MAU11201 – Calculus
Tutorial solutions #1

1. Find the domain and the range of the function f which is defined by

$$f(x) = \frac{4-3x}{6-5x}.$$

The domain consists of all points $x \neq 6/5$. To find the range, we note that

$$y = \frac{4-3x}{6-5x} \iff 6y - 5xy = 4 - 3x \iff 6y - 4 = 5xy - 3x$$
$$\iff x(5y-3) = 6y - 4 \iff x = \frac{6y-4}{5y-3}.$$

The rightmost formula determines the value of x that satisfies $y = f(x)$. Since the formula makes sense for any number $y \neq 3/5$, the range consists of all numbers $y \neq 3/5$.

2. Find the domain and the range of the function f which is defined by

$$f(x) = \sqrt{x - x^2}.$$

When it comes to the domain, one needs $x - x^2 = x(1-x)$ to be non-negative, so the factors $x, 1-x$ must have the same sign. If $x \geq 0$, then $1 - x \geq 0$ and this gives $0 \leq x \leq 1$. If $x \leq 0$, then $1 - x \leq 0$ and this gives $1 \leq x \leq 0$, which is absurd. Thus, only the former case may arise and the domain is $[0, 1]$. To find the range, we note that $y = f(x) \geq 0$ and

$$y^2 = x - x^2 \iff x^2 - x + y^2 = 0 \iff x = \frac{1 \pm \sqrt{1-4y^2}}{2}.$$

Since the rightmost formula is only defined when $4y^2 \leq 1$, the range is then $[0, 1/2]$.

3. Show that the function $f\colon (0,1) \to (0,\infty)$ is bijective in the case that

$$f(x) = \frac{1}{x} - 1.$$

To show that the given function is injective, we note that

$$f(x_1) = f(x_2) \implies \frac{1}{x_1} - 1 = \frac{1}{x_2} - 1 \implies \frac{1}{x_1} = \frac{1}{x_2} \implies x_1 = x_2.$$

To show that the given function is surjective, we note that

$$y = f(x) \iff y = \frac{1}{x} - 1 \iff \frac{1}{x} = y + 1 \iff x = \frac{1}{y+1}.$$

The rightmost formula determines the value of x such that $y = f(x)$ and we need to check that $0 < x < 1$ if and only if $y > 0$. When $y > 0$, we have $y + 1 > 1 > 0$, so $0 < x < 1$. When $0 < x < 1$, we have $0 < 1 < \frac{1}{x}$ and this gives $y > 0$, as needed.

4. Express the following polynomials as the product of linear factors.

$$f(x) = 2x^3 - 7x^2 + 9, \qquad g(x) = x^3 - \frac{3x}{4} - \frac{1}{4}.$$

The possible rational roots for the first polynomial are $\pm 1, \pm 3, \pm 9, \pm 1/2, \pm 3/2, \pm 9/2$. Checking the first few, one finds that $x = -1$ and $x = 3$ are both roots. This implies that both $x + 1$ and $x - 3$ must be factors, so it easily follows by division that

$$f(x) = (x+1)(2x^2 - 9x + 9) = (x+1)(x-3)(2x-3).$$

Let us now turn to the second polynomial and clear denominators to write

$$4g(x) = 4x^3 - 3x - 1.$$

The possible rational roots are $\pm 1, \pm 1/2, \pm 1/4$. Checking these possibilities, one finds that only $x = 1$ and $x = -1/2$ are actually roots. It easily follows by division that

$$4g(x) = (x-1)(4x^2 + 4x + 1) = (x-1)(2x+1)^2 \quad \Longrightarrow \quad g(x) = \frac{1}{4}(x-1)(2x+1)^2.$$

5. Use the addition formulas for sine and cosine to prove the identity

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha \cdot \tan\beta}.$$

By definition, the tangent of an angle is the quotient of its sine and cosine, so

$$\tan(\alpha \pm \beta) = \frac{\sin(\alpha \pm \beta)}{\cos(\alpha \pm \beta)} = \frac{\sin\alpha \cdot \cos\beta \pm \cos\alpha \cdot \sin\beta}{\cos\alpha \cdot \cos\beta \mp \sin\alpha \cdot \sin\beta}.$$

Once we now divide both the numerator and the denominator by $\cos\alpha \cdot \cos\beta$, we get

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha \cdot \tan\beta}.$$

6. Show that the function $f\colon (0, \infty) \to \mathbb{R}$ is injective in the case that

$$f(x) = \frac{2x-1}{3x+2}.$$

We assume that $f(x_1) = f(x_2)$ and we clear denominators to get

$$\begin{aligned} \frac{2x_1 - 1}{3x_1 + 2} = \frac{2x_2 - 1}{3x_2 + 2} \quad &\Longrightarrow \quad (2x_1 - 1)(3x_2 + 2) = (2x_2 - 1)(3x_1 + 2) \\ &\Longrightarrow \quad 6x_1x_2 - 3x_2 + 4x_1 - 2 = 6x_1x_2 - 3x_1 + 4x_2 - 2. \end{aligned}$$

Once we now cancel the common terms, we may easily conclude that

$$-3x_2 + 4x_1 = -3x_1 + 4x_2 \quad \Longrightarrow \quad 7x_1 = 7x_2 \quad \Longrightarrow \quad x_1 = x_2.$$

7. Find the roots of the polynomial $f(x) = x^3 + x^2 - 5x - 2$.

The only possible rational roots are $\pm 1, \pm 2$ and one may check each of those to see that only $x = 2$ is a root. This implies that $x - 2$ is a factor and division of polynomials gives

$$f(x) = (x - 2)(x^2 + 3x + 1).$$

To find the roots of the quadratic factor, one may use the quadratic formula to get

$$x = \frac{-3 \pm \sqrt{9 - 4}}{2} = \frac{-3 \pm \sqrt{5}}{2}.$$

8. Determine the range of the quadratic $f(x) = ax^2 + bx + c$ in the case that $a > 0$.

We use the standard approach and solve $y = f(x)$ in terms of x. This gives

$$y = ax^2 + bx + c \quad \Longrightarrow \quad ax^2 + bx + (c - y) = 0 \quad \Longrightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4a(c - y)}}{2a}$$

and we need the discriminant to be non-negative, so we need to have

$$b^2 - 4ac + 4ay \geq 0 \quad \Longrightarrow \quad 4ay \geq 4ac - b^2 \quad \Longrightarrow \quad y \geq \frac{4ac - b^2}{4a}.$$

In other words, the range of the quadratic has the form $[y_*, +\infty)$, where $y_* = \frac{4ac - b^2}{4a}$.

9. Relate the sines and the cosines of two angles θ_1, θ_2 whose sum is equal to 2π.

Since $\theta_1 + \theta_2 = 2\pi$ by assumption, the addition formulas for sine and cosine give

$$\begin{aligned} \sin \theta_2 &= \sin(2\pi - \theta_1) = \sin(2\pi) \cdot \cos \theta_1 - \cos(2\pi) \cdot \sin \theta_1, \\ \cos \theta_2 &= \cos(2\pi - \theta_1) = \cos(2\pi) \cdot \cos \theta_1 + \sin(2\pi) \cdot \sin \theta_1. \end{aligned}$$

On the other hand, $\sin(2\pi) = 0$ and $\cos(2\pi) = 1$ by definition, so it easily follows that

$$\sin \theta_2 = -\sin \theta_1, \qquad \cos \theta_2 = \cos \theta_1.$$

10. Determine all angles $0 \leq \theta \leq 2\pi$ such that $2\cos^2 \theta + 7\cos \theta = 4$.

Letting $x = \cos \theta$ for convenience, we get $2x^2 + 7x - 4 = 0$ and thus

$$x = \frac{-7 \pm \sqrt{49 + 4 \cdot 8}}{2 \cdot 2} = \frac{-7 \pm \sqrt{81}}{4} = \frac{-7 \pm 9}{4} \quad \Longrightarrow \quad x = \frac{1}{2}, -4.$$

Since $x = \cos \theta$ must lie between -1 and 1, the only relevant solution is $x = \cos \theta = \frac{1}{2}$. In view of the graph of the cosine function, there should be two angles $0 \leq \theta \leq 2\pi$ that satisfy this condition. The first one is $\theta_1 = \frac{\pi}{3}$ and the second one is $\theta_2 = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$.

