]$$

$$\tan^{-1} : \mathbb{R} \to (-\pi/2, \pi/2)$$

... as the inverses.

Thus, $\sin^{-1} \frac{1}{2}$ = angle whose sine is $\frac{1}{2} = \pi/6$

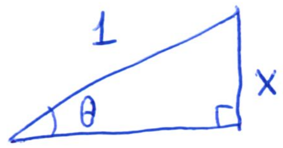
$\tan^{-1} 1$ = angle whose tangent is 1 = $\pi/4$

and $\sin^{-1} 2$ is not defined.

Example 1. We simplify $\tan(\sin^{-1} x)$.

Let $\theta = \sin^{-1} x$ for convenience. This is an angle with sine x.

⊙ When $x \geq 0$, this is depicted in the figure.



We need $\tan(\sin^{-1} x) = \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{\sqrt{1-x^2}}$

by Pythagoras' theorem.

⊙ To verify the formula when $x \leq 0$, we note that

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta.$$

Thus $\tan(\sin^{-1}(-x)) = -\tan(\sin^{-1} x)$ and we get

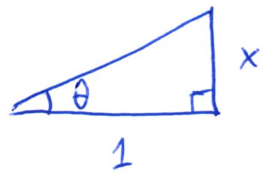
$$\tan(\sin^{-1} x) = -\tan(\sin^{-1}(-x)) = -\frac{-x}{\sqrt{1-(-x)^2}} \quad \text{by above}$$

so $\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}}$ in any case.

Example 2. We simplify $\sin(\tan^{-1} x)$ ... for any x.

Let $\theta = \tan^{-1} x$. Then θ = angle whose tangent is x.

⊙ When $x \geq 0$, we get such an angle as before.



Then $\sin(\tan^{-1} x) = \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{\sqrt{1+x^2}}$.

⊙ When $x \leq 0$, we establish a similar formula by noting that

$$\sin(\tan^{-1} x) = -\sin(\tan^{-1}(-x)) = -\frac{(-x)}{\sqrt{1+(-x)^2}} \quad \text{by above.}$$

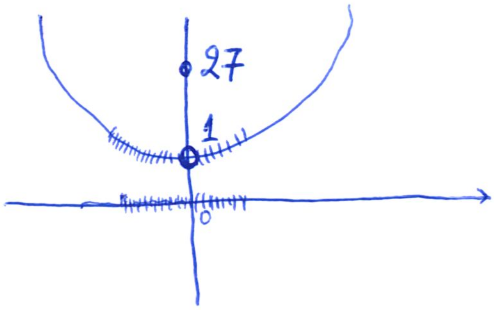
Thus $\sin(\tan^{-1} x) = \frac{x}{\sqrt{1+x^2}}$ in any case.

Introduction to limits We wish to study the values $f(x)$ as x approaches a fixed number x_0. We write $x \to x_0$ for simplicity. If the values $f(x)$ are also approaching a certain value L, we write $f(x) \to L$ as $x \to x_0$,

we call L the limit of $f(x)$ as $x \to x_0$

and we also write $\lim_{x \to x_0} f(x) = L$.

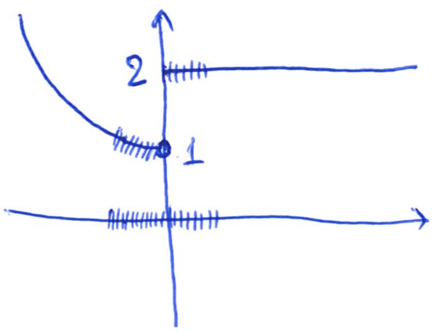
Example 1.



Consider $f(x) = \begin{cases} x^2+1 & \text{if } x \neq 0 \\ 27 & \text{if } x = 0 \end{cases}$.

In this case, as $x \to 0$ the corresponding values $f(x)$ are approaching 1. Note that $f(0)$ is irrelevant since $x \to 0$ means that x is approaching 0.

Example 2.



Consider $f(x) = \begin{cases} x^2+1 & \text{if } x \leq 0 \\ 2 & \text{if } x > 0 \end{cases}$.

In this case, $x \to 0$ refers to values x near 0 and the corresponding values $f(x)$ are approaching the value 1 from the left and 2 from the right. The limit does not exist.

Formal definition of limits

We wish to define $\lim_{x \to x_0} f(x) = L$ whenever the $f(x)$ values approach L as the x values approach x_0. Note that $|7-3| = |3-7|$ measures the distance between 3 and 7. We will keep track of the $|x-x_0|$ distance and also $|f(x)-L|$.

Definition (ε-δ definition of limits) We say that $\lim_{x \to x_0} f(x) = L$ if, given any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 \neq |x-x_0| < \delta \implies |f(x)-L| < \varepsilon .$$

- In practice, one starts with the assumption $0 \neq |x-x_0| < \delta$ and tries to estimate $|f(x)-L|$. The choice of δ becomes apparent later.

Example 1. (Constant functions) Suppose $f(x) = a$ for all x.
We claim that $\lim_{x \to x_0} f(x) = a$. We use the definition to prove this.
Let $\varepsilon > 0$ be given. Assuming that $0 \neq |x-x_0| < \delta$, we get

$$|f(x) - L| = |a-a| = 0 < \varepsilon \quad \text{for any choice of } \delta!$$

We may take $\delta = 1$, for instance.

Example 2. (Linear functions) Suppose $f(x) = ax+b$ with $a \neq 0$.
We claim that $\lim_{x \to x_0} f(x) = f(x_0) = ax_0 + b$ in this case.
Let $\varepsilon > 0$ be given. If $0 \neq |x-x_0| < \delta$, then

$$|f(x)-L| = |ax+b-(ax_0+b)| = |ax-ax_0| = |a| \cdot |x-x_0| < |a| \cdot \delta .$$

We choose $|a| \cdot \delta = \varepsilon$ or $\delta = \varepsilon/|a|$. This gives $|f(x)-L| < \varepsilon$, as needed.

(*) **Example 3.** (Piecewise linear) Suppose $f(x) = \begin{cases} 12x+6, & \text{if } x \leq 1 \\ 7x+11, & \text{if } x > 1 \end{cases}$.

We claim that $\lim_{x \to 1} f(x) = 18$ --- (obtained by taking $x=1$).
Let $\varepsilon > 0$ be given. If $0 \neq |x - \overset{1}{\cancel{x_0}}| < \delta$, then

$$|f(x)-L| = \begin{cases} |12x-12| & \text{if } x \leq 1 \\ |7x-7| & \text{if } x > 1 \end{cases} = \begin{cases} 12|x-1| & \text{if } x \leq 1 \\ 7|x-1| & \text{if } x > 1 \end{cases} \leq 12|x-1| < 12\delta ;$$

If we pick $12\delta = \varepsilon$ or $\delta = \varepsilon/12$, we get $|f(x)-L| < 12\delta = \varepsilon$, as needed.

Useful inequalities ① $|x-x_0|<\delta \iff -\delta < x-x_0 < \delta$

$\iff x_0-\delta < x < x_0+\delta$

② Triangle inequality $|x+y| \le |x|+|y|$ for all x,y.

This follows by squaring both sides:

$$|x+y|^2 = (x+y)^2 = x^2+y^2+2xy = |x|^2+|y|^2+2xy \le |x|^2+|y|^2+2|x|\cdot|y|$$

so that $|x+y|^2 \le (|x|+|y|)^2$.

(*) **Example 4.** (Quadratic) Consider $f(x)=3x^2+2x+5$. We show that $\lim_{x\to 2} f(x) = 3\cdot 2^2+2\cdot 2+5 = 21$. Let $\varepsilon>0$ be given. If $|x-2|<\delta$, then $|f(x)-21| = |3x^2+2x+5-21| = |3x^2+2x-16|$

$$= \underset{\sim\sim\sim}{|x-2|}\cdot\underset{\sim\sim\sim}{|3x+8|}.$$

For the first factor, we know $|x-2|<\delta$.

For the second factor, we are going to estimate $3x+8$.

<u>It generally helps to assume $\delta \le 1$</u>. Then $|x-2|<\delta\le 1$

so $-1 < x-2 < 1$ so $1<x<3$ so $3<3x<9$

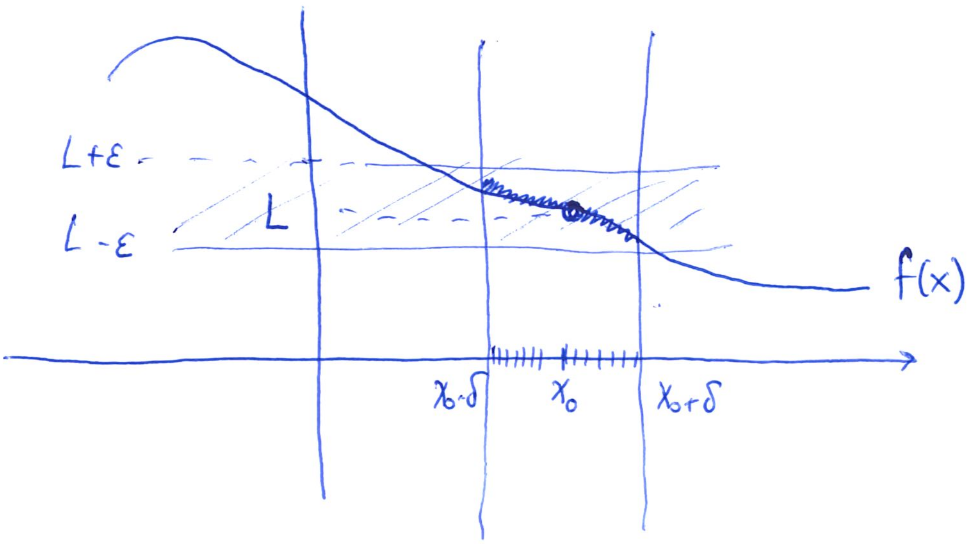
so $11 < 3x+8 < 17$.

This gives $|3x+8|<17$

and $|x-2|<\delta$ by above

so $|f(x)-21| = |x-2|\cdot|3x+8| < 17\delta \le \varepsilon$,

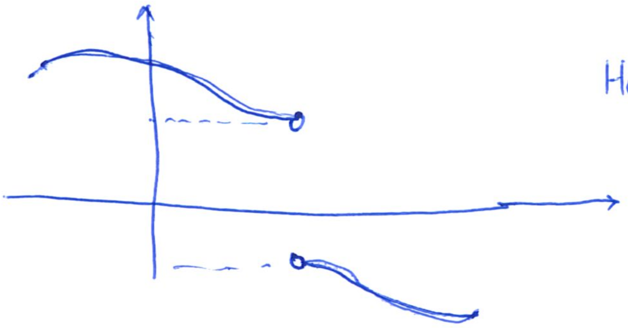
provided that $\delta\le 1$ and $\delta \le \varepsilon/17$. We may take $\delta = \min\{1, \frac{\varepsilon}{17}\}$.



$|x-x_0|<\delta$

$|f(x)-L|<\varepsilon$

for any given $\varepsilon>0$



Here $\lim_{x \to x_0} f(x)$ does not exis

Theorem 1. (Limits of sums and products)

The limit of a sum is the sum of the limits

$$\lim_{x \to x_0} [f(x) + g(x)] = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x).$$

The limit of a product is the product of the limits

$$\lim_{x \to x_0} [f(x) \cdot g(x)] = \lim_{x \to x_0} f(x) \cdot \lim_{x \to x_0} g(x).$$

Proof. We verify the ε-δ definition. For sums, we have

$\lim_{x \to x_0} f(x) = L$ ... so given $\varepsilon > 0$ there exists $\delta_1 > 0$ such that

$$0 \neq |x - x_0| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon$$

$\lim_{x \to x_0} g(x) = M$ .... so given $\varepsilon > 0$ there exists $\delta_2 > 0$ such that

$$0 \neq |x - x_0| < \delta_2 \Rightarrow |g(x) - M| < \varepsilon.$$

Let's show $\lim_{x \to x_0} [f(x) + g(x)] = L + M$. Let $\varepsilon > 0$ be given.

Then by above, there exist $\delta_1, \delta_2 > 0$ such that

$$0 \neq |x - x_0| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon/2$$

$$0 \neq |x - x_0| < \delta_2 \Rightarrow |g(x) - M| < \varepsilon/2.$$

Then $\delta = \min\{\delta_1, \delta_2\}$ gives

$$
\begin{aligned}
0 \neq |x - x_0| < \delta &\Rightarrow |f(x) - L| < \varepsilon/2 \text{ and } |g(x) - M| < \varepsilon/2 \\
&\Rightarrow |f(x) + g(x) - (L + M)| \\
&\quad = |f(x) - L + g(x) - M| \\
&\quad \leq |f(x) - L| + |g(x) - M| < \varepsilon.
\end{aligned}
$$

In particular, $\lim_{x \to x_0} [f(x) + g(x)] = L + M$.

⊙ For products, we have to show $\lim_{x \to x_0} f(x) \cdot g(x) = L \cdot M$.

Let $\varepsilon > 0$ be given. We need to estimate

$$|f(x)g(x) - LM| = |f(x)g(x) - Lg(x) + Lg(x) - LM|$$
$$\leq |g(x)| \cdot \underbrace{|f(x) - L|}_{\text{small}} + |L| \cdot \underbrace{|g(x) - M|}_{\text{small}}.$$

First of all, we estimate

$|g(x)| \leq |g(x) - M| + |M| \leq 1 + |M|$ by noting that given $\varepsilon = 1$ there exists $\delta_3 > 0$: $|x - x_0| < \delta_3 \Rightarrow |g(x) - M| \leq 1$.

We can then estimate

$$|g(x)| \cdot |f(x) - L| < (1 + |M|) \cdot |f(x) - L| < \cancel{(1+|M|)} \cdot \frac{\varepsilon}{2\cancel{(1+|M|)}} = \varepsilon/2$$

and $$|L| \cdot |g(x) - M| < (1 + |L|) \cdot |g(x) - M| < \cancel{(1+|L|)} \cdot \frac{\varepsilon}{2\cancel{(1+|L|)}} = \varepsilon/2.$$

Combining these estimates gives $|f(x)g(x) - LM| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. ▨

Theorem 2. (Limits of polynomials). If $f(x)$ is a polynomial, then $\lim_{x \to x_0} f(x) = f(x_0)$. That is, we can compute limits by taking $x = x_0$.

Proof. We assume $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$.

We can prove statements for polynomials by induction.

① We check the statement when $n = 1$.

$$\boxed{n=1} \to \boxed{n=2} \to \boxed{n=3}$$

② We assume the statement for <u>some</u> n, say $n = k$ and prove the statement for $n = k+1$ as well.

These two steps establish the statement for all n.

In our case, ① refers to $f(x) = a_0 + a_1 x$. Then

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} (a_0 + a_1 x) = \lim_{x \to x_0} a_0 + \lim_{x \to x_0} a_1 x = a_0 + a_1 x_0$$

by Example 2 for linear functions. Thus $\lim_{x \to x_0} f(x) = f(x_0)$.

② We assume $\lim_{x \to x_0} (a_0 + a_1 x + \dots + a_k x^k) = a_0 + a_1 x_0 + \dots + a_k x_0^k$.

We need $\lim\limits_{x \to x_0} \left(a_0 + a_1 x + \dots + a_k x^k + a_{k+1} x^{k+1} \right) = a_0 + a_1 x_0 + \dots + a_{k+1} x_0^{k+1}$.