MAU11201 – Calculus
Tutorial solutions #2

1. Determine the inverse function f^{-1} in each of the following cases.

$$f(x) = \log_3(2x-5) - 1, \qquad f(x) = \frac{2\cdot 5^x + 7}{3\cdot 5^x - 4}.$$

When it comes to the first case, one can easily check that

$$y + 1 = \log_3(2x-5) \quad\Longleftrightarrow\quad 3^{y+1} = 2x - 5 \quad\Longleftrightarrow\quad x = \frac{5+3^{y+1}}{2},$$

so the inverse function is defined by $f^{-1}(y) = \frac{5+3^{y+1}}{2}$. When it comes to the second case,

$$y = \frac{2\cdot 5^x + 7}{3\cdot 5^x - 4} \quad\Longleftrightarrow\quad 3y\cdot 5^x - 4y = 2\cdot 5^x + 7 \quad\Longleftrightarrow\quad 4y + 7 = 5^x(3y-2)$$

and this gives $5^x = \frac{4y+7}{3y-2}$, so the inverse function is defined by $f^{-1}(y) = \log_5 \frac{4y+7}{3y-2}$.

2. Simplify each of the following expressions.

$$\sec\left(\tan^{-1} x\right), \qquad \cos\left(\sin^{-1} x\right), \qquad \log_2 18 - 2\log_2 3.$$

To simplify the first expression, let $\theta = \tan^{-1} x$ and note that $\tan\theta = x$. When $x \geq 0$, the angle θ arises in a right triangle with an opposite side of length x and an adjacent side of length 1. It follows by Pythagoras' theorem that the hypotenuse has length $\sqrt{1+x^2}$, so the definition of secant gives

$$\sec(\tan^{-1} x) = \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent side}} = \sqrt{1+x^2}.$$

When $x \leq 0$, the last equation holds with $-x$ instead of x. This changes the term $\tan^{-1} x$ by a minus sign, but the secant remains unchanged, so the equation is still valid.

To simplify the second expression, one may use a similar approach or simply note that

$$\theta = \sin^{-1} x \quad\Longrightarrow\quad \sin\theta = x \quad\Longrightarrow\quad \cos^2\theta = 1 - \sin^2\theta = 1 - x^2.$$

Since $\theta = \sin^{-1} x$ lies between $-\pi/2$ and $\pi/2$ by definition, $\cos\theta$ is non-negative and

$$\cos^2\theta = 1 - x^2 \quad\Longrightarrow\quad \cos\theta = \sqrt{1-x^2}.$$

As for the third expression, the standard properties of the logarithmic function give

$$\log_2 18 - 2\log_2 3 = \log_2 18 - \log_2 3^2 = \log_2 \frac{18}{3^2} = \log_2 2^1 = 1.$$

3. Use the ε-δ definition of limits to compute $\lim_{x\to 3} f(x)$ in the case that

$$f(x) = \left\{ \begin{array}{ll} 3x-7 & \text{if } x \leq 3 \\ 8-2x & \text{if } x > 3 \end{array} \right\}.$$

Note that x is approaching 3 and that $f(x)$ is either $3x-7$ or $8-2x$. We thus expect the limit to be $L=2$. To prove this formally, we let $\varepsilon > 0$ and estimate the expression

$$|f(x)-2| = \left\{ \begin{array}{ll} |3x-9| & \text{if } x \leq 3 \\ |6-2x| & \text{if } x > 3 \end{array} \right\} = \left\{ \begin{array}{ll} 3|x-3| & \text{if } x \leq 3 \\ 2|x-3| & \text{if } x > 3 \end{array} \right\}.$$

If we assume that $0 \neq |x-3| < \delta$, then we may use the last equation to get

$$|f(x)-2| \leq 3|x-3| < 3\delta.$$

Since our goal is to show that $|f(x)-2| < \varepsilon$, an appropriate choice of δ is thus $\delta = \varepsilon/3$.

4. Compute each of the following limits.

$$L = \lim_{x\to 2} \frac{x^3-2x^2+5x-1}{x-3}, \qquad M = \lim_{x\to 2} \frac{x^3-3x^2+4x-4}{x-2}.$$

The first limit is the limit of a rational function which is defined at $x=2$, so

$$L = \frac{2^3 - 2\cdot 2^2 + 5\cdot 2 - 1}{2-3} = -9.$$

The second limit involves a rational function which can be simplified. In fact, one has

$$M = \lim_{x\to 2} \frac{(x-2)(x^2-x+2)}{x-2} = \lim_{x\to 2}\,(x^2-x+2) = 2^2-2+2 = 4.$$

5. Use the ε-δ definition of limits to compute $\lim_{x\to 3}\,(3x^2-7x+2)$.

Let $f(x) = 3x^2-7x+2$ for convenience. Then $f(3) = 8$ and one has

$$|f(x)-f(3)| = |3x^2-7x-6| = |x-3|\cdot|3x+2|.$$

The factor $|x-3|$ is related to our usual assumption that $0 \neq |x-3| < \delta$. To estimate the remaining factor $|3x+2|$, we assume that $\delta \leq 1$ for simplicity and note that

$$\begin{aligned} |x-3| < \delta \leq 1 \quad &\Longrightarrow \quad -1 < x-3 < 1 \\ &\Longrightarrow \quad 2 < x < 4 \quad \Longrightarrow \quad 8 < 3x+2 < 14. \end{aligned}$$

Combining the estimates $|x-3| < \delta$ and $|3x+2| < 14$, one may then conclude that

$$|f(x)-f(3)| = |x-3|\cdot|3x+2| < 14\delta \leq \varepsilon,$$

as long as $\delta \leq \varepsilon/14$ and $\delta \leq 1$. An appropriate choice of δ is thus $\delta = \min(\varepsilon/14, 1)$.

6. For which value of a does the limit $\lim_{x\to 2} f(x)$ exist? Explain.

$$f(x) = \left\{ \begin{array}{ll} 2x^2 - ax + 3 & \text{if } x \leq 2 \\ 4x^3 + 3x - a & \text{if } x > 2 \end{array} \right\}.$$

Since the given function is a polynomial on the interval $(-\infty, 2)$, its limit from the left is

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2x^2 - ax + 3) = 8 - 2a + 3 = 11 - 2a.$$

The same argument applies for the interval $(2, +\infty)$, so the limit from the right is

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (4x^3 + 3x - a) = 32 + 6 - a = 38 - a.$$

To ensure that the given function has a limit as x approaches 2, one must then have

$$11 - 2a = 38 - a \quad \Longleftrightarrow \quad a = -27.$$

7. Determine the inverse function f^{-1} in the case that $f\colon [2,\infty) \to [1,\infty)$ is defined by

$$f(x) = 2x^2 - 8x + 9.$$

Using the quadratic formula to solve the equation $y = f(x)$ for x, one finds that

$$2x^2 - 8x + (9 - y) = 0 \quad \Longrightarrow \quad x = \frac{8 \pm \sqrt{64 - 8(9-y)}}{4} = \frac{8 \pm \sqrt{8y - 8}}{4}.$$

Since $y \geq 1$, the square root is obviously defined. Since $x \geq 2$, however, one needs to have

$$x = \frac{8 + \sqrt{8y-8}}{4} = 2 + \frac{\sqrt{2y-2}}{2} \quad \Longrightarrow \quad f^{-1}(y) = 2 + \frac{\sqrt{2y-2}}{2}.$$

8. Compute each of the following limits.

$$L = \lim_{x\to 3} \frac{x^3 - 5x^2 + 7x - 3}{x - 3}, \qquad M = \lim_{x\to 3} \frac{2x^3 - 9x^2 + 27}{(x-3)^2}.$$

When it comes to the first limit, division of polynomials gives

$$L = \lim_{x\to 3} \frac{(x-3)(x^2 - 2x + 1)}{x - 3} = \lim_{x\to 3} (x^2 - 2x + 1) = 9 - 6 + 1 = 4.$$

When it comes to the second limit, division of polynomials gives

$$M = \lim_{x\to 3} \frac{(x^2 - 6x + 9)(2x + 3)}{x^2 - 6x + 9} = \lim_{x\to 3} (2x + 3) = 6 + 3 = 9.$$

9. Use the ε-δ definition of limits to compute $\lim_{x\to 2} \frac{1}{x}$.

To show that the limit is $L = \frac{1}{2}$, we let $\varepsilon > 0$ be given and we estimate the expression

$$|f(x) - L| = \left|\frac{1}{x} - \frac{1}{2}\right| = \frac{|x-2|}{2|x|}.$$

Assume that $0 \neq |x-2| < \delta$ and that $\delta \leq 1$ for simplicity. We must then have

$$\begin{aligned} |x-2| < \delta \leq 1 \quad &\Longrightarrow \quad -1 < x - 2 < 1 \\ &\Longrightarrow \quad 1 < x < 3 \quad \Longrightarrow \quad \frac{1}{2|x|} = \frac{1}{2x} < \frac{1}{2}. \end{aligned}$$

Once we now combine these estimates, we may actually conclude that

$$|f(x) - L| = \frac{|x-2|}{2|x|} < \frac{\delta}{2|x|} < \frac{\delta}{2} \leq \varepsilon,$$

as long as $\delta \leq 2\varepsilon$ and $\delta \leq 1$. An appropriate choice of δ is thus $\delta = \min(2\varepsilon, 1)$.

10. Use the ε-δ definition of limits to compute $\lim_{x\to 2} (4x^2 - 5x + 1)$.

Let $f(x) = 4x^2 - 5x + 1$ for convenience. Then $f(2) = 7$ and one has

$$|f(x) - f(2)| = |4x^2 - 5x - 6| = |x-2| \cdot |4x+3|.$$

The factor $|x-2|$ is related to our usual assumption that $0 \neq |x-2| < \delta$. To estimate the remaining factor $|4x+3|$, we assume that $\delta \leq 1$ for simplicity and we find that

$$\begin{aligned} |x-2| < \delta \leq 1 \quad &\Longrightarrow \quad -1 < x - 2 < 1 \\ &\Longrightarrow \quad 1 < x < 3 \quad \Longrightarrow \quad 7 < 4x + 3 < 15. \end{aligned}$$

Combining the estimates $|x-2| < \delta$ and $|4x+3| < 15$, one may now conclude that

$$|f(x) - f(2)| = |x-2| \cdot |4x+3| < 15\delta \leq \varepsilon,$$

as long as $\delta \leq \varepsilon/15$ and $\delta \leq 1$. An appropriate choice of δ is thus $\delta = \min(\varepsilon/15, 1)$.

MAU11201 – Calculus
Tutorial solutions #3

1. Show that there exists a real number $0 < x < \pi/2$ that satisfies the equation

$$x^3 \cos x + x^2 \sin x = 2.$$

Consider the function f which is defined by $f(x) = x^3 \cos x + x^2 \sin x - 2$. Being the sum of continuous functions, f is then continuous and one can easily check that

$$f(0) = -2 < 0, \qquad f(\pi/2) = \frac{\pi^2}{4} - 2 = \frac{\pi^2 - 8}{4} > 0.$$

In view of Bolzano's theorem, this already implies that f has a root $0 < x < \pi/2$.

2. For which values of a, b is the function f continuous at the point $x = 3$? Explain.

$$f(x) = \left\{ \begin{array}{cl} 2x^2 + ax + b & \text{if } x < 3 \\ 2a + b + 1 & \text{if } x = 3 \\ 5x^2 - bx + 2a & \text{if } x > 3 \end{array} \right\}.$$

Since f is a polynomial on the intervals $(-\infty, 3)$ and $(3, +\infty)$, it should be clear that

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (2x^2 + ax + b) = 3a + b + 18,$$
$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (5x^2 - bx + 2a) = 2a - 3b + 45.$$

In particular, the function f is continuous at the given point if and only if

$$3a + b + 18 = 2a - 3b + 45 = 2a + b + 1.$$

Solving this system of equations, one obtains a unique solution which is given by

$$45 - 3b = b + 1 \quad \Longrightarrow \quad 4b = 44 \quad \Longrightarrow \quad b = 11 \quad \Longrightarrow \quad a = 27 - 4b = -17.$$

In other words, f is continuous at the given point if and only if $a = -17$ and $b = 11$.

3. Show that $f(x) = x^3 - 3x^2 + 1$ has three roots in the interval $(-1, 3)$. Hint: you need only consider the values that are attained by f at the integers $-1 \leq x \leq 3$.

Being a polynomial, the given function is continuous and one can easily check that

$$f(-1) = -3, \qquad f(0) = 1, \qquad f(1) = -1, \qquad f(2) = -3, \qquad f(3) = 1.$$

Since the values $f(-1)$ and $f(0)$ have opposite signs, f has a root that lies in $(-1, 0)$. The same argument yields a second root in $(0, 1)$ and also a third root in $(2, 3)$.

4. Compute each of the following limits.

$$L = \lim_{x\to+\infty} \frac{2x^4 - 7x + 3}{3x^4 - 5x^2 + 1}, \qquad M = \lim_{x\to 2^-} \frac{2x^2 + 3x - 4}{3x^3 - 7x^2 + 4x - 4}.$$

Since the first limit involves infinite values of x, it should be clear that

$$L = \lim_{x\to+\infty} \frac{2x^4 - 7x + 3}{3x^4 - 5x^2 + 1} = \lim_{x\to+\infty} \frac{2x^4}{3x^4} = \frac{2}{3}.$$

For the second limit, the denominator becomes zero when $x = 2$, while the numerator is nonzero at that point. Thus, one needs to factor the denominator and this gives

$$M = \lim_{x\to 2^-} \frac{2x^2 + 3x - 4}{(x-2)(3x^2 - x + 2)} = \lim_{x\to 2^-} \frac{10}{12(x-2)} = -\infty.$$

5. Use the definition of the derivative to compute $f'(x_0)$ in each of the following cases.

$$f(x) = 3x^2, \qquad f(x) = 2/x, \qquad f(x) = (2x+3)^2.$$

The derivative of the first function is given by the limit

$$f'(x_0) = \lim_{x\to x_0} \frac{3x^2 - 3x_0^2}{x - x_0} = \lim_{x\to x_0} \frac{3(x-x_0)(x+x_0)}{x - x_0} = \lim_{x\to x_0} 3(x + x_0) = 6x_0.$$

To compute the derivative of the second function, we begin by writing

$$f(x) - f(x_0) = \frac{2}{x} - \frac{2}{x_0} = \frac{2(x_0 - x)}{xx_0}.$$

Once we now divide this expression by $x - x_0$, we may also conclude that

$$f'(x_0) = \lim_{x\to x_0} \frac{2(x_0 - x)}{(x - x_0)xx_0} = \lim_{x\to x_0} \frac{-2}{xx_0} = -\frac{2}{x_0^2}.$$

Finally, the derivative of the third function is given by the limit

$$f'(x_0) = \lim_{x\to x_0} \frac{(2x+3)^2 - (2x_0+3)^2}{x - x_0} = \lim_{x\to x_0} \frac{(2x + 2x_0 + 6)(2x - 2x_0)}{x - x_0} = 4(2x_0 + 3).$$

6. Show that there exists a real number $0 < x < \pi/2$ that satisfies the equation

$$x^2 + x - 1 = \sin x.$$

Consider the function f which is defined by $f(x) = x^2 + x - 1 - \sin x$. Being the sum of continuous functions, f is then continuous and one can easily check that

$$f(0) = -1 < 0, \qquad f(\pi/2) = \frac{\pi^2}{4} + \frac{\pi}{2} - 2 > \frac{\pi^2 - 8}{4} > 0.$$

In view of Bolzano's theorem, this already implies that f has a root $0 < x < \pi/2$.

7. Show that $f(x) = 3x^3 - 5x + 1$ has three roots in the interval $(-2, 2)$. Hint: you need only consider the values that are attained by f at the integers $-2 \leq x \leq 2$.

Being a polynomial, the given function is continuous and one can easily check that

$$f(-2) = -13, \qquad f(-1) = 3, \qquad f(0) = 1, \qquad f(1) = -1, \qquad f(2) = 15.$$

Since the values $f(-2)$ and $f(-1)$ have opposite signs, f has a root that lies in $(-2, -1)$. The same argument yields a second root in $(0, 1)$ and also a third root in $(1, 2)$.