In fact, $\lim\limits_{x \to x_0} \left(a_0 + a_1 x + \dots + a_k x^k + a_{k+1} x^{k+1} \right)$

$$= \lim_{x \to x_0} \left(a_0 + a_1 x + \dots + a_k x^k \right) + \lim_{x \to x_0} \left(a_{k+1} x^{k+1} \right) \quad \text{by Theorem 1}$$

$$= a_0 + a_1 x_0 + \dots + a_k x_0^k + \lim_{x \to x_0} \underbrace{a_{k+1} x^k}_{} \cdot \underbrace{x}_{}$$

$$= a_0 + a_1 x_0 + \dots + a_k x_0^k + a_{k+1} x_0^k \cdot x_0$$

by Example 2 and the statement for polynomials of degree k. ▨

<u>Example 5.</u> (Simple) $\lim\limits_{x \to 2} (3x^2 - 4x + 1) = 3 \cdot 2^2 - 4 \cdot 2 + 1 = 5$.

<u>Example 6.</u> (Division of polynomials) We compute $\lim\limits_{x \to 1} \frac{x^3 - 4x^2 + 3}{x - 1}$.

Note that the denominator becomes 0 when $x = 1$. We are interested in the limit as $x \to 1$. In this case, the numerator also becomes 0, so it has $x-1$ as a factor and the fraction can be simplified. We get

$$\lim_{x \to 1} \frac{x^3 - 4x^2 + 3}{x - 1} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x^2 - 3x - 3)}{\cancel{x-1}}$$

$$= \lim_{x \to 1} (x^2 - 3x - 3)$$

which is a limit of a polynomial

$$= 1^2 - 3 \cdot 1 - 3$$

$$= -5.$$

$$\begin{array}{r|l} & x^2 - 3x - 3 \\ \hline x - 1 & x^3 - 4x^2 + 3 \\ & x^3 - x^2 \\ \hline & -3x^2 + 3 \\ & -3x^2 + 3x \\ \hline & -3x + 3 \\ & -3x + 3 \\ \hline & 0 \end{array}$$

Theorem 3. (Limits of quotients and rational functions)

- If $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M \neq 0$, then $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{L}{M}$.
- If $f(x)$ is a polynomial or $f(x) = \frac{P(x)}{Q(x)}$ is a quotient of polynomials, then $\lim_{x\to x_0} f(x) = f(x_0)$.

Proof. For the first part, we show that $\lim_{x\to x_0} \frac{1}{g(x)} = \frac{1}{M}$.

Once we know this, then $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} f(x) \cdot \frac{1}{g(x)} = L \cdot \frac{1}{M} = L/M$.

Let $\varepsilon > 0$ be given. We estimate

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \left| \frac{M - g(x)}{g(x)\, M} \right| = \frac{|g(x) - M|}{|g(x)| \cdot |M|}.$$

For the denominator, $\cancel{|g(x)| = |g(x) - M + M| \le \underbrace{|g(x) - M|}_{\text{small}} + |M| < 1M}$

$$|M| = |M - g(x) + g(x)| \le \underbrace{|M - g(x)|}_{\text{small}} + |g(x)| \le \frac{|M|}{2} + |g(x)|$$

So that $|g(x)| \geq \frac{|M|}{2}$ for points x near x_0.

This gives $\frac{1}{|g(x)|} \le \frac{2}{|M|}$ so $\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \frac{|g(x) - M|}{|g(x)| \cdot |M|} \le \frac{2|g(x) - M|}{M^2}$

and thus $\left| \frac{1}{g(x)} - \frac{1}{M} \right| < \varepsilon$ for points x near x_0.

- For the second part, we know $\lim_{x\to x_0} f(x) = f(x_0)$ for polynomials.

If $f(x) = \frac{P(x)}{Q(x)}$ is a rational function, then the first part gives

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} \frac{P(x)}{Q(x)} = \frac{\lim_{x\to x_0} P(x)}{\lim_{x\to x_0} Q(x)} = \frac{P(x_0)}{Q(x_0)} = f(x_0).$$

∎

Example

$$\lim_{x\to 2} \frac{x^3 - 3x + 2}{2x - 1} = \frac{2^3 - 3\cdot 2 + 2}{2\cdot 2 - 1} = \frac{4}{3}$$

$$\lim_{x\to 1} \frac{x^2 - 3x + 2}{x - 1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x-2)}{\cancel{x-1}} = 1 - 2 = -1.$$

$\boxed{\text{One-sided limits}}$ We say that $f(x) \to L$ as x approaches x_0 <u>from the left</u> and we write $\lim_{x \to x_0^-} f(x) = L$ if, given any $\varepsilon > 0$ there exists $\delta > 0$ such that $x_0 - \delta < x < x_0 \Rightarrow |f(x) - L| < \varepsilon$.

We say that $f(x) \to L$ as $x \to x_0$ <u>from the right</u> and we write $\lim_{x \to x_0^+} f(x) = L$ if given $\varepsilon > 0$ there exists $\delta > 0$ with $x_0 < x < x_0 + \delta \Rightarrow |f(x) - L| < \varepsilon$

<u>Example</u> (Piecewise defined functions) Consider $f(x) = \begin{cases} 3x-1 & \text{if } x \le 2 \\ 4x-5 & \text{if } x > 2 \end{cases}$.

We wish to compute $\lim_{x \to 2} f(x)$. In this case

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (3x-1) = \text{limit of a polynomial} = 3 \cdot 2 - 1 = 5$$

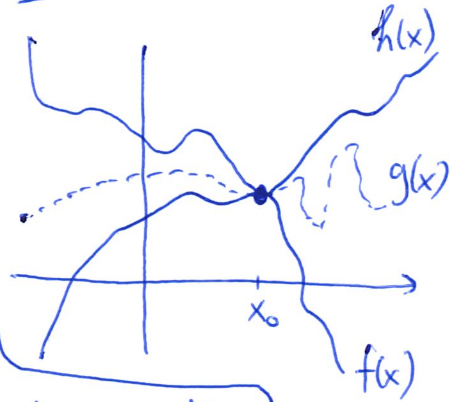
and $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (4x-5) = 4 \cdot 2 - 5 = 3$

are distinct, so the limit $\lim_{x \to 2} f(x)$ does <u>not</u> exist.

$\boxed{\text{Squeeze law}}$ Suppose $f(x) \le g(x) \le h(x)$ in some interval around x_0. If

$$\lim_{x \to x_0} f(x) = L \quad \text{and} \quad \lim_{x \to x_0} h(x) = L,$$

then it must be the case that $\lim_{x \to x_0} g(x) = L$ as well.



<u>Proof.</u> Let $\varepsilon > 0$ be given. We know

① there exists $\delta_1 > 0$ with $|x - x_0| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon$

$$\Rightarrow L - \varepsilon < f(x) < L + \varepsilon$$

② there exists $\delta_2 > 0$ with $|x - x_0| < \delta_2 \Rightarrow |h(x) - L| < \varepsilon$

$$\Rightarrow L - \varepsilon < h(x) < L + \varepsilon.$$

Letting $\delta =$ minimum of δ_1, δ_2 gives

$$|x - x_0| < \delta \Rightarrow L - \varepsilon < f(x) \le g(x) \le h(x) < L + \varepsilon$$

$$\Rightarrow L - \varepsilon < g(x) < L + \varepsilon$$

$$\Rightarrow |g(x) - L| < \varepsilon. \quad \blacksquare$$

Theorem (Limits of trigonometric and exponential functions)

- When $f(x)$ is a trigonometric or exponential function, one has $\lim_{x\to x_0} f(x) = f(x_0)$. We call these functions are continuous.

Proof. We check this when $f(x) = \sin x$, $f(x) = \cos x$, $f(x) = a^x$. Those are all similar. We can reduce them to the case $x_0 = 0$. Namely,

$$\lim_{x\to x_0} \sin x = \lim_{x\to x_0} \sin(x - x_0 + x_0)$$

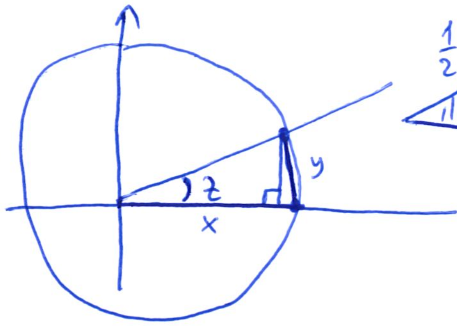
$$= \lim_{x\to x_0} \sin(x-x_0)\cdot\cos x_0 + \sin x_0 \cdot \cos(x-x_0)$$

$$= \boxed{\lim_{x\to x_0} \sin(x-x_0)} \cdot \cos x_0 + \sin x_0 \cdot \boxed{\lim_{x\to x_0} \cos(x-x_0)}$$

We claim $\lim_{z\to 0} \sin z = 0$ and $\lim_{z\to 0} \cos z = 1$.

Assuming this, we get $\lim_{x\to x_0} \sin x = \sin x_0$, as needed.

In fact, we have



$$0 \le \text{Area of large triangle} \le \text{Area of sector}$$

$$0 \le \frac{1}{2} \times 1 \times \sin z \le \frac{1}{2} z$$

$$0 \le \frac{1}{2}\sin z \le \frac{1}{2} z.$$

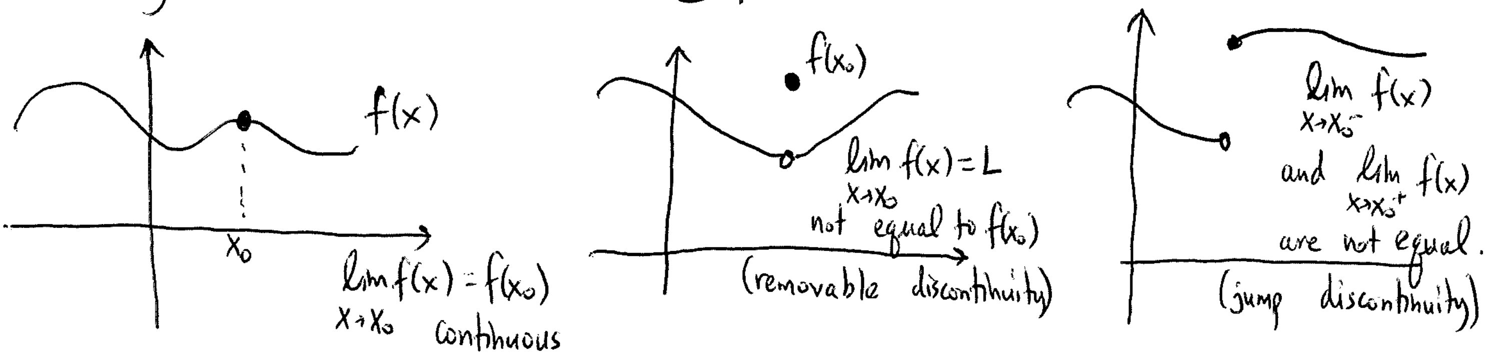
By the squeeze law, we get $\lim_{z\to 0} \sin z = 0$.

It follows that $\lim_{z\to 0} \cos z = 1$ (because $\sin^2 + \cos^2 = 1$ and $\cos$ is positive).

Continuous functions We say that a function f is continuous at the point x_0 .... if ① $\lim\limits_{x \to x_0} f(x) = f(x_0)$.

In terms of ε & δ, this means .... ② given any $\varepsilon > 0$ there exists $\delta > 0$ with $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

In some cases, we can check continuity using ①. In other cases, we may need to resort to ②.



Examples of continuous functions The following are continuous.

① Polynomials and rational functions throughout their domains

② Trigonometric and exponential functions. For exponential functions,

$$\lim_{x \to x_0} a^x = \lim_{x \to x_0} a^{x - x_0} \cdot a^{x_0} = \lim_{z \to 0} a^z \cdot a^{x_0} = a^{x_0} \quad \text{because}$$

one can show a^z approaches 1 as z approaches 0.

③ Sums/products/quotients of continuous functions are continuous, as

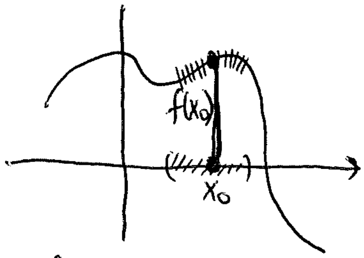
$$\lim_{x \to x_0} f(x) \cdot g(x) = \lim_{x \to x_0} f(x) \cdot \lim_{x \to x_0} g(x) = f(x_0) \cdot g(x_0), \text{ for instance.}$$

④ Square roots are continuous. Consider $f(x) = \sqrt{x}$. We need to show $\lim\limits_{x \to x_0} \sqrt{x} = \sqrt{x_0}$. Let $\varepsilon > 0$ be given. We need to show there exists $\delta > 0$ with $|x - x_0| < \delta \Rightarrow |\sqrt{x} - \sqrt{x_0}| < \varepsilon$. In this case,

$$|\sqrt{x} - \sqrt{x_0}| = \frac{|\sqrt{x} - \sqrt{x_0}| \cdot |\sqrt{x} + \sqrt{x_0}|}{\sqrt{x} + \sqrt{x_0}} = \frac{|x - x_0|}{\sqrt{x} + \sqrt{x_0}} < \frac{\delta}{\sqrt{x} + \sqrt{x_0}} \le \frac{\delta}{\sqrt{x_0}} = \varepsilon$$

by choosing our δ so that $\delta = \varepsilon \sqrt{x_0}$. We thus get continuity whenever $x_0 > 0$. When $x_0 = 0$, the argument is simpler.

Continuity and positivity



Suppose f is <u>continuous</u> at the point x_0.

ⓐ If $f(x_0) > 0$, then there is a whole interval $(x_0-\delta, x_0+\delta)$ such that $f(x) > 0$ on that interval.

ⓑ If $f(x_0) < 0$, then $f(x) < 0$ on a whole interval $(x_0-\delta, x_0+\delta)$.

<u>Proof</u>. We prove ⓐ. Since $\lim_{x\to x_0} f(x) = f(x_0)$, given any $\varepsilon > 0$ there exists $\delta > 0$ with $|x-x_0| < \delta \Rightarrow |f(x)-f(x_0)| < \varepsilon$.

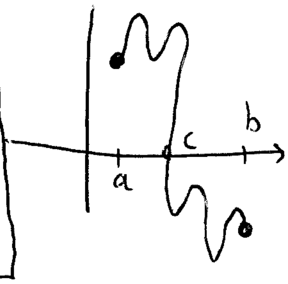
Take $\varepsilon = f(x_0)$ to get

$$x_0-\delta < x < x_0+\delta \Rightarrow f(x_0)-\varepsilon < f(x) < f(x_0)+\varepsilon$$
$$\Rightarrow 0 < f(x) < 2f(x_0),$$

as needed. ▨

Bolzano's theorem | Existence of roots

Suppose f is <u>continuous</u> on some interval $[a,b]$ with $f(a), f(b)$ having opposite signs. Then there exists a point $a < c < b$ such that $f(c) = 0$.



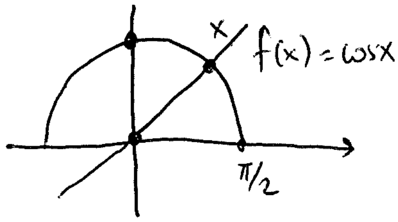
<u>Example 1</u>. Consider $f(x) = x^n + x - 1$ for any positive integer n. We know this function is continuous on $[0,1]$.