8. Compute each of the following limits.

$$L = \lim_{x \to -\infty} \frac{6x^3 - 5x^2 + 7}{5x^4 - 3x + 1}, \qquad M = \lim_{x \to 2^+} \frac{x^3 + x^2 - 5x - 2}{x^3 - 5x^2 + 8x - 4}.$$

Since the first limit involves infinite values of x, it should be clear that

$$L = \lim_{x \to -\infty} \frac{6x^3 - 5x^2 + 7}{5x^4 - 3x + 1} = \lim_{x \to -\infty} \frac{6x^3}{5x^4} = \lim_{x \to -\infty} \frac{6}{5x} = 0.$$

For the second limit, both the numerator and the denominator become zero when $x = 2$, so one needs to factor each of these expressions. Using division of polynomials, we get

$$M = \lim_{x \to 2^+} \frac{(x-2)(x^2 + 3x + 1)}{(x-2)^2(x-1)} = \lim_{x \to 2^+} \frac{11}{x - 2} = +\infty.$$

9. Use the Squeeze Theorem to show that $\lim_{x \to 0} x^2 \sin(1/x) = 0$.

Since $-1 \leq \sin x \leq 1$ for all x, one has $-1 \leq \sin(1/x) \leq 1$ for all $x \neq 0$ and

$$-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2.$$

On the other hand, both $-x^2$ and x^2 approach zero as $x \to 0$, so this also implies

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0.$$

10. Suppose that f is continuous with $f(0) < 1$. Show that there exists some $\delta > 0$ such that $f(x) < 1$ for all $-\delta < x < \delta$. Hint: use the ε-δ definition for some suitable ε.

Since $\varepsilon = 1 - f(0)$ is positive by assumption, there exists some $\delta > 0$ such that

$$|x - 0| < \delta \quad \Longrightarrow \quad |f(x) - f(0)| < \varepsilon \quad \Longrightarrow \quad |f(x) - f(0)| < 1 - f(0).$$

Rearranging terms to simplify this equation, one may thus conclude that

$$-\delta < x < \delta \quad \Longrightarrow \quad f(0) - 1 < f(x) - f(0) < 1 - f(0) \quad \Longrightarrow \quad f(x) < 1.$$

MAU11201 – Calculus
Tutorial solutions #4

1. Compute the derivative $y' = \frac{dy}{dx}$ in each of the following cases.

$$y = \ln(\sec x) + e^{\tan x}, \qquad y = \sin(\sec^2(4x)).$$

When it comes to the first function, one may use the chain rule to get

$$y' = \frac{1}{\sec x} \cdot \sec x \tan x + e^{\tan x} \sec^2 x = \tan x + e^{\tan x} \sec^2 x.$$

When it comes to the second function, one similarly finds that

$$\begin{aligned} y' &= \cos(\sec^2(4x)) \cdot [\sec^2(4x)]' \\ &= \cos(\sec^2(4x)) \cdot 2\sec(4x) \cdot [\sec(4x)]' \\ &= \cos(\sec^2(4x)) \cdot 2\sec(4x) \cdot 4\sec(4x)\tan(4x) \\ &= 8\cos(\sec^2(4x)) \cdot \sec^2(4x) \cdot \tan(4x). \end{aligned}$$

2. Compute the derivative $y' = \frac{dy}{dx}$ in the case that $x^2 \sin y = y^2 e^x$.

We differentiate both sides of the equation and then rearrange terms. This gives

$$\begin{aligned} 2x\sin y + x^2 y' \cos y = 2yy'e^x + y^2 e^x \quad &\Longrightarrow \quad (x^2\cos y - 2ye^x)y' = y^2 e^x - 2x\sin y \\ &\Longrightarrow \quad y' = \frac{y^2 e^x - 2x\sin y}{x^2\cos y - 2ye^x}. \end{aligned}$$

3. Compute the derivative $y' = \frac{dy}{dx}$ in each of the following cases.

$$y = x^2 \cdot \tan^{-1}(2x), \qquad y = (x \cdot \sin x)^x.$$

When it comes to the first function, we use the product rule and the chain rule to get

$$y' = 2x \cdot \tan^{-1}(2x) + x^2 \cdot \frac{2}{(2x)^2 + 1} = 2x \cdot \tan^{-1}(2x) + \frac{2x^2}{4x^2 + 1}.$$

When it comes to the second function, logarithmic differentiation gives

$$\begin{aligned} \ln y = x\ln(x \cdot \sin x) \quad &\Longrightarrow \quad \frac{y'}{y} = \ln(x\sin x) + x \cdot \frac{1}{x\sin x} \cdot (\sin x + x\cos x) \\ &\Longrightarrow \quad y' = y \cdot (\ln(x\sin x) + 1 + x\cot x) \\ &\Longrightarrow \quad y' = (x \cdot \sin x)^x \cdot (\ln(x\sin x) + 1 + x\cot x). \end{aligned}$$

4. Compute the derivative $f'(x_0)$ in the case that

$$f(x) = \frac{(x^3 + 5x^2 + 2)^3 \cdot e^{\sin x}}{\sqrt{x^2 + 4x + 1}}, \qquad x_0 = 0.$$

First, we use logarithmic differentiation to determine $f'(x)$. In this case, we have

$$\begin{aligned} \ln|f(x)| &= \ln|x^3 + 5x^2 + 2|^3 + \ln e^{\sin x} - \ln|x^2 + 4x + 1|^{1/2} \\ &= 3\ln|x^3 + 5x^2 + 2| + \sin x - \frac{1}{2}\ln|x^2 + 4x + 1|. \end{aligned}$$

Differentiating both sides of this equation, one easily finds that

$$\frac{f'(x)}{f(x)} = \frac{3(3x^2 + 10x)}{x^3 + 5x^2 + 2} + \cos x - \frac{2x + 4}{2(x^2 + 4x + 1)}.$$

To compute the derivative $f'(0)$, one may then substitute $x = 0$ to conclude that

$$\frac{f'(0)}{f(0)} = 0 + \cos 0 - \frac{4}{2} = -1 \quad\Longrightarrow\quad f'(0) = -f(0) = -8.$$

5. Compute the derivative $y' = \frac{dy}{dx}$ in the case that

$$y = \sin^{-1} u, \qquad u = \ln(2z^2 + 3z + 1), \qquad z = \frac{3x - 1}{2x + 5}.$$

Differentiating the given equations, one easily finds that

$$\frac{dy}{du} = \frac{1}{\sqrt{1 - u^2}}, \qquad \frac{du}{dz} = \frac{4z + 3}{2z^2 + 3z + 1}, \qquad \frac{dz}{dx} = \frac{3(2x + 5) - 2(3x - 1)}{(2x + 5)^2} = \frac{17}{(2x + 5)^2}.$$

According to the chain rule, the derivative $\frac{dy}{dx}$ is the product of these factors, namely

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dz}\frac{dz}{dx} = \frac{1}{\sqrt{1 - u^2}} \cdot \frac{4z + 3}{2z^2 + 3z + 1} \cdot \frac{17}{(2x + 5)^2}.$$

6. Compute the derivative $y' = \frac{dy}{dx}$ in each of the following cases.

$$y = (e^{2x} + x^3)^4, \qquad y = \tan(x \sin x).$$

When it comes to the first function, one may use the chain rule to get

$$y' = 4(e^{2x} + x^3)^3 \cdot (e^{2x} + x^3)' = 4(e^{2x} + x^3)^3 \cdot (2e^{2x} + 3x^2).$$

When it comes to the second function, one similarly finds that

$$y' = \sec^2(x \sin x) \cdot (x \sin x)' = \sec^2(x \sin x) \cdot (\sin x + x \cos x).$$

7. Compute the derivative $y' = \frac{dy}{dx}$ in the case that $x^2 + y^2 = \sin(xy)$.

Differentiating both sides of the given equation, one finds that

$$2x + 2yy' = \cos(xy) \cdot (y + xy') = y\cos(xy) + xy'\cos(xy).$$

Once we now rearrange terms and solve for y', we may conclude that

$$(2y - x\cos(xy)) \cdot y' = y\cos(xy) - 2x \quad \Longrightarrow \quad y' = \frac{y\cos(xy) - 2x}{2y - x\cos(xy)}.$$

8. Compute the derivative $f'(x_0)$ in the case that

$$f(x) = \frac{(x^2 + 3x + 1)^4 \cdot \sqrt{2x + \cos x}}{(e^x + x)^3}, \qquad x_0 = 0.$$

First, we use logarithmic differentiation to determine $f'(x)$. In this case, we have

$$\begin{aligned} \ln|f(x)| &= \ln|x^2 + 3x + 1|^4 + \ln|2x + \cos x|^{1/2} - \ln|e^x + x|^3 \\ &= 4\ln|x^2 + 3x + 1| + \frac{1}{2}\ln|2x + \cos x| - 3\ln|e^x + x|. \end{aligned}$$

Differentiating both sides of this equation, one may use the chain rule to get

$$\frac{f'(x)}{f(x)} = \frac{4(2x + 3)}{x^2 + 3x + 1} + \frac{2 - \sin x}{2(2x + \cos x)} - \frac{3(e^x + 1)}{e^x + x}.$$

To compute the derivative $f'(0)$, one may then substitute $x = 0$ to conclude that

$$\frac{f'(0)}{f(0)} = 4 \cdot 3 + \frac{2}{2} - 3 \cdot 2 = 7 \quad \Longrightarrow \quad f'(0) = 7f(0) = 7.$$

9. Compute the derivative $y' = \frac{dy}{dx}$ in the case that

$$y = \frac{2u - 1}{3u + 1}, \qquad u = \sin(e^z), \qquad z = \tan^{-1}(x^2).$$

Differentiating the given equations, one easily finds that

$$\frac{dy}{du} = \frac{2(3u + 1) - 3(2u - 1)}{(3u + 1)^2} = \frac{5}{(3u + 1)^2}, \qquad \frac{du}{dz} = e^z\cos(e^z), \qquad \frac{dz}{dx} = \frac{2x}{x^4 + 1}.$$

According to the chain rule, the derivative $\frac{dy}{dx}$ is the product of these factors, namely

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dz}\frac{dz}{dx} = \frac{5}{(3u + 1)^2} \cdot e^z\cos(e^z) \cdot \frac{2x}{x^4 + 1}.$$

10. Compute the derivative $f'(1)$ in the case that $x^2 f(x) + xf(x)^3 = 2$ for all x.

Letting $y = f(x)$ for convenience, we get $x^2y + xy^3 = 2$ and this implies that

$$\begin{aligned} 2xy + x^2y' + y^3 + 3xy^2y' = 0 \quad &\implies \quad (x^2 + 3xy^2)y' = -2xy - y^3 \\ &\implies \quad y' = -\frac{y(2x + y^2)}{x(x + 3y^2)}. \end{aligned}$$

We need to evaluate this expression at the point $x = 1$. At that point, one has

$$x^2y + xy^3 = 2 \quad \implies \quad y + y^3 = 2 \quad \implies \quad y^3 + y - 2 = 0.$$

It is easy to see that $y = 1$ is a solution. In fact, it is the only real solution because

$$y^3 + y - 2 = (y - 1)(y^2 + y + 2)$$

and the quadratic factor has no real roots. This gives $y = 1$ at the point $x = 1$, so

$$f'(x) = -\frac{y(2x + y^2)}{x(x + 3y^2)} \quad \implies \quad f'(1) = -\frac{3}{4}.$$

MAU11201 – Calculus
Tutorial solutions #5

1. Show that the polynomial $f(x) = x^3 - 5x^2 - 8x + 1$ has exactly one root in $(0, 1)$.

Being a polynomial, f is continuous on the interval $[0, 1]$ and we also have

$$f(0) = 1, \qquad f(1) = 1 - 5 - 8 + 1 = -11.$$

Since $f(0)$ and $f(1)$ have opposite signs, f must have a root that lies in $(0, 1)$. To show it is unique, suppose that f has two roots in $(0, 1)$. Then f' must have a root in this interval by Rolle's theorem. On the other hand, it is easy to check that

$$f'(x) = 3x^2 - 10x - 8 = (3x + 2)(x - 4).$$

Since f' has no roots in $(0, 1)$, we conclude that f has exactly one root in $(0, 1)$.

2. Let $b > 1$ be a given constant. Use the mean value theorem to show that

$$1 - \frac{1}{b} < \ln b < b - 1.$$

Since $f(x) = \ln x$ is differentiable with $f'(x) = 1/x$, the mean value theorem gives

$$\frac{f(b) - f(1)}{b - 1} = f'(c) = \frac{1}{c}$$

for some point $1 < c < b$. Using the fact that $\frac{1}{b} < \frac{1}{c} < 1$, one may thus conclude that

$$\frac{1}{b} < \frac{\ln b - \ln 1}{b - 1} < 1 \quad \Longrightarrow \quad 1 - \frac{1}{b} < \ln b < b - 1.$$

3. Compute each of the following limits.

$$L_1 = \lim_{x\to 2} \frac{2x^3 - 5x^2 + 5x - 6}{3x^3 - 5x^2 - 4}, \qquad L_2 = \lim_{x\to\infty} \frac{\ln x}{x^2}, \qquad L_3 = \lim_{x\to 0} (x + \cos x)^{1/x}.$$

The first limit has the form $0/0$, so one may use L'Hôpital's rule to get

$$L_1 = \lim_{x\to 2} \frac{6x^2 - 10x + 5}{9x^2 - 10x} = \frac{24 - 20 + 5}{36 - 20} = \frac{9}{16}.$$

The second limit has the form ∞/∞, so L'Hôpital's rule is still applicable and

$$L_2 = \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0.$$

The third limit involves a non-constant exponent which can be eliminated by writing

$$\ln L_3 = \ln \lim_{x\to 0} (x+\cos x)^{1/x} = \lim_{x\to 0} \ln(x+\cos x)^{1/x} = \lim_{x\to 0} \frac{\ln(x+\cos x)}{x}.$$

This gives a limit of the form 0/0, so one may use L'Hôpital's rule to find that

$$\ln L_3 = \lim_{x\to 0} \frac{1-\sin x}{x+\cos x} = \frac{1-0}{0+1} = 1.$$

Since $\ln L_3 = 1$, the original limit L_3 is then equal to $L_3 = e^{\ln L_3} = e$.

4. For which values of x is $f(x) = (\ln x)^2$ increasing? For which values is it concave up?

To say that $f(x)$ is increasing is to say that $f'(x) > 0$. Let us then compute

$$f'(x) = 2\ln x \cdot (\ln x)' = \frac{2\ln x}{x}.$$

Since the given function is only defined at points $x > 0$, it is increasing if and only if

$$\ln x > 0 \iff x > e^0 \iff x > 1.$$

To say that $f(x)$ is concave up is to say that $f''(x) > 0$. According to the quotient rule,

$$f''(x) = \frac{(2/x)\cdot x - 2\ln x}{x^2} = \frac{2(1-\ln x)}{x^2}.$$

Since the denominator is always positive, $f(x)$ is then concave up if and only if

$$1 - \ln x > 0 \iff \ln x < 1 \iff 0 < x < e.$$

5. Find the intervals on which f is increasing/decreasing and the intervals on which f is concave up/down. Use this information to sketch the graph of f.

$$f(x) = \frac{x^2}{x^2+3}.$$

To say that $f(x)$ is increasing is to say that $f'(x) > 0$. In this case, we have

$$f'(x) = \frac{2x\cdot(x^2+3) - 2x\cdot x^2}{(x^2+3)^2} = \frac{6x}{(x^2+3)^2},$$

so it is clear that $f(x)$ is increasing if and only if $x > 0$. To say that $f(x)$ is concave up is to say that $f''(x) > 0$. Using both the quotient rule and the chain rule, we get

$$f''(x) = \frac{6(x^2+3)^2 - 2(x^2+3)\cdot 2x\cdot 6x}{(x^2+3)^4} = \frac{6(x^2+3) - 24x^2}{(x^2+3)^3} = \frac{18(1-x^2)}{(x^2+3)^3}.$$

Since the denominator is always positive, $f(x)$ is then concave up if and only if

$$1 - x^2 > 0 \iff x^2 < 1 \iff -1 < x < 1.$$

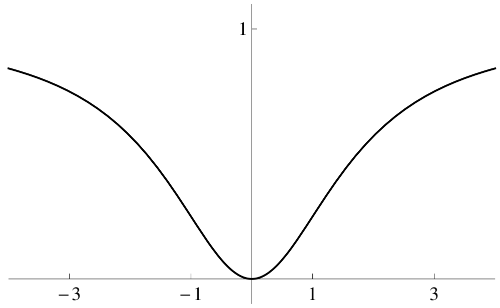

Figure 1: The graph of $f(x) = \dfrac{x^2}{x^2+3}$.