We check $f(0) = 0 + 0 - 1 < 0$

and $f(1) = 1 + 1 - 1 > 0$.

According to Bolzano's theorem, we have $f(x) = 0$ for some $0 < x < 1$.

<u>Example 2</u>. We show that $\cos x = x$ has a solution $0 < x < \pi/2$. Define <u>$g(x) = \cos x - x$</u>, the function that needs to vanish. This is the sum of continuous functions $\Rightarrow$ continuous.

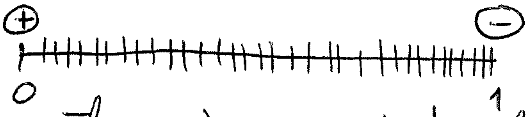


We check $g(0) = \cos 0 - 0 = 1 > 0$

and $g(\pi/2) = \cos\frac{\pi}{2} - \frac{\pi}{2} = -\pi/2 < 0$.

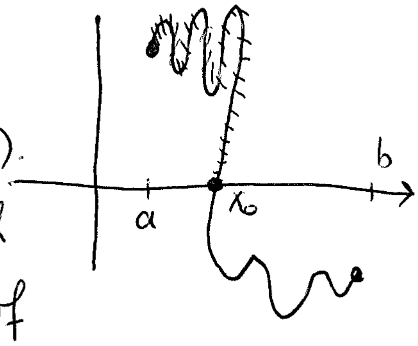
By Bolzano's theorem, we get $g(x) = 0$ at some point, so $\cos x = x$.

Digression: The root can be approximated to any degree of accuracy as follows. Suppose, for instance, that $f(0) > 0 > f(1)$. Then we get a root $0 < x < 1$. Subdivide this interval into 10^6 intervals of the same length and compute the values $f(x_k)$ at each point.



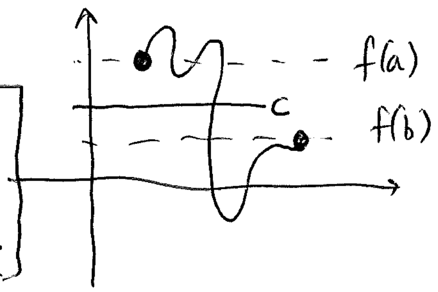
There is a subinterval of length 10^{-6} that gives a change of signs. We thus get a root with 10^{-6} accuracy.

Proof of Bolzano's theorem



Suppose f is continuous on $[a,b]$ with $f(a) > 0 > f(b)$. We consider the <u>largest</u> possible interval on which f is positive. What happens at the endpoint x_0 of this interval?? Can $f(x_0)$ be positive? If it were, f would be positive a little bit further, contradicting the largest interval assumption. Can $f(x_0)$ be negative? If it were, f would be negative a little bit to the left of x_0, still f is positive on the left. Thus $f(x_0) = 0$. ▨

Intermediate value theorem



Suppose f is continuous on $[a,b]$.
Then f attains <u>ALL</u> values between $f(a)$ and $f(b)$.

<u>Proof</u>. We have $f(a) > c > f(b)$ and we need to show $f(x) = c$ at some point. To prove this, define $g(x) = f(x) - c$, the function to vanish. Then g is continuous with

$$g(a) = f(a) - c > 0$$
$$g(b) = f(b) - c < 0$$

So Bolzano's theorem gives $g(x) = 0$ at a point $a < x < b$, hence $f(x) = c$ at that point. ▨

Theorem (Composition of continuous functions) The composition of continuous functions is continuous. Namely,

$$\left.\begin{array}{l} f \text{ is continuous at } x_0 \\ g \text{ is continuous at } f(x_0) \end{array}\right\} \Rightarrow g(f(x)) \text{ is continuous at } x_0.$$

Proof. To check continuity at x_0, let $\varepsilon > 0$ be given. We need some $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |g(f(x)) - g(f(x_0))| < \varepsilon$.

(a) We are assuming g continuous at $f(x_0)$, so there exists $\delta_1 > 0$ with $|z - f(x_0)| < \delta_1 \Rightarrow |g(z) - g(f(x_0))| < \varepsilon$

(b) We are assuming f continuous at x_0, so there exists $\delta_2 > 0$ with $|x - x_0| < \delta_2 \Rightarrow |f(x) - f(x_0)| < \delta_1$.

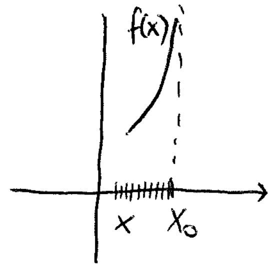
Then $|x - x_0| < \delta_2 \Rightarrow |f(x) - f(x_0)| < \delta_1$

$$\Rightarrow |g(f(x)) - g(f(x_0))| < \varepsilon. \quad \blacksquare$$

Example. $\displaystyle\lim_{x\to 1} \sqrt{\frac{x^3 + 3x + 1}{2x + 1}} = \sqrt{\frac{1^3 + 3\cdot 1 + 1}{2\cdot 1 + 1}} = \sqrt{5/3}$.

Here, $\sqrt{\ }$ and rational functions continuous $\Rightarrow$ composition continuous.

$\boxed{\text{Infinite limits}}$ We write $\displaystyle\lim_{x\to x_0^-} f(x) = +\infty$ when the values $f(x)$ become arbitrarily large as $x \to x_0$ from the left. Formally, given any (large) $N > 0$ there exists $\delta > 0$ with $x_0 - \delta < x < x_0 \Rightarrow f(x) > N$. The proofs for these limits are similar to former proofs.



- We define $\displaystyle\lim_{x\to x_0^+} f(x) = +\infty$ similarly and also $\displaystyle\lim_{x\to x_0^+} f(x) = -\infty$, $\displaystyle\lim_{x\to x_0^-} f(x) = -\infty$.
- Infinite limits usually arise due to zero denominators. In those cases, the sign of the denominator is important.

Example 1. $\lim\limits_{x\to 3^-} \dfrac{1}{x-3} = \dfrac{1}{\text{small negative}} = -\infty$.

Example 2. $\lim\limits_{x\to 3^-} \dfrac{3x+11}{x-3} = \lim\limits_{x\to 3^-} \dfrac{20}{x-3} = -\infty$.

Example 3. $\lim\limits_{x\to 2^+} \dfrac{4+3x}{2-x} = \dfrac{10}{\text{small negative}} = -\infty$.

Example 4. $\lim\limits_{x\to 1^-} \dfrac{6x^3+x+4}{2x^3-5x^2+2x+1} = \lim\limits_{x\to 1^-} \dfrac{11}{2x^3-5x^2+2x+1}$ .... let's factor!!

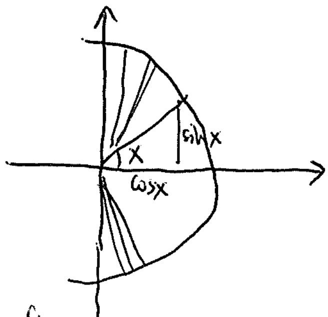
$$= \lim_{x\to 1^-} \frac{11}{(x-1)(2x^2-3x-1)} = \lim_{x\to 1^-} \frac{11}{(x-1)(-2)}$$

$$= \frac{-11/2}{\text{small negative}} = +\infty.$$

$$\begin{array}{r|l} & 2x^2-3x-1 \\ \hline x-1 & 2x^3-5x^2+2x+1 \\ & 2x^3-2x^2 \\ \hline & -3x^2+2x+1 \\ & -3x^2+3x \\ \hline & -x+1 \\ & -x+1 \\ \hline & 0 \end{array}$$

Example 5. Consider the tangent function $f(x) = \tan x = \dfrac{\sin x}{\cos x}$.

Then $\lim\limits_{x\to \pi/2^-} \tan x = \lim\limits_{x\to \pi/2^-} \dfrac{\sin x}{\cos x} = \lim\limits_{x\to \pi/2^-} \dfrac{1}{\cos x} = +\infty$



and $\lim\limits_{x\to -\pi/2^+} \tan x = \lim\limits_{x\to -\pi/2^+} \dfrac{\sin x}{\cos x} = \lim\limits_{x\to -\pi/2^+} \dfrac{-1}{\cos x} = -\infty$.

It follows by the IVT (intermediate value theorem) since $f(x)$ is continuous throughout $(-\pi/2, \pi/2)$, all values between $-\infty$ and $+\infty$ are attained. Thus $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is surjective.

Warning! The usual rules of arithmetic do not necessarily apply for the symbols $\pm\infty$. It is true that $(-\infty)(-\infty) = +\infty$ but one does not have $0 \cdot \infty = 0$ or $\dfrac{\infty}{\infty} = 1$ or $\infty - \infty = 0$ or $1^\infty = 1$ necessarily. These are not valid in the context of limits!!

$\boxed{\text{Limits at infinity}}$ We studied the case $\lim_{x \to x_0^{\pm}} f(x) = \pm\infty$ in which the $f(x)$ values become arbitrarily large. We can also introduce $\lim_{x \to +\infty} f(x) = L$ and $\lim_{x \to -\infty} f(x) = L$.

<u>Theorem 1.</u> One has the following facts.

① $\lim_{x \to +\infty} x^p = +\infty$ for any positive power p.

② $\lim_{x \to -\infty} x^n = \begin{cases} +\infty & \text{for any } \underline{\text{even}} \text{ integer } n > 0 \\ -\infty & \text{for any } \underline{\text{odd}} \text{ integer } n > 0 \end{cases}$

③ $\lim_{x \to \pm\infty} \frac{1}{x^n} = 0$ for any integer n

④ $\lim_{x \to \pm\infty} (a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n) = \lim_{x \to \pm\infty} (a_n x^n)$ for any polynomial function.

<u>Example</u> $\lim_{x \to \pm\infty} \frac{3x^2 - 2x + 6}{5x^2 - 4x + 3} = \lim_{x \to \pm\infty} \frac{x^2\left(3 - \cancel{\frac{2}{x}}^{0} + \cancel{\frac{6}{x^2}}\right)}{x^2\left(5 - \cancel{\frac{4}{x}} + \cancel{\frac{3}{x^2}}\right)} = \frac{3}{5}$.

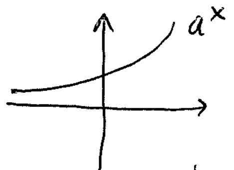
<u>Example</u> $\lim_{x \to \pm\infty} \frac{3x^2 - 2x + 6}{4x^3 - x^2 + 10} = \lim_{x \to \pm\infty} \frac{3x^2}{4x^3} = \lim_{x \to \pm\infty} \frac{3}{4x} = 0$.

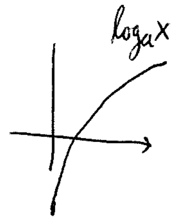
<u>Theorem 2.</u> One has the following facts.

① Suppose $a > 1$. Then

$\lim_{x \to +\infty} a^x = +\infty$, $\lim_{x \to -\infty} a^x = 0$,

$\lim_{x \to +\infty} \log_a x = +\infty$, $\lim_{x \to 0^+} \log_a x = -\infty$

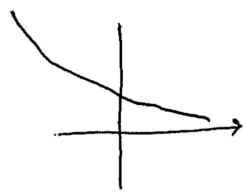


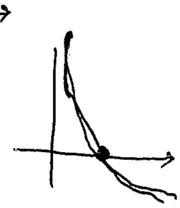


② Suppose $0 < a < 1$. Then

$\lim_{x \to +\infty} a^x = 0$, $\lim_{x \to -\infty} a^x = +\infty$

$\lim_{x \to 0^+} \log_a x = +\infty$, $\lim_{x \to +\infty} \log_a x = -\infty$





Example (Analysis of a typical function) Let $f(x) = \dfrac{3x-1}{2x-4}$.

- The domain consists of all points with $2x \neq 4$, namely $x \neq 2$. We look at the limits $x \to 2^-$ and $x \to 2^+$. We have

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{3x-1}{2(x-2)} = \lim_{x\to 2^-} \frac{5}{2(x-2)} = -\infty \quad \text{and}$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{3x-1}{2(x-2)} = \lim_{x\to 2^+} \frac{5}{2(x-2)} = +\infty.$$

- The range can be determined by solving for x. We get

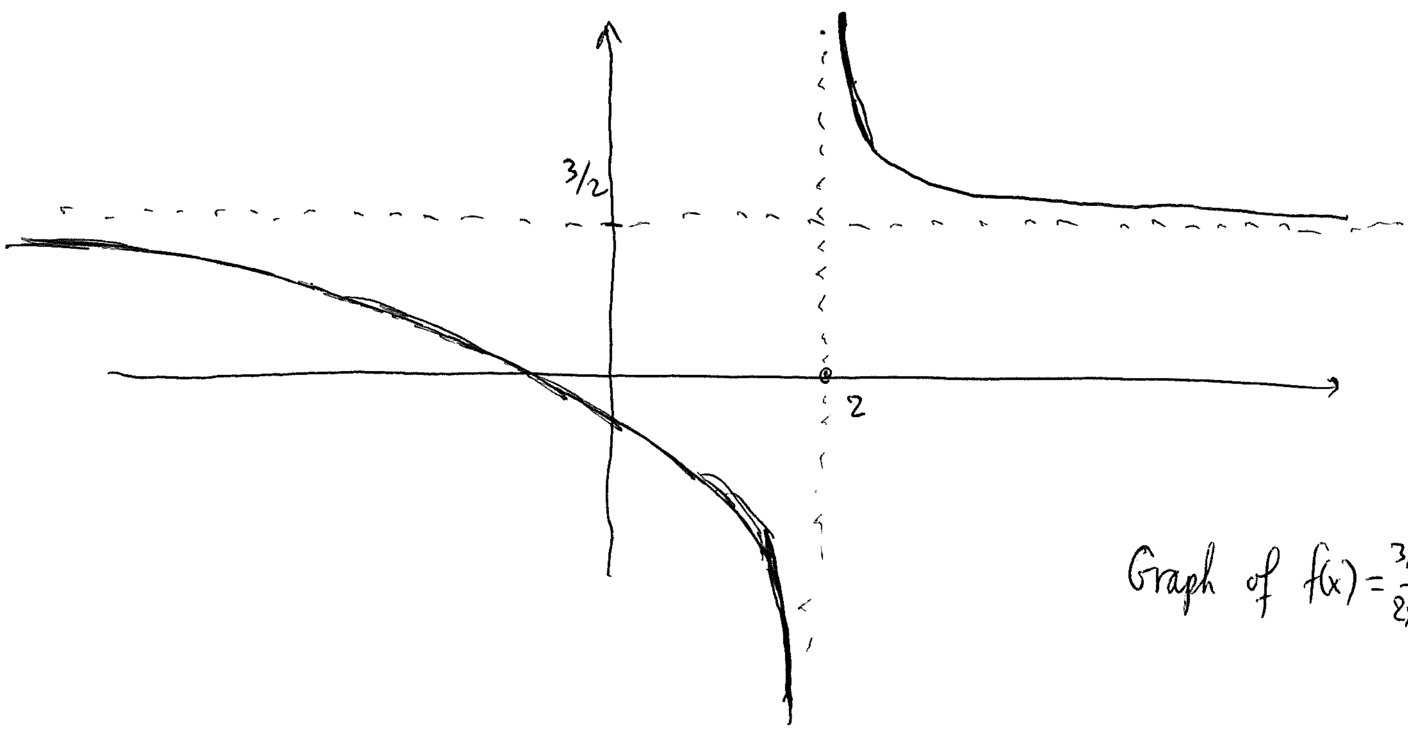
$$y = \frac{3x-1}{2x-4} \Leftrightarrow 2xy - 4y = 3x-1 \Leftrightarrow 2xy - 3x = 4y-1 \Leftrightarrow x = \frac{4y-1}{2y-3}.$$

The range consists of all $y \neq 3/2$.

- Worrying about the limits at infinity $x \to \pm\infty$, we get

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{3x-1}{2x-4} = \lim_{x\to\pm\infty} \frac{3x}{2x} = 3/2.$$

This value is <u>approached</u> as $x \to \pm\infty$ (but never attained)



Graph of $f(x) = \dfrac{3x-1}{2x-4}$

$\boxed{\text{Derivatives}}$ We wish to analyse a function $f(x)$ and the rate at which its values are increasing or decreasing.

<u>Average rate of change</u> If we concentrate on an interval $[x_0, x_1]$ we can define the average rate of change as $\dfrac{f(x_1)-f(x_0)}{x_1-x_0}$.

<u>Instantaneous rate of change.</u> If we concentrate on a specific point x_0, we can define the rate of change at x_0 as

$$\lim_{x_1 \to x_0} \frac{f(x_1)-f(x_0)}{x_1-x_0}, \text{ provided that the limit exists.}$$

If this limit exists, it is called the derivative $f'(x_0)$ and we say that f is differentiable at x_0.

<u>Example 1.</u> Consider a linear function, say $f(x) = ax+b$.
We compute $f'(x_0)$ at a given point x_0. In this case,

$$f'(x_0) = \lim_{x_1 \to x_0} \frac{f(x_1)-f(x_0)}{x_1-x_0} = \lim_{x_1 \to x_0} \frac{ax_1+b-(ax_0+b)}{x_1-x_0}$$

$$= \lim_{x_1 \to x_0} \frac{ax_1-ax_0}{x_1-x_0} = \lim_{x_1 \to x_0} \frac{a\cancel{(x_1-x_0)}}{\cancel{x_1-x_0}} = a.$$

Thus, the derivative is the same at all points.

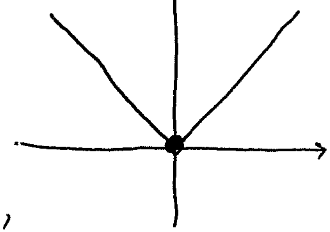
<u>Example 2.</u> Consider a quadratic function, say $f(x) = x^2$.
Then $f'(x_0) = \lim_{x_1 \to x_0} \dfrac{f(x_1)-f(x_0)}{x_1-x_0} = \lim_{x_1 \to x_0} \dfrac{x_1^2-x_0^2}{x_1-x_0}$

$$= \lim_{x_1 \to x_0} \frac{\cancel{(x_1-x_0)}(x_1+x_0)}{\cancel{x_1-x_0}} = \lim_{x_1 \to x_0} (x_1+x_0) = x_0+x_0 = 2x_0.$$

In other words, $f'(x) = 2x$ for all x.
We will sometimes write $(x^2)' = 2x$ for simplicity.

Example 3. (An example of a non-differentiable function)

Consider the absolute value $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$.



In this case, f is not differentiable at the point $x_0 = 0$ (because of the corner). More precisely,

$$\lim_{x_1 \to x_0^-} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \to 0^-} \frac{f(x_1) - \cancel{f(0)}}{x_1 - \cancel{0}} = \lim_{x_1 \to 0^-} \frac{-\cancel{x_1}}{\cancel{x_1}} = -1$$

$$\lim_{x_1 \to x_0^+} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \to 0^+} \frac{f(x_1) - \cancel{f(0)}}{x_1 - \cancel{0}} = \lim_{x_1 \to 0^+} \frac{x_1}{x_1} = 1.$$

These do not agree, so $f'(x_0) = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$ does not exist.

Theorem (Differentiable implies continuous) If a function f is differentiable at x_0, then f is continuous at x_0.

Proof. Differentiability means $\lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(x_0)$ exists.

Continuity means $\lim_{x_1 \to x_0} f(x_1) = f(x_0)$.

Assuming the former, we get

$$\lim_{x_1 \to x_0} f(x_1) = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} \cdot (x_1 - x_0) + f(x_0)$$

$$= \underbrace{\lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}}_{f'(x_0)} \cdot \underbrace{\lim_{x_1 \to x_0} (x_1 - x_0)}_{0} + f(x_0) = f(x_0).$$

Example 4. We compute the derivative of $f(x) = 1/x$. We have

$$f'(x_0) = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{\frac{1}{x_1} - \frac{1}{x_0}}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{\overset{-1}{\cancel{x_0 - x_1}}}{x_0 x_1 \cancel{(x_1 - x_0)}}$$

$$= \lim_{x_1 \to x_0} \frac{-1}{x_1 x_0} = -\frac{1}{x_0^2} \quad \text{as long as } x_0 \neq 0.$$

This gives $f'(x) = -1/x^2$ or just $(1/x)' = -1/x^2$.

Example 5. We compute the derivative of $f(x)=\sqrt{x}$. We have

$$f'(x_0) = \lim_{x_1\to x_0} \frac{f(x_1)-f(x_0)}{x_1-x_0} = \lim_{x_1\to x_0} \frac{\sqrt{x_1}-\sqrt{x_0}}{x_1-x_0} \cdot \frac{\sqrt{x_1}+\sqrt{x_0}}{\sqrt{x_1}+\sqrt{x_0}}$$

$$= \lim_{x_1\to x_0} \frac{\cancel{x_1-x_0}}{(\cancel{x_1-x_0})(\sqrt{x_1}+\sqrt{x_0})}$$

$$= \lim_{x_1\to x_0} \frac{1}{\sqrt{x_1}+\sqrt{x_0}} = \frac{1}{\sqrt{x_0}+\sqrt{x_0}} = \frac{1}{2\sqrt{x_0}}.$$

This proves that $f'(x) = \frac{1}{2\sqrt{x}}$ or just $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$.

Theorem (Derivatives of sums and constant multiples)

ⓐ The derivative of a sum is the sum of the derivatives

$$[f(x)+g(x)]' = f'(x) + g'(x)$$

ⓑ The derivative of a constant multiple is

$$[c\cdot f(x)]' = c\cdot f'(x) \quad \text{.... for any constant } c.$$

Proof. For the first part, let $h(x) = f(x)+g(x)$. Then

$$h'(x_0) = \lim_{x_1\to x_0} \frac{h(x_1)-h(x_0)}{x_1-x_0} = \lim_{x_1\to x_0} \frac{f(x_1)+g(x_1)-f(x_0)-g(x_0)}{x_1-x_0}$$

$$= \lim_{x_1\to x_0} \frac{f(x_1)-f(x_0)}{x_1-x_0} + \lim_{x_1\to x_0} \frac{g(x_1)-g(x_0)}{x_1-x_0}$$

$$= f'(x_0) + g'(x_0).$$

This proves $h'(x) = f'(x)+g'(x)$. For the second part, let $B(x) = c\cdot f(x)$. Then $B'(x_0) = \lim_{x_1\to x_0} \frac{B(x_1)-B(x_0)}{x_1-x_0} = \lim_{x_1\to x_0} \frac{cf(x_1)-cf(x_0)}{x_1-x_0}$

so $B'(x_0) = c\cdot \lim_{x_1\to x_0} \frac{f(x_1)-f(x_0)}{x_1-x_0} = c\,f'(x_0)$ by definition. ▨

Example $(3x^2+4x-6)' = (3x^2)'+(4x-6)' = 3\cdot 2x + 4 = 6x+4$

and $\left(5\sqrt{x}+\frac{2}{x}\right)' = 5(\sqrt{x})' + 2\left(\frac{1}{x}\right)' = \frac{5}{2\sqrt{x}} - \frac{2}{x^2}$.

Product rule Suppose f, g are differentiable at x_0. Then so is their product $f \cdot g$ and its derivative is

$$[f(x)\, g(x)]' = f'(x)\, g(x) + f(x)\, g'(x).$$

Proof. Define $H(x) = f(x)\, g(x)$. We compute $H'(x_0)$ as

$$H'(x_0) = \lim_{x_1 \to x_0} \frac{H(x_1) - H(x_0)}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{f(x_1)\, g(x_1) - f(x_0) g(x_1) + f(x_0) g(x_1) - f(x_0)\, g(x_0)}{x_1 - x_0}$$

$$= \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} \cdot g(x_1) + \frac{g(x_1) - g(x_0)}{x_1 - x_0} \cdot f(x_0)$$

$$= f'(x_0) \cdot \lim_{x_1 \to x_0} g(x_1) + g'(x_0) \cdot f(x_0)$$

$$= f'(x_0)\; g(x_0) + g'(x_0)\, f(x_0)$$

since g is differentiable and thus continuous. ▨

Derivatives of powers We have $(x^n)' = n \cdot x^{n-1}$ for each integer $n \geq 1$.

One checks this using induction. When $n = 1$, we have

$$(x^1)' = x' = 1 = 1x^0$$

and the result holds.

Assume it holds for some n, say $n = k$.

$n=1 \quad n=2 \quad n=3 \quad n=4$

We have to check this for $k+1$ and

$$(x^{k+1})' = (x^k \cdot x)' = (x^k)' \cdot x + x' \cdot x^k$$

$$= kx^{k-1} \cdot x + 1 \cdot x^k = (k+1)x^k, \text{ as needed.}$$

Quotient rule Suppose f, g are differentiable at x_0. Then f/g is also differentiable and

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)\, g(x) - f(x)\, g'(x)}{g(x)^2}.$$

Proof. Suppose we can show $\left[\frac{1}{g(x)}\right]' = -\frac{g'(x)}{g(x)^2}$. Then we can already conclude that

$$\left[\frac{f(x)}{g(x)}\right]' = \left[f(x) \cdot \frac{1}{g(x)}\right]' = f'(x) \cdot \frac{1}{g(x)} + \left(-\frac{g'(x)}{g(x)^2}\right) \cdot f(x)$$

... by the product rule so the result follows.

Let's worry about $H(x) = \frac{1}{g(x)}$ then. We get

$$H'(x_0) = \lim_{x_1 \to x_0} \frac{H(x_1) - H(x_0)}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{\frac{1}{g(x_1)} - \frac{1}{g(x_0)}}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{g(x_0) - g(x_1)}{g(x_1)g(x_0)(x_1 - x_0)}$$

$$= \lim_{x_1 \to x_0} \frac{g(x_1) - g(x_0)}{x_1 - x_0} \cdot \frac{-1}{g(x_1)g(x_0)} = \frac{-g'(x_0)}{g(x_0)^2}, \text{ as needed. } \blacksquare$$

Example
$$\left(\frac{x^2+1}{3x-2}\right)' = \frac{(x^2+1)' \cdot (3x-2) - (3x-2)' \cdot (x^2+1)}{(3x-2)^2}$$

$$= \frac{2x(3x-2) - 3(x^2+1)}{(3x-2)^2} = \frac{3x^2 - 4x - 3}{(3x-2)^2}$$

$\boxed{\text{Derivatives of trigonometric functions}}$ All trigonometric functions are differentiable and one has:

- $(\sin x)' = \cos x$, $(\tan x)' = \sec^2 x$, $(\sec x)' = \sec x \tan x$
- $(\cos x)' = -\sin x$, $(\cot x)' = -\csc^2 x$, $(\csc x)' = -\csc x \cot x$

Note. The main ones are sine and cosine as usual.

Suppose $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$. Then

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cos x - (\cos x)' \sin x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x,$$

and
$$(\sec x)' = \left(\frac{1}{\cos x}\right)' = \frac{\cancelto{0}{1' \cos x} - (\cos x)' \cdot 1}{\cos^2 x} = \frac{\sin x}{\cos x \cos x} = \sec x \tan x.$$

The other two formulas follow similarly.

Derivatives of sine/cosine. Let us worry about $f(x) = \sin x$.

We have
$$f'(x_0) = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{\sin(x_1) - \sin(x_0)}{x_1 - x_0}$$

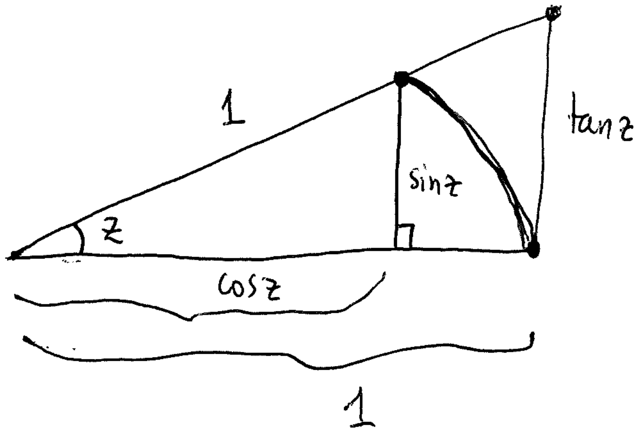
$z = x_1 - x_0$

$$= \lim_{z \to 0} \frac{\sin(z + x_0) - \sin x_0}{z}$$

$$= \lim_{z \to 0} \frac{\sin z \cos x_0 + \sin x_0 \cos z - \sin x_0}{z}.$$

This gives $f'(x_0) = \cos x_0 \boxed{\lim_{z\to 0} \frac{\sin z}{z}} + \sin x_0 \cdot \boxed{\lim_{z\to 0} \frac{\cos z - 1}{z}}$.

We need to show the limits are 1 and 0 (so that $f'(x_0) = \cos x_0$).



Consider the case $z \geq 0$.

Area of small triangle $\leq$ Area of sector $\leq$ Area of large triangle

$$\frac{1}{2} \sin z \cos z \leq \frac{1}{2} z \leq \frac{1}{2} \tan z$$

This gives $\sin z \cos z \leq z \leq \frac{\sin z}{\cos z} \Rightarrow \cos z \leq \boxed{\frac{z}{\sin z}} \leq \frac{1}{\cos z}$.

As $z \to 0$, $\cos z \to 1$ and $\frac{1}{\cos z} \to 1$ so $\frac{z}{\sin z} \to 1$ as well.

Thus $\frac{\sin z}{z} \to 1$ as well. We proved this for $z \geq 0$. It follows for $z \leq 0$ as well. And then

$$\lim_{z\to 0} \frac{\cos z - 1}{z} \cdot \frac{\cos z + 1}{\cos z + 1} = \lim_{z\to 0} \frac{\overbrace{-\sin z}^{\to 1} \overbrace{\sin z}^{\to 0}}{z \cdot (\cos z + 1)} = 0.$$

$\boxed{\text{Exponential functions}}$ Consider $f(x) = a^x$ for some $a>0$.

Then $f'(x_0) = \lim\limits_{x\to x_0} \dfrac{f(x)-f(x_0)}{x-x_0} = \lim\limits_{x\to x_0} \dfrac{a^x - a^{x_0}}{x-x_0}$ and $z = x - x_0 \to 0$

so $f'(x_0) = \lim\limits_{z\to 0} \dfrac{a^{x_0+z} - a^{x_0}}{z} = \lim\limits_{z\to 0} \dfrac{a^{x_0}a^z - a^{x_0}}{z} = a^{x_0} \boxed{\lim\limits_{z\to 0} \dfrac{a^z-1}{z}}$.