6. Show that the polynomial $f(x) = x^3 + x^2 - 5x + 1$ has exactly two roots in $(0, 2)$.

To prove existence using Bolzano's theorem, we note that f is continuous with

$$f(0) = 1, \qquad f(1) = 1 + 1 - 5 + 1 = -2, \qquad f(2) = 8 + 4 - 10 + 1 = 3.$$

In view of Bolzano's theorem, f must then have a root in $(0, 1)$ and another root in $(1, 2)$, so it has two roots in $(0, 2)$. Suppose that it has three roots in $(0, 2)$. Then f' must have two roots in this interval by Rolle's theorem. On the other hand, it is easy to check that

$$f'(x) = 3x^2 + 2x - 5 = (3x + 5)(x - 1).$$

Since f' has only one root in $(0, 2)$, we conclude that f has only two roots in $(0, 2)$.

7. Use the mean value theorem for the case $f(x) = \sqrt{x+4}$ to show that

$$2 + \frac{1}{2} < \sqrt{7} < 2 + \frac{3}{4}.$$

According to the mean value theorem, there exists a point $0 < c < 3$ such that

$$\frac{f(3) - f(0)}{3 - 0} = f'(c) \quad \Longrightarrow \quad \frac{\sqrt{7} - \sqrt{4}}{3} = \frac{1}{2\sqrt{c+4}}.$$

To estimate the square root on the right hand side, we note that

$$0 < c < 3 \quad \Longrightarrow \quad 4 < c + 4 < 7 < 9 \quad \Longrightarrow \quad 2 < \sqrt{c+4} < 3.$$

Once we now combine the last two equations, we may easily conclude that

$$\frac{1}{3} < \frac{1}{\sqrt{c+4}} < \frac{1}{2} \quad \Longrightarrow \quad \frac{1}{2} < \sqrt{7} - 2 < \frac{3}{4}.$$

8. Compute each of the following limits.

$$L_1 = \lim_{x\to 2} \frac{x^3 - 5x^2 + 8x - 4}{x^3 - 3x^2 + 4}, \qquad L_2 = \lim_{x\to 1} \frac{\ln x}{x^4 - 1}, \qquad L_3 = \lim_{x\to 0^+} \frac{\ln(\sin x)}{\ln(\tan x)}.$$

The first limit has the form $0/0$, so one may use L'Hôpital's rule to find that

$$L_1 = \lim_{x\to 2} \frac{3x^2 - 10x + 8}{3x^2 - 6x}.$$

Since the limit on the right hand side is still a limit of the form $0/0$, one has

$$L_1 = \lim_{x\to 2} \frac{6x - 10}{6x - 6} = \frac{12 - 10}{12 - 6} = \frac{1}{3}.$$

The second limit is also of the form $0/0$ and an application of L'Hôpital's rule gives

$$L_2 = \lim_{x\to 1} \frac{1/x}{4x^3} = \frac{1}{4}.$$

The third limit has the form ∞/∞, so it follows by L'Hôpital's rule that

$$L_3 = \lim_{x\to 0^+} \frac{(\sin x)^{-1} \cdot \cos x}{(\tan x)^{-1} \cdot \sec^2 x}.$$

Since both $\cos x$ and $\sec x$ are approaching 1 as x approaches zero, we conclude that

$$L_3 = \lim_{x\to 0^+} \frac{\tan x}{\sin x} = \lim_{x\to 0^+} \frac{1}{\cos x} = 1.$$

9. For which values of x is $f(x) = e^{-2x^2}$ increasing? For which values is it concave up?

To say that $f(x)$ is increasing is to say that $f'(x) > 0$. Let us then compute

$$f'(x) = e^{-2x^2} \cdot (-2x^2)' = -4xe^{-2x^2}.$$

Since the exponential factor is always positive, $f(x)$ is increasing if and only if $x < 0$. To say that $f(x)$ is concave up is to say that $f''(x) > 0$. In this case, we have

$$f''(x) = -4e^{-2x^2} - 4x \cdot (-4x)e^{-2x^2} = (16x^2 - 4)e^{-2x^2} = 4(2x-1)(2x+1)e^{-2x^2}.$$

It easily follows that $f(x)$ is concave up if and only if $x \in (-\infty, -1/2) \cup (1/2, +\infty)$.

10. Show that there exists a unique number $1 < x < \pi$ such that $x^3 = 3\sin x + 1$.

It is clear that $f(x) = x^3 - 3\sin x - 1$ is continuous on $[1, \pi]$ and we also have

$$f(1) = -3\sin 1 < 0, \qquad f(\pi) = \pi^3 - 1 > 0.$$

Since $f(1)$ and $f(\pi)$ have opposite signs, f must have a root that lies in $(1, \pi)$. To show it is unique, suppose f has two roots in $(1, \pi)$. Then f' must have a root in this interval by Rolle's theorem. On the other hand, it is easy to check that

$$f'(x) = 3x^2 - 3\cos x > 3 - 3\cos x = 3(1 - \cos x) \geq 0$$

for all $x > 1$. In particular, f' has no roots in $(1, \pi)$ and f has exactly one root in $(1, \pi)$.

MAU11201 – Calculus
Tutorial solutions #6

1. Let $a_1, a_2, \ldots, a_n$ be some given constants and let f be the function defined by

$$f(x) = (x - a_1)^2 + (x - a_2)^2 + \ldots + (x - a_n)^2.$$

Show that $f(x)$ becomes minimum when x is equal to $\overline{x} = (a_1 + a_2 + \ldots + a_n)/n$.

The derivative of the given function can be expressed in the form

$$f'(x) = 2(x - a_1) + 2(x - a_2) + \ldots + 2(x - a_n) = 2(nx - n\overline{x}) = 2n(x - \overline{x}).$$

This means that $f'(x)$ is negative when $x < \overline{x}$ and positive when $x > \overline{x}$. In particular, $f(x)$ is decreasing when $x < \overline{x}$ and increasing when $x > \overline{x}$, so it becomes minimum when $x = \overline{x}$.

2. Find the global minimum and the global maximum values that are attained by

$$f(x) = 3x^4 - 16x^3 + 18x^2 - 1, \qquad 0 \leq x \leq 2.$$

The derivative of the given function can be expressed in the form

$$f'(x) = 12x^3 - 48x^2 + 36x = 12x(x^2 - 4x + 3) = 12x(x - 1)(x - 3).$$

Thus, the only points at which the minimum/maximum value may occur are the points

$$x = 0, \qquad x = 2, \qquad x = 1, \qquad x = 3.$$

We exclude the rightmost point, as it does not lie in the given interval, and we compute

$$f(0) = -1, \qquad f(2) = 48 - 128 + 72 - 1 = -9, \qquad f(1) = 3 - 16 + 18 - 1 = 4.$$

This means that the minimum value is $f(2) = -9$ and the maximum value is $f(1) = 4$.

3. Find the linear approximation to the function f at the point x_0 in the case that

$$f(x) = \frac{(x^2 + 1)^4 \cdot e^{x^2 - 1}}{\sqrt{3x + 1}}, \qquad x_0 = 1.$$

First, we use logarithmic differentiation to compute the derivative $f'(x)$. Let us write

$$\begin{aligned} \ln f(x) &= \ln(x^2 + 1)^4 + \ln e^{x^2 - 1} - \ln(3x + 1)^{1/2} \\ &= 4\ln(x^2 + 1) + x^2 - 1 - \frac{1}{2}\ln(3x + 1). \end{aligned}$$

Differentiating both sides of this equation, one may use the chain rule to find that

$$\frac{f'(x)}{f(x)} = \frac{4 \cdot 2x}{x^2+1} + 2x - \frac{3}{2(3x+1)}.$$

In our case, we have $f(1) = \frac{2^4 e^0}{\sqrt{4}} = 8$, so one may substitute $x = 1$ to conclude that

$$\frac{f'(1)}{f(1)} = \frac{4 \cdot 2}{1+1} + 2 - \frac{3}{2 \cdot 4} \quad \Longrightarrow \quad f'(1) = 8\left(6 - \frac{3}{8}\right) = 48 - 3 = 45.$$

Since $f(1) = 8$ and $f'(1) = 45$, the linear approximation at the given point is thus

$$L(x) = f'(1) \cdot (x-1) + f(1) = 45(x-1) + 8 = 45x - 37.$$

4. The top of a 5m ladder is sliding down a wall at the rate of 0.25 m/sec. How fast is the base sliding away from the wall when the top lies 3 metres above the ground?