This gives $f'(x_0) = \text{constant} \cdot f(x_0)$ for some constant.

The formula simplifies when the constant is 1, namely when $\dfrac{a^z-1}{z} \approx 1$ as $z\to 0$, namely when $a^z \approx 1+z$, namely when $\boxed{a = \lim\limits_{z\to 0} (1+z)^{1/z}}$. Loosely speaking,

$z = -1/2$ gives $(1+z)^{1/z} = (1/2)^{-2} = 2^2 = 4$

and $z = 1/2$ gives $(1+z)^{1/z} = (3/2)^2 = 9/4 = 2.25$. This special value of a is actually $e = \lim\limits_{z\to 0} (1+z)^{1/z} = 2.71828\ldots$

<u>The exponential function</u> is defined by $f(x) = e^x$ and it satisfies $f'(x) = f(x) = e^x$. <u>The logarithmic function</u> is the inverse of $f(x) = e^x$ and it is denoted by $g(x) = \ln x$. Every other logarithmic function can be expressed in terms of ln because

$$\log_a x = \log_a e^{\ln x} = \ln x \cdot \log_a e \quad \text{and}$$
$$\ln x = \ln a^{\log_a x} = \log_a x \cdot \ln a.$$

$\boxed{\text{Derivatives of inverse functions}}$ Suppose $f: A \to B$ is differentiable and bijective. Let $g: B\to A$ be its inverse $g = f^{-1}$. Then g is differentiable and its derivative is

$$\boxed{g'(x) = \frac{1}{f'(g(x))} \quad \text{for all } x.}$$

Example 1. We compute the derivative of $g(x) = \ln x$, the inverse function of $f(x) = e^x$. The formula gives

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{f(g(x))} = \frac{1}{x},$$

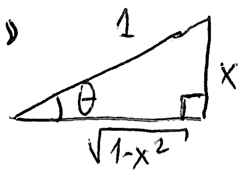
so $(\ln x)' = 1/x$.

(*) Example 2. We compute the derivative of the inverse sine function. Let $f(x) = \sin x$ and $g(x) = \sin^{-1} x$. The formula gives

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\cos(\sin^{-1} x)}$$

$f(x) = \sin x$
$f'(x) = \cos x$

and we can simplify this as follows. Let $\theta = \sin^{-1} x$, an angle whose sine is x. When $x \geq 0$, geometry gives



$$\cos(\sin^{-1} x) = \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \sqrt{1-x^2}.$$

When $x \leq 0$, the same formula applies and $g'(x) = \dfrac{1}{\sqrt{1-x^2}}$.

Example 3. (Homework) One has $(\tan^{-1} x)' = \dfrac{1}{1+x^2}$ for all x.

Proof of the formula for $g'(x)$. We have $f(x)$ and its inverse function $g(y)$. Let $y = f(x)$ and $y_0 = f(x_0)$ for convenience. We compute

$$g'(y_0) = \lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = \lim_{y \to y_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)}$$

to get

$$g'(y_0) = \lim_{y \to y_0} \frac{x - x_0}{f(x) - f(x_0)} = \lim_{y \to y_0} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}}.$$

If we had the limit as $x \to x_0$, this would be

$$g'(y_0) = \lim_{x \to x_0} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} = \frac{1}{f'(x_0)} = \frac{1}{f'(g(y_0))}.$$

It remains to check that $y \to y_0$ whenever $x \to x_0$, namely that $f(x) \to f(x_0)$ whenever $x \to x_0$. This is because f is differentiable and thus continuous. $\blacksquare$

$(x^n)' = nx^{n-1}$	$(1/x)' = -1/x^2$	$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$	
$(\sin x)' = \cos x$	$(\cos x)' = -\sin x$	$(\tan x)' = \sec^2 x$	$(\sec x)' = \sec x \tan x$
$(e^x)' = e^x$	$(\ln x)' = 1/x$	$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$	$(\tan^{-1} x)' = \frac{1}{1+x^2}$

Chain rule. The derivative of a composite function $f(g(x))$ is given by $[f(g(x))]' = f'(g(x)) \cdot g'(x)$.

In practice, one has

$$[\sin g(x)]' = \cos(g(x)) \cdot g'(x)$$

and $$[\ln g(x)]' = \frac{1}{g(x)} \cdot g'(x)$$

and $$[\sin^{-1} g(x)]' = \frac{1}{\sqrt{1-g(x)^2}} \cdot g'(x)$$

and $$[g(x)^n]' = n\, g(x)^{n-1} \cdot g'(x).$$

Example 1. Take $f(x) = \sin(\tan x)$. Then

$$f'(x) = \cos(\tan x) \cdot (\tan x)' = \cos(\tan x) \cdot \sec^2 x.$$

Example 2. Take $f(x) = \ln(\ln x)$. Then

$$f'(x) = \frac{1}{\ln x} \cdot (\ln x)' = \frac{1}{\ln x} \cdot \frac{1}{x} = \frac{1}{x \ln x}.$$

Example 3. Take $f(x) = \sin^{-1}(\sqrt{x})$. Then

$$f'(x) = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot (\sqrt{x})' = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}.$$

$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$
$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$

Example 4. Take $f(x) = \sec(\tan x)$. Then

$$f'(x) = \sec(\tan x) \cdot \tan(\tan x) \cdot (\tan x)'$$
$$= \sec(\tan x) \cdot \tan(\tan x) \cdot \sec^2 x.$$

$(\sec x)' = \sec x \tan x$
$(\tan x)' = \sec^2 x$

Example 5. Take $f(x) = \ln(\sinh^{-1}(e^{\cos x}))$. Then

$$f'(x) = \frac{1}{\sinh^{-1}(e^{\cos x})} \cdot \left[\sinh^{-1}(e^{\cos x})\right]'$$

$$= \frac{1}{\sinh^{-1}(e^{\cos x})} \cdot \frac{1}{\sqrt{1-(e^{\cos x})^2}} \cdot (e^{\cos x})'$$

$(e^x)' = e^x$

$$= \frac{1}{\sinh^{-1}(e^{\cos x})} \cdot \frac{1}{\sqrt{1-e^{2\cos x}}} \cdot e^{\cos x} \cdot (-\sin x).$$

Proof of chain rule. We have a composition $y = f(g(x)) = f(u)$.

The derivative of y is then

x
$u = g(x)$
$y = f(u)$

$$\lim_{x\to x_0} \frac{f(g(x)) - f(g(x_0))}{x - x_0} = \lim_{x\to x_0} \frac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)} \cdot \frac{g(x) - g(x_0)}{x - x_0}$$

$$= \lim_{x\to x_0} \frac{f(u) - f(u_0)}{u - u_0} \cdot \frac{g(x) - g(x_0)}{x - x_0}.$$

As $x \to x_0$, and g is continuous, $g(x) \to g(x_0)$ so $u \to u_0$.

Taking limits, we end up with $f'(u_0) \cdot g'(x_0) = f'(g(x_0)) \cdot g'(x_0)$. ▨

$\boxed{\text{Leibniz notation}}$ When one is dealing with several variables at the same time, the notation f' is a bit ambiguous. One may denote by $\frac{dy}{dx}$ the derivative of y as a function of x and by $\frac{dy}{du}$ the derivative of y as a function of u.

In this notation, the chain rule $[f(g(x))]' = f'(g(x)) \cdot g'(x)$ takes the form

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

x
$u = g(x)$
$y = f(u)$

Example 1. Take $y = 3u^2$, $u = \cos x$.

Then $\frac{dy}{du} = 6u$, $\frac{du}{dx} = -\sin x$

so $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -6u \sin x$$

Example 2. Take $y = \sqrt{\dfrac{\tan(\sec x) + 1}{\tan(\sec x) - 1}}$, for instance.

We can write this as $y = \sqrt{\dfrac{u+1}{u-1}}$ ~~and~~ with $u = \tan(\sec x)$.

Then $y' = \dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$. In this case,

- $\dfrac{dy}{du} = \dfrac{1}{2\sqrt{\frac{u+1}{u-1}}} \cdot \left(\dfrac{u+1}{u-1}\right)'$ ---- by the chain rule $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$

$$= \frac{1}{2}\sqrt{\frac{u-1}{u+1}} \; \frac{u-1-(u+1)}{(u-1)^2} \quad \text{---- by the quotient rule}$$

$$= -\sqrt{\frac{u-1}{u+1}} \cdot \frac{1}{(u-1)^2}$$

- $\dfrac{du}{dx} = \sec^2(\sec x) \cdot (\sec x)' = \sec^2(\sec x) \cdot \sec x \cdot \tan x$.

$\boxed{\text{Implicit differentiation}}$ Suppose that the variables x, y are related through some equation such as $x^3 y + xy^2 = 1$.

In those cases, we may either solve for y and then differentiate or differentiate the equation directly.

The latter approach is called implicit differentiation. It does not assume an explicit equation $y = f(x)$, but one needs to recall that y is still a function of x.

Example 1. Suppose $x^2 + y^2 = 1$. (a) Solving for y gives

$$y^2 = 1 - x^2 \Rightarrow y = \pm\sqrt{1-x^2} \Rightarrow \frac{dy}{dx} = \pm \frac{1}{2\sqrt{1-x^2}} \cdot (1-x^2)'$$

$$\Rightarrow \frac{dy}{dx} = \pm \frac{1}{2\sqrt{1-x^2}}(-2x) = \mp \frac{x}{\sqrt{1-x^2}}.$$

(b) We compute $\frac{dy}{dx}$ by differentiating directly. We get

$$x^2+y^2=1 \implies 2x + \underbrace{2y\cdot y'}_{\text{by chain rule}} = 0$$

$$\implies x+yy'=0 \implies y' = -\frac{x}{y}.$$

This gives the same answer but it does directly.

Example 2. Suppose $x^3 + \sin y = y^4 - \sec x$.

One cannot solve this equation for y in terms of x!!!

We get
$$3x^2 + \cos y \cdot y' = 4y^3 \cdot y' - \sec x \tan x$$
by the chain rule, so
$$\cos y \cdot y' - 4y^3 \cdot y' = -3x^2 - \sec x \tan x$$
so
$$(\cos y - 4y^3)\, y' = -3x^2 - \sec x \tan x$$
so
$$y' = -\frac{3x^2 + \sec x \tan x}{\cos y - 4y^3}.$$

Example 3. Suppose $\sin(xy) + \ln(xy^2) = xy$.

We wish to compute $y' = \frac{dy}{dx}$ using implicit differentiation.

We get $\cos(xy)\cdot(xy)' + \frac{1}{xy^2}\cdot(xy^2)' = (xy)'$ by the chain rule,

so $\cos(xy)\cdot(y+xy') + \frac{1}{xy^2}(y^2 + x\cdot 2yy') = y + xy'$ by the product/chain rules. This gives

$$y\cos(xy) + xy'\cos(xy) + \frac{1}{x} + \frac{2y'}{y} = y + xy'$$

$$xy'\cos(xy) + \frac{2y'}{y} - xy' = y - \frac{1}{x} - y\cos(xy)$$

$$\left(x\cos(xy) + \frac{2}{y} - x\right) y' = y - 1/x - y\cos(xy)$$

$$y' = \frac{y - 1/x - y\cos(xy)}{x\cos(xy) + 2/y - x}.$$

$\boxed{\text{Logarithmic differentiation}}$ We can use logarithms to simplify

products .... $\ln(xy) = \ln x + \ln y$

quotients ..... $\ln \frac{x}{y} = \ln x - \ln y$

exponents ...... $\ln x^y = y \ln x$.

We may use these properties to differentiate messy expressions.

<u>Example</u>. We show that $(x^n)' = nx^{n-1}$ for <u>any</u> power n.

Take $y = x^n \implies \ln y = \ln x^n = n \cdot \ln x$

$$\implies \frac{1}{y} \cdot y' = n \cdot \frac{1}{x}$$

$$\implies y' = \frac{n}{x} \cdot y = \frac{n}{x} \cdot x^n = nx^{n-1}.$$

<u>Example</u>. We compute $(a^x)'$ for any base $a > 0$.

Take $y = a^x \implies \ln y = \ln a^x = x \ln a$

$$\implies \frac{1}{y} \cdot y' = \ln a \implies y' = y \ln a = a^x \ln a.$$

We derived this formula yesterday by showing $(a^x)' = a^x \cdot \lim_{z \to 0} \frac{a^z - 1}{z}$.

<u>Note</u>. Strictly speaking, $\ln x$ is only defined when $x > 0$.
In some cases, it is better to use $\ln |x|$ which is defined when $x \neq 0$. The derivative of $\ln x$ is $1/x$.

The derivative of $\ln |x|$ is still $1/x$ because

$$[\ln(-x)]' = \frac{1}{-x} \cdot (-x)' = \frac{1}{-x}(-1) = 1/x$$

by the chain rule.

Thus, we can simplify expressions by taking absolute values and then differentiating.

<u>Example</u>. Consider $y = \dfrac{(x^2+1)^3 \cdot \sqrt{x^4+3x} \cdot \sqrt[5]{\ln(\sec x)}}{(3\cos x - 4\tan x)^{75}}$.

Let's take absolute values and then ln to simplify. We get

$$y = (x^2+1)^3 \cdot (x^4+3x)^{1/2} \cdot (\ln(\sec x))^{1/5} \cdot (3\cos x - 4\tan x)^{-75}$$

$$\Rightarrow |y| = |x^2+1|^3 \cdot |x^4+3x|^{1/2} \cdot |\ln(\sec x)|^{1/5} \cdot |3\cos x - 4\tan x|^{-75}$$

$$\Rightarrow \ln|y| = \ln|x^2+1|^3 + \ln|x^4+3x|^{1/2} + \ln|\ln(\sec x)|^{1/5} + \ln|3\cos x - 4\tan x|^{-75}$$

$$\Rightarrow \ln|y| = 3\ln|x^2+1| + \tfrac{1}{2}\ln|x^4+3x| + \tfrac{1}{5}\ln|\ln(\sec x)| - 75\ln|3\cos x - 4\tan x|$$

$$\Rightarrow \frac{1}{y}y' = \frac{3}{x^2+1}\cdot 2x + \frac{1}{2}\,\frac{1}{x^4+3x}\cdot(4x^3+3) \qquad [\ln|x|]' = 1/x$$

$$+ \frac{1}{5}\,\frac{1}{\ln(\sec x)}\cdot\frac{1}{\cancel{\sec x}}\cdot\cancel{\sec x}\tan x$$

$$-\frac{75}{3\cos x - 4\tan x}\cdot(-3\sin x - 4\sec^2 x)$$

$$\Rightarrow y' = y\left(\frac{6x}{x^2+1} + \frac{4x^3+3}{2(x^4+3x)} + \frac{\tan x}{5\ln(\sec x)} + \frac{75(3\sin x + 4\sec^2 x)}{3\cos x - 4\tan x}\right).$$

<u>Example</u> Consider $y = x^x$. Note that neither base nor exponent is constant. Let us introduce ln once again. Here $x > 0$ so absolute values are redundant. We get

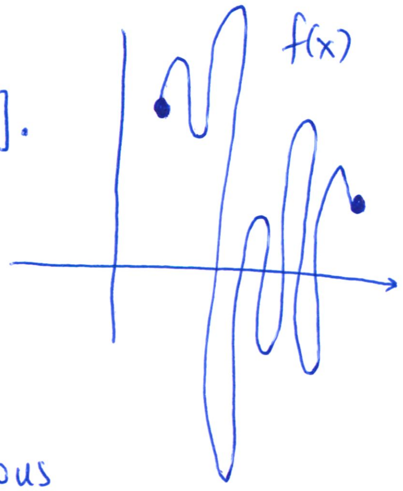
$$y = x^x \Rightarrow \ln y = \ln x^x = x\cdot\ln x$$

$$\Rightarrow \frac{1}{y}\cdot y' = 1\cdot\ln x + x\cdot\frac{1}{x} \quad \text{by the product rule}$$

$$\Rightarrow y' = y(\ln x + 1) = x^x(\ln x + 1).$$

The rule $(x^n)' = nx^{n-1}$ is <u>only</u> valid when n is constant!!!

Extreme value theorem Suppose that $f(x)$ is a continuous function defined on some <u>closed</u> interval $[a,b]$. Then $f(x)$ necessarily attains a minimum value and a maximum value on $[a,b]$. The proof will be given in MAU11204 next term.



<u>Theorem</u> (Rolle's theorem) Suppose that f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). If $f(a)=f(b)$, then there exists some $a<c<b$ such that $f'(c)=0$.

- We have established Bolzano's theorem which is related to <u>existence</u> of roots (f <u>continuous</u> on $[a,b]$ with $f(a), f(b)$ of opposite sign, then $f(c)=0$ at some point $a<c<b$).
- Rolle's theorem can be used to study <u>uniqueness</u> of roots. If we assume f has two roots x_1, x_2 then $f(x_1)=0=f(x_2)$ and Rolle's theorem implies that f' has a root in between.

<u>Example 1</u>. We show $f(x)=x^5-5x^3+1$ has a <u>unique</u> root in $(0,1)$.

Existence: f polynomial $\Rightarrow$ f continuous on $[0,1]$.

We compute $f(0)=1>0$

and $f(1)=1-5+1<0$.

By Bolzano's theorem, f has a root $0<c<1$.

Uniqueness: Suppose f has two roots in $(0,1)$, say $0<x_1<x_2<1$.

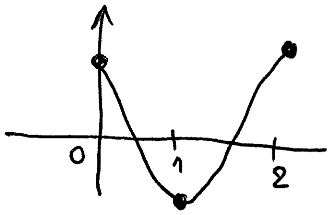
Then $f(x_1)=0=f(x_2)$, so Rolle's theorem gives

$f'(x_3)=0$ for some $0<x_1<x_3<x_2<1$.

In this case, $f'(x)=5x^4-15x^2=5x^2(x^2-3)$ has roots $0, \pm\sqrt{3}$ and none of those lie in $(0,1)$. We thus have a contradiction which proves the root is unique.

Example 2. We show $f(x) = 2x^3 - x^2 - 4x + 1$ has <u>exactly</u> <u>two</u> roots in the interval $(0,2)$. To prove existence, we compute

$$f(0) = 1 > 0$$



$$f(1) = 2 - \cancel{1} - 4 \cancel{+1} < 0$$

$$f(2) = 16 - 4 - 8 + 1 > 0 .$$

Then f has a root in $(0,1)$ and another root in $(1,2)$. This gives two roots in $(0,2)$. We need to show that f has <u>exactly</u> two roots in $(0,2)$. Suppose it had three roots,

$$0 < x_1 < x_2 < x_3 < 2 .$$

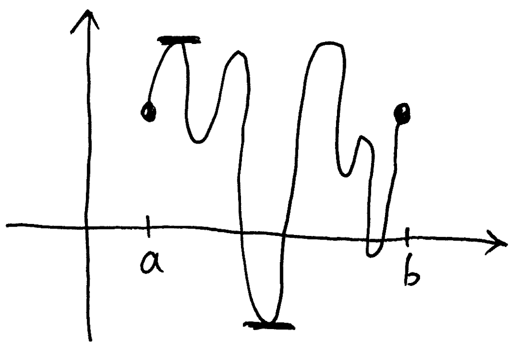
Then f' has a root in (x_1, x_2) by Rolle's theorem and a root in (x_2, x_3) for similar reasons. Thus f' has 2 roots in $(0,2)$. On the other hand,

$$f'(x) = 6x^2 - 2x - 4 = 2(3x^2 - x - 2)$$

has roots $x = \dfrac{1 \pm \sqrt{1+24}}{6} = \dfrac{1 \pm 5}{6}$ $\to 6/6 = 1$, $\to -4/6 = -2/3$.

Only one of those lies in $(0,2)$, a contradiction. This proves that f has exactly 2 roots in $(0,2)$.

<u>Proof of Rolle's theorem.</u>



We assume f continuous on $[a,b]$ and f differentiable on (a,b) with $f(a) = f(b)$.

This implies f attains a minimum value & a maximum value. We consider the points x_{min}, x_{max} at which those values occur. We show that $f'(x_{min}) = 0$, unless x_{min} is an endpoint.

⊙ Indeed, if $f'(x_{min}) > 0$, then $\lim_{x \to x_{min}} \dfrac{f(x) - f(x_{min})}{x - x_{min}} > 0$

so $\dfrac{f(x)-f(x_{min})}{x-x_{min}} > 0$ at points near x_{min}.

This gives $f(x)-f(x_{min}) < 0$ when $x - x_{min} < 0$

namely $f(x) < f(x_{min})$ when $x < x_{min}$, a contradiction.

⊙ Similarly, if $f'(x_{min}) < 0$, then $\dfrac{f(x)-f(x_{min})}{x-x_{min}} < 0$ at points near x_{min}. This gives $f(x)-f(x_{min}) < 0$ when $x-x_{min} > 0$, namely $f(x) < f(x_{min})$ when $x > x_{min}$, a contradiction.

This argument works at all interior points (but not the endpoints). If both the minimum & the maximum are attained at the endpoints a, b so minimum value = maximum value $= f(a) = f(b)$

so $f(x) =$ constant on $[a,b]$

so $f'(x) = 0$ on $[a,b]$. ∎

Example 3. Consider $f(x) = x^3 + (a+1)x + a^2 - 1$ for some $a > 0$. This can be graphed only for particular values of a. However, Bolzano's / Rolle's theorem are still applicable.

Consider the interval $(-a, a+1)$. Since f is continuous on $[-a, a+1]$ with

$$f(-a) = -a^3 + (a+1)(-a) + a^2 - 1$$
$$= -a^3 - \cancel{a^2} - a + \cancel{a^2} - 1 < 0$$

and

$$f(a+1) = (a+1)^3 + (a+1)^2 + a^2 - 1$$
$$= (a+1)^3 + (a+1)^2 + (a+1)(a-1) = (a+1)\left((a+1)^2 + a\cancel{+1} + a\cancel{-1}\right)$$
$$= (a+1)\left((a+1)^2 + 2a\right) > 0.$$

We get a root in $(-a, a+1)$ by Bolzano's theorem. To prove uniqueness of this root, suppose f has 2 roots in $(-a, a+1)$. Then f' has 1 root in $(-a, a+1)$ by Rolle's theorem, still $f'(x) = 3x^2 + a + 1$ is clearly positive so that is a contradiction.

Rolle's theorem	Mean value theorem
Suppose f is continuous on $[a,b]$ and differentiable on (a,b) with $f(a)=f(b)$. Then $f'(c)=0$ at some point $a<c<b$.  	Suppose f is continuous on $[a,b]$ and differentiable on (a,b). Then there exists a point $a<c<b$ with $f'(c) = \dfrac{f(b)-f(a)}{b-a}$. Thus, instantaneous rate of change = average rate of change at some point.

__Example 1.__ We use the mean value theorem to estimate $|f(b)-f(a)|$ for the functions $f(x)=\sin x$, $f(x)=\cos x$ and $f(x)=\tan^{-1}x$.

- In the first case, $\dfrac{|f(b)-f(a)|}{|b-a|} = |f'(c)|$ at some point c

  so $\dfrac{|\sin b - \sin a|}{|b-a|} = |\cos c|$ at some point c.

  This gives $\dfrac{|\sin b - \sin a|}{|b-a|} \leq 1$ and thus $|\sin b - \sin a| \leq |b-a|$ for all a,b.

- In the second case, we get

  $\dfrac{|\cos b - \cos a|}{|b-a|} = |f'(c)| = |\sin c| \leq 1$ once again

  so $|\cos b - \cos a| \leq |b-a|$ for all a,b.

- In the third case,

  $\dfrac{|\tan^{-1} b - \tan^{-1} a|}{|b-a|} = |f'(c)| = \dfrac{1}{1+c^2} \leq 1$ so $|\tan^{-1}b - \tan^{-1}a| \leq |b-a|$ as well.

__Example 2.__ We use the mean value theorem to approximate $\sqrt{2}$.

We know $\sqrt{1}=1$ and we need to approximate $\sqrt{2}$.

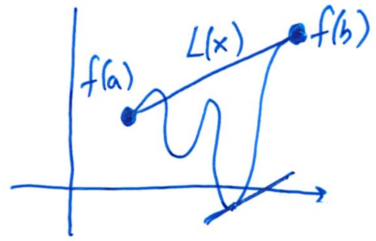
Take $f(x)=\sqrt{x}$ on the interval $[1,2]$. Then $f'(x) = \dfrac{1}{2\sqrt{x}}$ and

$$\frac{f(2)-f(1)}{2-1} = \frac{\sqrt{2}-1}{2-1} = f'(c) = \frac{1}{2\sqrt{c}} \text{ at \underline{some} point } 1<c<2.$$

Since $1<c<2$, $1<\sqrt{c}<\sqrt{2}<2$ and $2<2\sqrt{c}<4$ and $\dfrac{1}{2} > \dfrac{1}{2\sqrt{c}} > \dfrac{1}{4}$.

This gives $\dfrac{1}{2} > \dfrac{\sqrt{2}-1}{\cancel{2-1}} > \dfrac{1}{4}$ or simply $1.5 > \sqrt{2} > 1.25$.

Proof of mean value theorem



We compare $f(x)$ to the line $L(x)$ that passes through the points $(a, f(a))$ and $(b, f(b))$. Note that the line has slope $\dfrac{f(b)-f(a)}{b-a}$.

Consider the line ---- $g(x) = \dfrac{f(b)-f(a)}{b-a} \cdot (x-a) + f(a)$

and the difference between the two, namely $H(x) = f(x) - g(x)$.

We have $H(a) = H(b) = 0$ because both functions agree at a, b.

Also f, g are continuous on $[a,b] \Rightarrow H$ is also

f, g are differentiable on $(a,b) \Rightarrow H$ is also.

It follows by Rolle's theorem that $H'(c) = 0$ at some point.

This gives $0 = H'(c) = f'(c) - g'(c) = f'(c) - \dfrac{f(b)-f(a)}{b-a}$, as needed ∎

L'Hôpital's Rule for limits 0/0 and ∞/∞

Consider the case $\lim\limits_{x\to a} f(x) = \lim\limits_{x \to a} g(x) = L$ with $L = 0, \infty, -\infty$.

Then one has $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$. The same applies for limits as $x \to a^+$, $x \to a^-$, $x \to +\infty$, $x \to -\infty$ as well.

<u>Example 1.</u> $\lim\limits_{x\to 1} \dfrac{x^2-3x+2}{x^2-5x+4}$.

Our old approach gives ----- $\lim\limits_{x\to 1} \dfrac{\cancel{(x-1)}(x-2)}{\cancel{(x-1)}(x-4)} = \dfrac{1-2}{1-4} = 1/3$

after factoring numerator & denominator.

Our new approach using L'Hôpital's rule ...

$$\lim_{x\to 1} \frac{x^2-3x+2}{x^2-5x+4} \left(\frac{0}{0}\right) = \lim_{x\to 1} \frac{2x-3}{2x-5} = \frac{2-3}{2-5} = 1/3.$$

Example 2. $$\lim_{x\to 2} \frac{x^4-16}{3x-6} \left(\frac{0}{0}\right) = \lim_{x\to 2}\frac{4x^3}{3} = \frac{32}{3}.$$

Example 3. $$\lim_{x\to\infty}\frac{\ln x}{x^2+1}\left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty}\frac{1/x}{2x} = \lim_{x\to\infty}\frac{1}{2x^2} = 0.$$

Example 4. $$\lim_{x\to\infty}\frac{x^2+4x+3}{e^x}\left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty}\frac{2x+4}{e^x}\left(\frac{\infty}{\infty}\right)$$
$$= \lim_{x\to\infty}\frac{2}{e^x} = 0.$$

Example 5. $$\lim_{x\to 0}\frac{e^x-1}{\sin(2x)}\left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{e^x}{2\cos(2x)} = 1/2.$$

Proof of L'Hôpital's Rule for $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$.

We have to prove this for $0/0$, ∞/∞ limits when $x\to a^+$, $x\to a^-$, $x\to\infty$, $x\to-\infty$.

- We look at the case $x\to 0^+$ first. The others follow from this case:

$$\lim_{x\to a^+}\frac{f(x)}{g(x)} = \lim_{z\to 0^+}\frac{f(a+z)}{g(a+z)} = \lim_{z\to 0^+}\frac{f'(a+z)}{g'(a+z)} = \lim_{x\to a^+}\frac{f'(x)}{g'(x)} \qquad z = x-a$$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{z\to 0^+}\frac{f(1/z)}{g(1/z)} = \lim_{z\to 0^+}\frac{f'(1/z)\cdot\cancel{(1/z)'}}{g'(1/z)\cdot\cancel{(1/z)'}} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)} \qquad z = 1/x$$

$$\lim_{x\to-\infty}\frac{f(x)}{g(x)} = \lim_{z\to 0^+}\frac{f(-1/z)}{g(-1/z)} = \lim_{z\to 0^+}\frac{f'(-1/z)\cdot(-1/z)'}{g'(-1/z)\cdot(-1/z)'} = \lim_{x\to-\infty}\frac{f'(x)}{g'(x)}. \qquad z=-1/x$$

- To prove it for the case $x\to 0^+$, fix some small $x_0 > 0$.
  We consider $H(x) = f(x)\,g(x_0) - f(x_0)\,g(x)$ on $[0,x_0]$.
  This is differentiable with $H'(x) = f'(x)\,g(x_0) - f(x_0)\,g'(x)$.
  When $f(0)=0$ and $g(0)=0$, we get $H(0) = f(0)\,g(x_0) - f(x_0)\,g(0) = 0$
  $H(x_0) = f(x_0)\,g(x_0) - f(x_0)\,g(x_0) = 0.$
  By Rolle's theorem, $H'(c)=0$ at some point $0<c<x_0$
  $f'(c)\,g(x_0) = f(x_0)\,g'(c)$ at some point c
  $$\frac{f'(c)}{g'(c)} = \frac{f(x_0)}{g(x_0)} \text{ at some point } 0<c<x_0.$$

When $x_0 \to 0$, we have $c \to 0$ as well and

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x_0 \to 0} \frac{f(x_0)}{g(x_0)} = \lim_{x_0\to 0} \frac{f'(c)}{g'(c)} = \lim_{c\to 0} \frac{f'(c)}{g'(c)}.$$

This settles the case $0/0$ when $f(0) = g(0) = 0$. The other case follows from this by noting that

$$\lim_{x\to a} \frac{f(x)}{g(x)} \left(\frac{\infty}{\infty}\right) \quad \text{can be written as} \quad \lim_{x\to a} \frac{1/g(x)}{1/f(x)} \left(0/0\right)$$

so our previous work becomes applicable. $\blacksquare$

$\boxed{\text{Computation of limits}}$

- $0/0$ limits and ∞/∞ limits $\longrightarrow$ L'Hôpital's rule
- $0 \cdot \infty$ limits $\longrightarrow$ rearrange terms to write as either $0/0$ or ∞/∞.