Let x be the horizontal distance between the base of the ladder and the wall, and let y be the vertical distance between the top of the ladder and the floor. We must then have

$$x(t)^2 + y(t)^2 = 5^2 \quad \Longrightarrow \quad 2x(t)x'(t) + 2y(t)y'(t) = 0.$$

At the given moment, $y'(t) = -1/4$ and $y(t) = 3$, so it easily follows that

$$x'(t) = -\frac{y(t)y'(t)}{x(t)} = -\frac{y(t)y'(t)}{\sqrt{5^2 - y(t)^2}} = \frac{3/4}{\sqrt{5^2 - 3^2}} = \frac{3}{16}.$$

5. Let $n > 0$ be a given constant. Show that $x^n \ln x \geq -\frac{1}{ne}$ for all $x > 0$.

Setting $f(x) = x^n \ln x$ for convenience, one may use the product rule to find that

$$f'(x) = nx^{n-1} \cdot \ln x + x^n \cdot x^{-1} = x^{n-1}(n \ln x + 1).$$

Since $x > 0$ by assumption, the derivative $f'(x)$ is negative if and only if

$$n \ln x + 1 < 0 \quad \Longleftrightarrow \quad \ln x < -1/n \quad \Longleftrightarrow \quad 0 < x < e^{-1/n}.$$

It easily follows that $f(x)$ is decreasing when $0 < x < e^{-1/n}$ and increasing when $x > e^{-1/n}$. In particular, the minimum value of $f(x)$ is attained at the point $x = e^{-1/n}$ and

$$f(x) \geq f(e^{-1/n}) = \left(e^{-1/n}\right)^n \cdot \ln e^{-1/n} \quad \Longrightarrow \quad f(x) \geq -\frac{e^{-1}}{n} = -\frac{1}{ne}.$$

6. Find the global minimum and the global maximum values that are attained by

$$f(x) = x^2 \cdot e^{4-2x}, \qquad -1 \le x \le 2.$$

Using both the product rule and the chain rule, one may differentiate $f(x)$ to get

$$f'(x) = 2x \cdot e^{4-2x} + x^2 \cdot e^{4-2x} \cdot (-2) = 2xe^{4-2x} \cdot (1-x).$$

Thus, the only points at which the minimum/maximum value may occur are the points

$$x = -1, \qquad x = 2, \qquad x = 0, \qquad x = 1.$$

The corresponding values that are attained by $f(x)$ are easily found to be

$$f(-1) = e^6, \qquad f(2) = 4e^0 = 4, \qquad f(0) = 0, \qquad f(1) = e^2.$$

In particular, the minimum value is $f(0) = 0$ and the maximum value is $f(-1) = e^6$.

7. Find the point on the graph of $y = 2\sqrt{x}$ which lies closest to the point $(2, 1)$.

The distance between the point (x, y) and the point $(2, 1)$ is given by the formula

$$d(x) = \sqrt{(x-2)^2 + (y-1)^2} = \sqrt{(x-2)^2 + (2\sqrt{x}-1)^2}.$$

The value of x that minimises this expression is the value of x that minimises its square

$$f(x) = d(x)^2 = (x-2)^2 + (2\sqrt{x}-1)^2.$$

Let us then worry about $f(x)$, instead. Using the chain rule, one finds that

$$f'(x) = 2(x-2) + 2(2\sqrt{x}-1) \cdot \frac{2}{2\sqrt{x}} = 2\left(x - 2 + 2 - \frac{1}{\sqrt{x}}\right) = \frac{2(x^{3/2}-1)}{\sqrt{x}}.$$

This means that $f'(x)$ is negative when $0 < x < 1$ and positive when $x > 1$, so $f(x)$ attains its minimum value when $x = 1$. Thus, the closest point is the point $(x, y) = (1, 2)$.

8. If a right triangle has a hypotenuse of length $a > 0$, how large can its perimeter be?

Let x, y be the other two sides of the triangle. Then $x^2 + y^2 = a^2$ and the perimeter is

$$f(x) = a + x + y = a + x + \sqrt{a^2 - x^2}, \qquad 0 \le x \le a.$$

We need to determine the maximum value that is attained by this function. Since

$$f'(x) = 1 + \frac{1}{2\sqrt{a^2-x^2}} \cdot (a^2 - x^2)' = 1 - \frac{x}{\sqrt{a^2-x^2}},$$

it is easy to check that

$$f'(x) = 0 \quad \Longleftrightarrow \quad x = \sqrt{a^2 - x^2} \quad \Longleftrightarrow \quad x^2 = a^2 - x^2 \quad \Longleftrightarrow \quad 2x^2 = a^2.$$

In particular, the only points at which the maximum value may occur are the points

$$x = a/\sqrt{2}, \qquad x = 0, \qquad x = a.$$

The corresponding values are $f(0) = a + \sqrt{a^2} = 2a$, $f(a) = a + a = 2a$ and

$$f(a/\sqrt{2}) = a + \frac{a}{\sqrt{2}} + \sqrt{a^2 - \frac{a^2}{2}} = a + \frac{2a}{\sqrt{2}} = (1 + \sqrt{2})a.$$

Noting that $1 + \sqrt{2} > 2$, we conclude that the largest possible perimeter is $(1 + \sqrt{2})a$.

9. Two cars are driving in opposite directions along two parallel roads which are 300m apart. If one is driving at 50 m/sec and the other is driving at 30 m/sec, how fast is the distance between them changing 5 seconds after they pass one another?

Let us denote by x and y the displacements of the two cars after they pass one another. Then $x + y$ and 300 are the sides of a right triangle whose hypotenuse is the distance z between the two cars. In view of Pythagoras' theorem, we must then have

$$z(t)^2 = (x(t) + y(t))^2 + 300^2 \quad \Longrightarrow \quad 2z(t)z'(t) = 2(x(t) + y(t)) \cdot (x'(t) + y'(t)).$$

At the given moment, $x'(t) = 50$, $y'(t) = 30$ and $x(t) + y(t) = 5 \cdot 50 + 5 \cdot 30 = 400$, so

$$z'(t) = \frac{400 \cdot 80}{\sqrt{400^2 + 300^2}} = \frac{400 \cdot 80}{500} = \frac{320}{5} = 64.$$

10. Show that $f(x) = x^4 + 5x - 1$ has a unique root in $(0, 1)$ and use Newton's method with initial guess $x_1 = 0$ to approximate this root within two decimal places.

The existence of a root in $(0, 1)$ follows by Bolzano's theorem, as f is continuous with

$$f(0) = -1, \qquad f(1) = 1 + 5 - 1 = 5.$$

Moreover, the root is unique because $f'(x) = 4x^3 + 5$ is positive on $(0, 1)$, so f is increasing on this interval. To use Newton's method, we repeatedly apply the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 + 5x_n - 1}{4x_n^3 + 5}.$$

Starting with the initial guess $x_1 = 0$, one obtains the approximations

$$x_1 = 0, \qquad x_2 = 0.2, \qquad x_3 = 0.1996820350, \qquad x_4 = 0.1996820302.$$

This suggests that the unique root in $(0, 1)$ is roughly 0.1996820 to seven decimal places.

MAU11201 – Calculus
Tutorial solutions #7

1. Find the area of the region enclosed by the graphs of $f(x) = 3x^2$ and $g(x) = x + 4$.

The graph of the parabola $f(x) = 3x^2$ meets the graph of the line $g(x) = x + 4$ when

$$3x^2 = x + 4 \iff 3x^2 - x - 4 = 0 \iff (3x - 4)(x + 1) = 0.$$

Since the line lies above the parabola at the points $-1 \leq x \leq 4/3$, the area is then

$$\int_{-1}^{4/3} [g(x) - f(x)]\, dx = \int_{-1}^{4/3} [x + 4 - 3x^2]\, dx = \left[\frac{x^2}{2} + 4x - x^3\right]_{-1}^{4/3} = \frac{343}{54}.$$

2. Compute the volume of the solid that is obtained when the graph of $f(x) = x^2 + 3$ is rotated around the x-axis over the interval $[0, 2]$.