  In other words, one has $f \cdot g = \dfrac{g}{1/f} = \dfrac{f}{1/g}$.
- 0^∞, 0^0, 1^∞, ∞^0 $\longrightarrow$ one may use $\ln$ to eliminate the exponents.

Example 1. Consider $\lim_{x\to 0^+} x \cdot \ln x$. This has the form $0 \cdot (-\infty)$.

We can write

$$\lim_{x\to 0^+} x \cdot \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} \quad \left(\frac{-\infty}{+\infty}\right)$$

$$= \lim_{x\to 0^+} \frac{(\ln x)'}{(1/x)'} \quad \text{by L'Hôpital's rule}$$

$$= \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} \frac{x}{-1} = 0.$$

Example 2. Consider $L = \lim_{x\to 0^+} (1+ax)^{1/x}$ with $a>0$ fixed.

This has the form 1^∞, so we try to eliminate the exponent.

Theorem. (Interchanging functions and limits) When f is a continuous function,
$$f\left(\lim_{x\to a} g(x)\right) = \lim_{x\to a} f(g(x)).$$
In particular,
$$\sqrt{\lim_{x\to a} g(x)} = \lim_{x\to a}\sqrt{g(x)}, \qquad \ln\left(\lim_{x\to a} g(x)\right) = \lim_{x\to a}\ln g(x) \quad \text{etc.}$$
This is because $g(x)\to L$ implies $f(g(x))\to f(L)$ by continuity.

To compute $L = \lim_{x\to 0^+}(1+ax)^{1/x}$, we write
$$\Rightarrow \ln L = \ln \lim_{x\to 0^+}(1+ax)^{1/x} = \lim_{x\to 0^+}\ln(1+ax)^{1/x}$$
$$\Rightarrow \ln L = \lim_{x\to 0^+}\frac{1}{x}\cdot\ln(1+ax) \quad \cdots (\infty\cdot 0)$$
$$\Rightarrow \ln L = \lim_{x\to 0^+}\frac{\ln(1+ax)}{x} \quad \left(\frac{0}{0}\right)$$
$$\Rightarrow \ln L = \lim_{x\to 0^+}\frac{\ln(1+ax)'}{x'} = \lim_{x\to 0^+}\frac{\frac{1}{1+ax}\cdot(1+ax)'}{1}$$
$$\Rightarrow \ln L = \lim_{x\to 0^+}\frac{a}{1+ax} = a$$
$$\Rightarrow \cancel{e}^{\ln L} = e^a \Rightarrow L = e^a.$$

$\boxed{\text{Monotonicity}}$ We say that <u>f is increasing</u> in some interval I if $a<b$ implies $f(a)<f(b)$ in that interval.
We say that <u>f is decreasing</u> in I, if $a<b$ implies $f(a)>f(b)$.

$\boxed{\text{Monotonicity theorem}}$ Suppose f is differentiable on some interval I.

(a) If $f'(x)>0$ for all x in I, then f is increasing in I.

(b) If $f'(x)<0$ for all x in I, then f is decreasing in I.

<u>Relation to inequalities</u> If $a<b$, then $5a<5b$.

This is because $f(x)=5x$ is increasing, namely $f'(x)=5$ is positive.

On the other hand, $f(x)=-3x$ satisfies $f'(x)=-3<0$

so $f(x)$ is decreasing and $a<b \Rightarrow -3a>-3b$.

⊙ Similarly, $a<b \Rightarrow a^2<b^2$ is not true, in general.

In this case, $f(x)=x^2$ and $f'(x)=2x$ is positive for $x>0$
but negative for $x<0$.

Thus, $0<a<b \Rightarrow a^2<b^2$

and $a<b<0 \Rightarrow a^2>b^2$.

<u>Proof of monotonicity theorem.</u> We need to relate $a<b$ and $f(a)<f(b)$.

We know $\dfrac{f(b)-f(a)}{b-a}=f'(c)$ at some $a<c<b$ by Mean Value.

If a,b are in the interval, then so is c.

For part ⓐ, $\dfrac{f(b)-f(a)}{b-a}=f'(c)>0$ so $f(b)-f(a)>0$
so $f(b)>f(a)$.

For part ⓑ, $\dfrac{f(b)-f(a)}{b-a}=f'(c)<0$ so $f(b)-f(a)<0$
so $f(b)<f(a)$. ▨

Monotonicity theorem

ⓐ If $f'(x) > 0$ in some interval I, then $f(x)$ is increasing within I.

ⓑ If $f'(x) < 0$ in some interval I, then $f(x)$ is decreasing within I.

Example 1. Consider $f(x) = x^4 \cdot \ln x$ with $x > 0$.

We check for monotonicity. We have

$$f'(x) = (x^4)' \cdot \ln x + x^4 \cdot (\ln x)' = 4x^3 \ln x + x^3 = x^3(4\ln x + 1).$$

⊙ To say $f'(x) > 0$ is to say $\cancel{x^3(\ln x + 1)}$ $x^3(4\ln x + 1) > 0$,

namely $4\ln x + 1 > 0$.

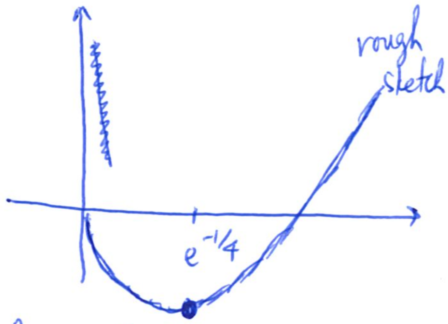
Since $4\ln x > -1 \iff \ln x > -1/4 \iff e^{\ln x} > e^{-1/4} \iff x > e^{-1/4}$,

we find $f(x)$ increasing on $(e^{-1/4}, \infty)$

and also decreasing on $(0, e^{-1/4})$ for similar reasons.

In our case

$$f(e^{-1/4}) = (e^{-1/4})^4 \ln e^{-1/4} = e^{-1}(-1/4) = -\frac{1}{4e}$$



and

$$\lim_{x\to 0^+} x^4 \ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-4}} \quad \left(\frac{-\infty}{+\infty}\right)$$

$$= \lim_{x\to 0^+} \frac{x^{-1}}{-4x^{-5}} = \lim_{x\to 0^+} \frac{x^5}{-4x} = -\lim_{x\to 0^+} \frac{x^4}{4} = 0,$$

while $\lim_{x\to\infty} x^4 \ln x = +\infty$ obviously.

Example 2. Consider $f(x) = x^4 - 8x^2 + 6$ for any x.

Then $f'(x) = 4x^3 - 16x$ ..... factor to determine sign

$$= 4x(x^2 - 4)$$

$$= 4x\,(x+2)(x-2).$$

This becomes zero when $x = -2$, $x = 0$, $x = 2$.

By the table,

$f'(x) > 0 \iff x \in (-2, 0) \cup (2, +\infty)$

and $f'(x) < 0 \iff x \in (-\infty, -2) \cup (0, 2)$

		-2		0		2	
$4x$	$-$		$-$		$+$		$+$
$x+2$	$-$		$+$		$+$		$+$
$x-2$	$-$		$-$		$-$		$+$
$f'(x)$	$-$		$+$		$-$		$+$

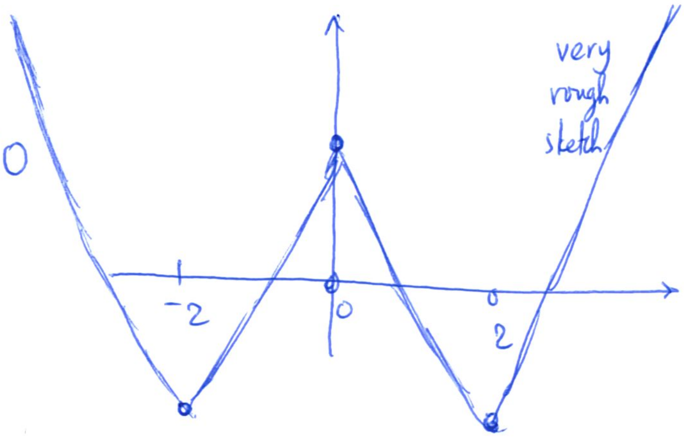
We check the values at the relevant points. We have

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x^4 = \infty$$

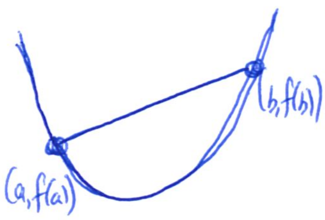
$$f(\pm 2) = 16 - 8\times 4 + 6 = -10$$

$$f(0) = 6$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x^4 = +\infty .$$



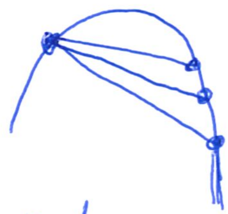
Concavity We say f is <u>concave up</u> in some interval if any line connecting $(a, f(a))$ and $(b, f(b))$ lies <u>above</u> the graph of f. This means that



$$(*) \quad f(x) < \frac{f(b)-f(a)}{b-a}(x-a) + f(a) \quad \text{for all } a<x<b .$$

We say f is <u>concave down</u> in some interval, if

$$f(x) > \frac{f(b)-f(a)}{b-a}(x-a) + f(a) \quad \text{for all } a<x<b \text{ in the interval.}$$



The definition is related to derivatives because $(*)$ reads

$$\frac{f(x)-f(a)}{x-a} < \frac{f(b)-f(a)}{b-a}$$

and the Mean Value theorem becomes applicable.

Concavity theorem

(a) If $f''(x) > 0$ in some interval I, then f is concave up in I.

(b) If $f''(x) < 0$ in some interval I, then f is concave down in I.

<u>Example</u>. Consider $f(x) = ax^2 + bx + c$. Then $f'(x) = 2ax + b$ and $f''(x) = 2a$. When $a > 0$, we get $f'' > 0$ at all points and the graph looks like $\cup$. When $a<0$, we get $\cap$.

<u>Example</u>. Consider $f(x) = x^3 - 3x + 1$.

Then $f'(x) = 3x^2 - 3 = 3(x^2-1) = 3(x+1)(x-1)$

and $f''(x) = 6x$.

$\odot$ Monotonicity is determined by the sign of f'

We get f increasing on $(-\infty, -1)$
decreasing on $(-1, 1)$
increasing on $(1, \infty)$.

		-1		1	
$3(x+1)$	$-$		$+$		$+$
$x-1$	$-$		$-$		$+$
$f'(x)$	$+$		$-$		$+$

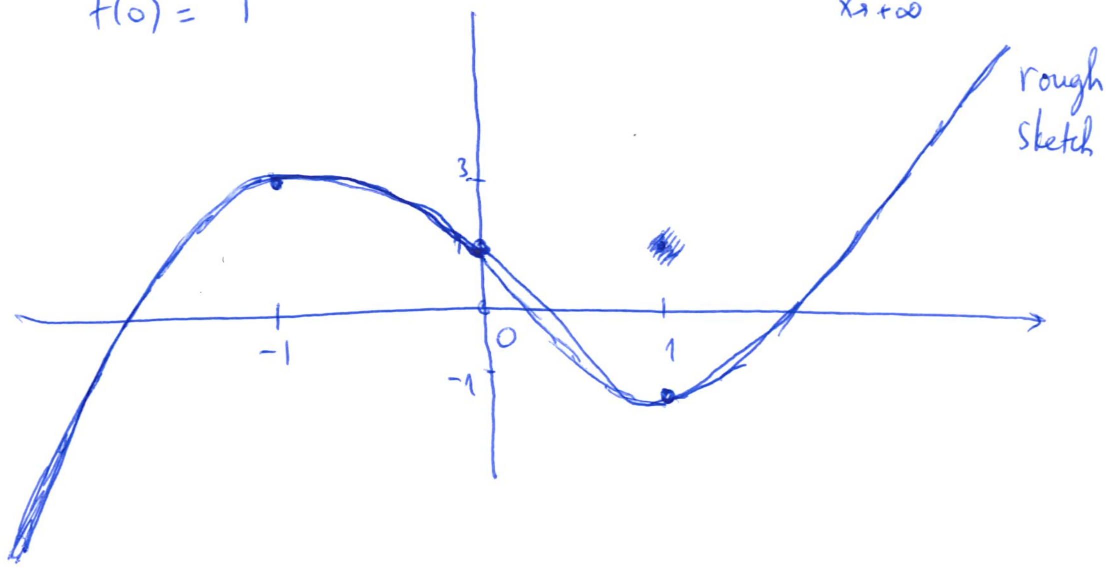
$\odot$ Concavity is determined by the sign of f''.

We get f concave down on $(-\infty, 0)$
concave up on $(0, \infty)$.

$\odot$ Compute $f(-1) = -1+3+1 = 3$
$f(1) = 1-3+1 = -1$
$f(0) = 1$

and $\lim\limits_{x\to-\infty} f(x) = -\infty$

$\lim\limits_{x\to+\infty} f(x) = +\infty$.



<u>Proof of concavity theorem.</u> We assume $f''(x) > 0$ in some interval.

We need to show f is concave up, namely

$$f(x) < \frac{f(b)-f(a)}{b-a}(x-a) + f(a) \quad \text{for all } a<x<b$$

or just ① $$\frac{f(x)-f(a)}{x-a} < \frac{f(b)-f(a)}{b-a} \quad \text{for all } a<x<b.$$

The mean value theorem gives $\dfrac{f(x)-f(a)}{x-a} = f'(c)$ for <u>some</u> $a<c<x$

and $\dfrac{f(b)-f(x)}{b-x} = f'(d)$ for <u>some</u> $x<d<b$.

Now, $f''>0$ means f' increasing so

$$\begin{array}{ccccccc} & & c & & d & & \\ \vdash & & + & \vdash & + & & \dashv \\ a & & & x & & & b \end{array}$$

$f'(c) < f'(d)$ so $\dfrac{f(x)-f(a)}{x-a} < \dfrac{f(b)-f(x)}{b-x}$. ②

We established ② but we need ①. These are actually equivalent. Starting with ② we get

$$\frac{f(x)-f(a)}{x-a} < \frac{f(b)-f(x)}{b-x} \iff \underline{bf(x)} - bf(a) - \cancel{xf(x)} + xf(a) < xf(b) - \cancel{xf(x)} - af(b) + \underline{\underline{af(x)}}$$

$$\iff (b-a)f(x) + \underline{xf(a)} < xf(b) + \underline{bf(a)} - af(b)$$

$$\cancel{\iff (b-a)f(x) + xf(a) < xf(b) + (b}$$

$$\iff (b-a)f(x) + (x-a+a-b)f(a) < (x-a)f(b)$$

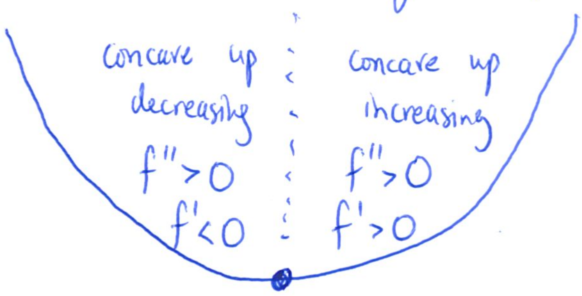
$$\iff (b-a)f(x) + (x-a)f(a) + (a-b)f(a) < (x-a)f(b)$$

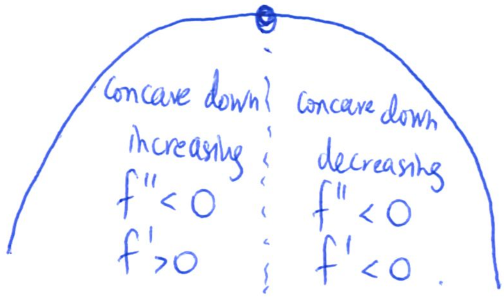
$$\iff (b-a)\big(f(x)-f(a)\big) < (x-a)\big(f(b)-f(a)\big)$$

$$\iff \frac{f(x)-f(a)}{x-a} < \frac{f(b)-f(a)}{b-a}$$, as needed. ▨

Monotonicity, concavity & graphing

There are 4 different types of behaviour for a function f:





<u>Example 1.</u> Consider $f(x) = x^3 - 3x^2 - 9x$, for instance.

Then $f'(x) = 3x^2 - 6x - 9 = 3(x^2-2x-3) = 3(x+1)(x-3)$

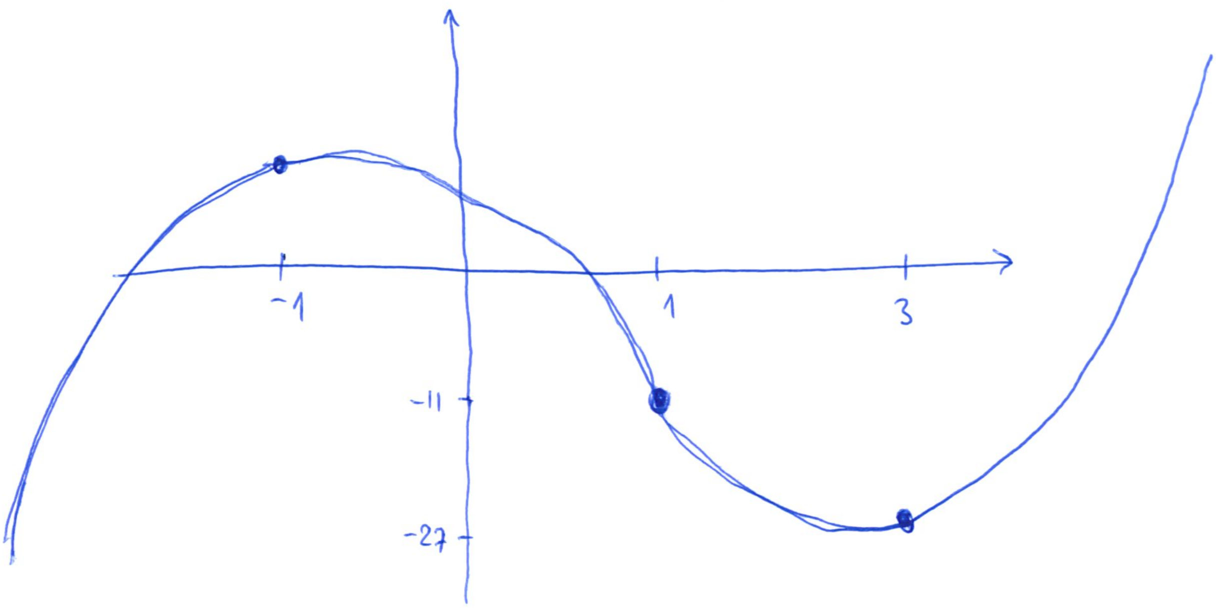
and $f''(x) = 6x - 6 = 6(x-1)$.

Points of interest .... $-1, 3$ because of f' and 1 because of f''.

	-1	1	3	
$3(x+1)$	$-$	$+$	$+$	$+$
$x-3$	$-$	$-$	$-$	$+$
$f'(x)$	$+$	$-$	$-$	$+$
$f''=6(x-1)$	$-$	$-$	$+$	$+$
f	↗	↘	↘	↗

We compute $f(-1) = 5$, $f(1) = -11$, $f(3) = -27$

$$\lim_{x\to-\infty} f(x) = -\infty, \qquad \lim_{x\to+\infty} f(x) = +\infty.$$



Example 2. Consider $f(x) = \dfrac{x^2+3}{1-x}$ for any $x \neq 1$.

Then $$f'(x) = \frac{2x(1-x) + x^2+3}{(1-x)^2} = \frac{2x-2x^2+x^2+3}{(1-x)^2} = \frac{-x^2+2x+3}{(1-x)^2}$$

and $$f''(x) = \frac{(-2x+2)\cdot(1-x)^{\cancel{2}} + 2\cancel{(1-x)}(-x^2+2x+3)}{(1-x)^{\cancel{4}\,3}}$$

$$= \frac{\cancel{-2x}+\cancel{2x^2}+2\cancel{-2x}\;\cancel{-2x^2}+\cancel{4x}+6}{(1-x)^3} = \frac{8}{(1-x)^3}.$$

The sign of f'' is easy. The sign of f' is determined through

$$f'(x) = -\frac{x^2-2x-3}{(1-x)^2} = -\frac{(x+1)(x-3)}{(1-x)^2}.$$

Points of interest ..... $-1, 3$ because of f', 1 because of f.

	-1	1	3	
$-(x+1)$	$+$	$-$	$-$	$-$
$(x-3)$	$-$	$-$	$-$	$+$
$f'(x)$	$-$	$+$	$+$	$-$
$8/(1-x)^3 = f''(x)$	$+$	$+$	$-$	$-$
$f(x)$				

We compute $f(x) = \dfrac{x^2+3}{1-x} \implies$ $f(-1) = 2$

$f(1)$ not defined

$f(3) = -6$

and

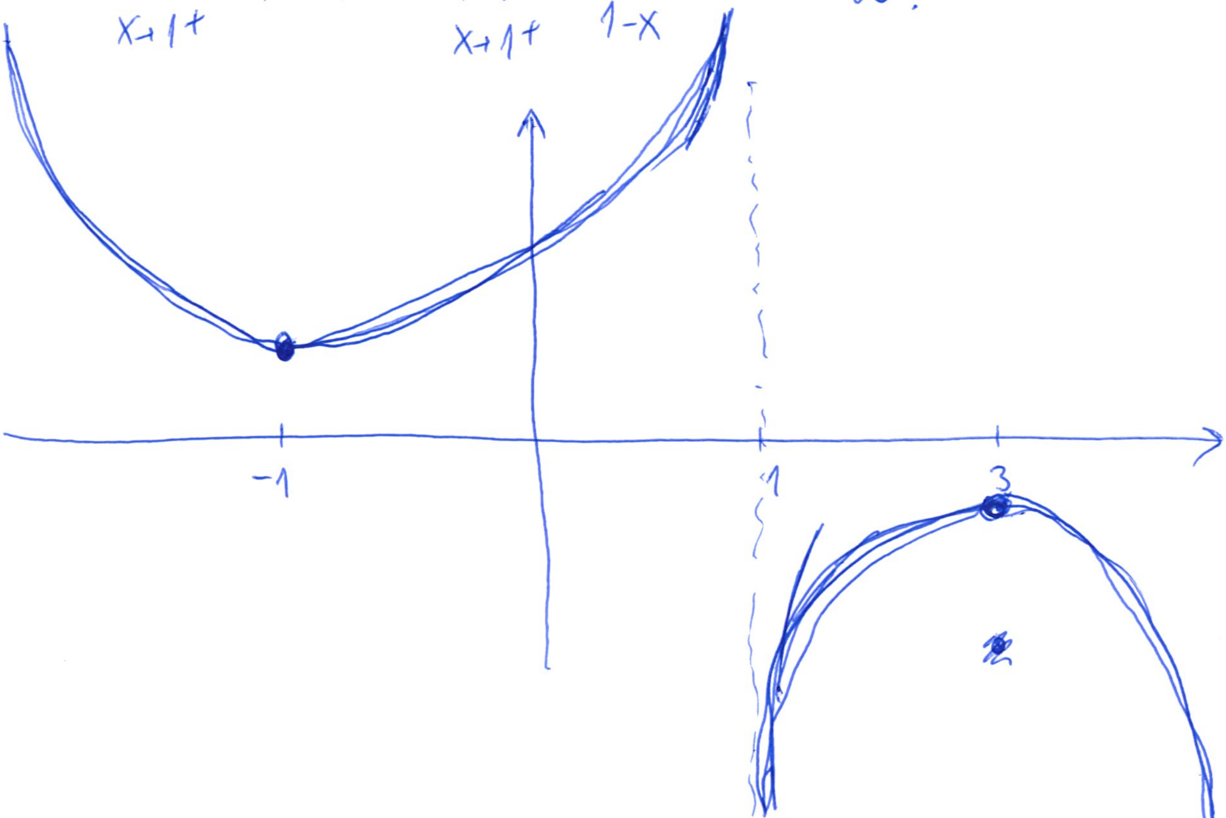
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2}{-x} = \lim_{x\to-\infty} (-x) = +\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^2}{-x} = \lim_{x\to+\infty} (-x) = -\infty$$

and

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{4}{1-x} = +\infty$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{4}{1-x} = -\infty.$$



$\boxed{\text{Local maximum/minimum}}$

We call x_0 a point of a <u>local maximum</u> if $f(x)$ only attains smaller values at nearby points: $f(x) \leq f(x_0)$ for x near x_0.

We call x_0 a point of a <u>local minimum</u> if $f(x) \geq f(x_0)$ at nearby points. There are 2 different tests for checking that x_0 is a local min/max using either f' or f''.

<u>Local minimum at x_0</u> means $f(x) \geq f(x_0)$ for all points x near x_0.

<u>Local maximum at x_0</u> means $f(x) \leq f(x_0)$ for all points x near x_0.

First derivative test Suppose that $f'(x)$ changes sign at x_0.

ⓐ If it changes from ⊖ to ⊕, then f has a <u>local</u> <u>min</u> at x_0.

ⓑ If it changes from ⊕ to ⊖, then f has a <u>local</u> <u>max</u> at x_0.

Second derivative test Suppose that $f'(x_0) = 0$.

ⓐ If $f''(x_0) > 0$, then f has a <u>local</u> <u>min</u> at x_0.

ⓑ If $f''(x_0) < 0$, then f has a <u>local</u> <u>max</u> at x_0.

<u>Proofs.</u> For the first derivative test, part ⓐ

f' is negative on the left of x_0, so f decreases there and $f(x) \geq f(x_0)$ there

and f' is positive on the right of x_0, so $f(x) \nearrow$ and $f(x) \geq f(x_0)$.

For the second derivative test, part ⓐ

$f''(x_0) > 0$ means f' increasing at x_0

Since $f' = 0$ at x_0 & increasing at x_0,

we get f' changing from ⊖ to ⊕

and the first derivative test becomes applicable. $\blacksquare$

<u>Example 1.</u> Consider $f(x) = x(x+3)^2$.

Then $f'(x) = (x+3)^2 + 2(x+3) \cdot x = (x+3)(x+3+2x)$

so $f'(x) = 3(x+3)(x+1)$.

To determine the sign of f', we use the table. By the first derivative test ... $x=-3$ gives a local max

$x=-1$ gives a local min

		-3		-1	
$3(x+3)$	$-$		$+$		$+$
$x+1$	$-$		$-$		$+$
$f'(x)$	$+$		$-$		$+$

(ii) Let's use the second derivative test, instead.

We get $f'(x_0)=0 \Rightarrow x_0=-3$ and $x_0=-1$.

Since $f''(x) = 3(x+1)+3(x+3) = 3(2x+4) = 6(x+2)$

$f''(-3) = 6(-1) < 0$ ... and $x_0=-3$ local max

and $f''(-1) = 6(1) > 0$ ... and $x_0=-1$ local min

Example 2. Consider $f(x) = \dfrac{x^2}{x^4+4}$.

Then $f'(x) = \dfrac{2x(x^4+4) - 4x^3 \cdot x^2}{(x^4+4)^2} = \dfrac{8x-2x^5}{(x^4+4)^2}$

so $f'(x) = \dfrac{2x(4-x^4)}{(x^4+4)^2} = \dfrac{2x(2-x^2)(2+x^2)}{(x^4+4)^2}$

So $f'(x) = \dfrac{2x(\sqrt{2}-x)(\sqrt{2}+x)(2+x^2)}{(x^4+4)^2}$.

To use the 2nd derivative test, one would need f'' which gets messy.

To use the 1st derivative test, we use a table

		$-\sqrt{2}$		0		$\sqrt{2}$	
$2x$	$-$		$-$		$+$		$+$
$\sqrt{2}-x$	$+$		$+$		$+$		$-$
$\sqrt{2}+x$	$-$		$+$		$+$		$+$
$2+x^2$	$+$		$+$		$+$		$+$
$f'(x)$	$+$		$-$		$+$		$-$
		local max		local min		local max	

$\boxed{\text{Global or Absolute Min / Max}}$ We say $f(x)$ has a <u>global</u> <u>min</u> at x_0, if $f(x) \geqslant f(x_0)$ for all x in the domain of f.

We say f has <u>globax</u> <u>max</u> at x_0, if $f(x) \leq f(x_0)$.

- Most functions do not have a global max/min. Namely, $f(x)$ can become arbitrarily large positive / negative.

$\boxed{\text{Case 1.} \mid \text{We get a global min/max when } f' \text{ changes sign once}}$

<u>Example A.</u> Consider $f(x) = x^4 - 4x$. We find the global min.

We have $f'(x) = 4x^3 - 4 = 4(x^3 - 1)$

$$\Rightarrow \quad f'(x) = 4(x-1)(x^2+x+1).$$

The quadratic x^2+x+1 has discriminant $b^2 - 4ac = 1 - 4 < 0$, so it cannot be factored and it has no real roots.

(The quadratic does <u>not</u> change sign.
It is positive when $x = 0 \Rightarrow$ positive for all x)

		1	
$4(x-1)$	$-$		$+$
x^2+x+1	$+$		$+$
$f'(x)$	$-$		$+$
$f(x)$	$\searrow$		$\nearrow$

The table makes $f(1) = 1 - 4 = -3$ the <u>global</u> <u>min</u> value attained by f.

<u>Example B.</u> Consider $f(x) = xe^{-x}$ for all x.

Then $f'(x) = e^{-x} + x(e^{-x})' = e^{-x} + x\, e^{-x}(-1)$

so $f'(x) = (1-x)e^{-x}$. In this case, e^{-x} is always positive (as an exponential). Thus, $f(x)$ has a global max which is

$$f(1) = e^{-1} = 1/e.$$

		1	
$1-x$	$+$		$-$
e^{-x}	$+$		$+$
$f'(x)$	$+$		$-$
$f(x)$	$\nearrow$		$\searrow$

We find that $xe^{-x} \leq 1/e$ for all x.

We have shown that $\dfrac{x}{e^x} \leq \dfrac{1}{e}$ for all x.