The volume of the resulting solid is the integral of $\pi f(x)^2$ and this is equal to

$$\pi \int_0^2 (x^2 + 3)^2\, dx = \pi \int_0^2 (x^4 + 6x^2 + 9)\, dx = \pi \left[\frac{x^5}{5} + 2x^3 + 9x\right]_0^2 = \frac{202\pi}{5}.$$

3. Compute the length of the graph of $f(x) = \frac{1}{3}(x^2 + 2)^{3/2}$ over the interval $[1, 3]$.

The length of the graph is given by the integral of $\sqrt{1 + f'(x)^2}$. In this case,

$$f'(x) = \frac{1}{3} \cdot \frac{3}{2} \cdot (x^2 + 2)^{1/2} \cdot 2x = x(x^2 + 2)^{1/2},$$

so the expression $1 + f'(x)^2$ can be written in the form

$$1 + f'(x)^2 = 1 + x^2(x^2 + 2) = 1 + x^4 + 2x^2 = (1 + x^2)^2.$$

Taking the square root of both sides, we conclude that the length of the graph is

$$\int_1^3 \sqrt{1 + f'(x)^2}\, dx = \int_1^3 (1 + x^2)\, dx = \left[x + \frac{x^3}{3}\right]_1^3 = \frac{32}{3}.$$

4. Find both the mass and the centre of mass for a thin rod whose density is given by

$$\delta(x) = x^2 + 4x + 1, \qquad 0 \leq x \leq 2.$$

The mass of the rod is merely the integral of its density function, namely

$$M = \int_0^2 \delta(x)\,dx = \int_0^2 (x^2 + 4x + 1)\,dx = \left[\frac{x^3}{3} + 2x^2 + x\right]_0^2 = \frac{38}{3}.$$

The centre of mass is given by a similar formula and one finds that

$$\overline{x} = \frac{1}{M}\int_0^2 x\delta(x)\,dx = \frac{3}{38}\int_0^2 (x^3 + 4x^2 + x)\,dx = \frac{3}{38}\left[\frac{x^4}{4} + \frac{4x^3}{3} + \frac{x^2}{2}\right]_0^2 = \frac{25}{19}.$$

5. A chain that is 4m long has a uniform density of 3kg/m. If the chain is hanging from the top of a tall building, then how much work is needed to pull it up to the top?

Consider an arbitrarily small part of the chain, say one of length dx, which lies x metres from the top. The work that is needed to pull this part to the top is then

$$\text{Work} = \text{Force} \cdot \text{Displacement} = mg \cdot x = (3\,dx)g \cdot x.$$

Summing up these expressions over all possible values of $0 \leq x \leq 4$, we conclude that

$$\text{Work} = 3g\int_0^4 x\,dx = 3g\left[\frac{x^2}{2}\right]_0^4 = 24g.$$

6. Find the area of the region enclosed by the graphs of $f(x)$ and $g(x)$ in the case that

$$f(x) = \sin x, \qquad g(x) = \cos x, \qquad 0 \leq x \leq \pi/2.$$

The two functions are both non-negative on the interval $[0, \pi/2]$ and one has

$$f(x) \leq g(x) \quad\Longleftrightarrow\quad \sin x \leq \cos x \quad\Longleftrightarrow\quad \tan x \leq 1 \quad\Longleftrightarrow\quad x \in [0, \pi/4].$$

In other words, $f(x) \leq g(x)$ when $0 \leq x \leq \pi/4$ and $g(x) \leq f(x)$ when $\pi/4 \leq x \leq \pi/2$, so

$$\begin{aligned}
\text{Area} &= \int_0^{\pi/4} [\cos x - \sin x]\,dx + \int_{\pi/4}^{\pi/2} [\sin x - \cos x]\,dx \\
&= \Big[\sin x + \cos x\Big]_0^{\pi/4} + \Big[-\cos x - \sin x\Big]_{\pi/4}^{\pi/2} \\
&= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0 - 1 - 0 - 1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 2\sqrt{2} - 2.
\end{aligned}$$

7. The graph of $f(x) = 2e^{6x}$ is rotated around the x-axis over the interval $[0, a]$. If the volume of the resulting solid is equal to π, then what is the value of a?

The volume of the resulting solid is the integral of $\pi f(x)^2$ and this is given by

$$\text{Volume} = \pi \int_0^a 4e^{12x}\, dx = 4\pi \left[\frac{e^{12x}}{12} \right]_0^a = \frac{\pi}{3}(e^{12a} - 1).$$

Since the volume must be equal to π by assumption, it easily follows that

$$e^{12a} - 1 = 3 \quad \Longrightarrow \quad e^{12a} = 4 \quad \Longrightarrow \quad 12a = \ln 4 \quad \Longrightarrow \quad a = \frac{\ln 2^2}{12} = \frac{\ln 2}{6}.$$

8. Compute the length of the graph of $f(x) = x^{3/2} - \frac{1}{3}x^{1/2}$ over the interval $[0, 2]$.

The length of the graph is given by the integral of $\sqrt{1 + f'(x)^2}$. In this case,

$$f'(x) = \frac{3}{2}x^{1/2} - \frac{1}{6}x^{-1/2} \quad \Longrightarrow \quad 1 + f'(x)^2 = 1 + \frac{9}{4}x + \frac{1}{36x} - \frac{1}{2}$$

and one may use a common denominator to write this expression in the form

$$1 + f'(x)^2 = \frac{18x + (9x)^2 + 1}{36x} = \frac{(9x+1)^2}{36x}.$$

Taking the square root of both sides, we conclude that the length of the graph is

$$\int_0^2 \frac{9x+1}{6\sqrt{x}}\, dx = \frac{1}{6}\int_0^2 \left(9x^{1/2} + x^{-1/2}\right)\, dx = \frac{1}{6}\left[\frac{9x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2}\right]_0^2 = \frac{7}{3}\sqrt{2}.$$

9. Show that the function f is integrable on $[0, 1]$ for any given constants a, b when

$$f(x) = \left\{ \begin{array}{ll} a & \text{if } x \neq 0 \\ b & \text{if } x = 0 \end{array} \right\}.$$

Let $x_0, x_1, \ldots, x_n$ be the points that divide the interval $[0, 1]$ into n subintervals of equal length. To show that f is integrable on $[0, 1]$, we need to compute the limit

$$\int_0^1 f(x)\, dx = \lim_{n\to\infty} \sum_{k=1}^n f(x_k^*)\, \Delta x$$

for any choice of points $x_k^* \in [x_{k-1}, x_k]$. When $x_1^* > 0$, we have $x_k^* > 0$ for all $k \geq 1$ and so

$$\int_0^1 f(x)\, dx = \lim_{n\to\infty} \sum_{k=1}^n \frac{a}{n} = \lim_{n\to\infty} n \cdot \frac{a}{n} = a.$$

When $x_1^* = 0$, on the other hand, we have $x_k^* > 0$ for all $k \geq 2$ and the limit is still

$$\int_0^1 f(x)\, dx = \lim_{n\to\infty} \left[\frac{b}{n} + \sum_{k=2}^n \frac{a}{n}\right] = \lim_{n\to\infty} \left[\frac{b}{n} + \frac{(n-1)a}{n}\right] = a.$$

10. Compute each of the following improper integrals.

$$I_1 = \int_2^\infty \frac{dx}{(x-1)^5}, \qquad I_2 = \int_2^3 \frac{dx}{\sqrt[4]{x-2}}, \qquad I_3 = \int_0^\infty \frac{dx}{x^2+1}.$$

When it comes to the first integral, one easily finds that

$$I_1 = \lim_{L\to\infty} \int_2^L (x-1)^{-5}\,dx = \lim_{L\to\infty} \left[-\frac{1}{4}(x-1)^{-4} \right]_2^L = \frac{1}{4}.$$

When it comes to the second integral, one similarly finds that

$$I_2 = \lim_{a\to 2^+} \int_a^3 (x-2)^{-1/4}\,dx = \lim_{a\to 2^+} \left[\frac{4}{3}(x-2)^{3/4} \right]_a^3 = \frac{4}{3}.$$

Finally, the third integral is related to the inverse tangent function and one has

$$I_3 = \lim_{L\to\infty} \int_0^L \frac{dx}{x^2+1} = \lim_{L\to\infty} (\tan^{-1} L - \tan^{-1} 0) = \frac{\pi}{2}.$$

MAU11201 – Calculus
Tutorial solutions #8

1. Compute each of the following indefinite integrals.

$$\int \frac{x^2}{x^3+1}\,dx, \qquad \int \frac{x^2}{x+1}\,dx.$$

For the first integral, we use the substitution $u = x^3+1$. Since $du = 3x^2\,dx$, we get

$$\int \frac{x^2}{x^3+1}\,dx = \frac{1}{3}\int \frac{du}{u} = \frac{\ln|u|}{3} + C = \frac{\ln|x^3+1|}{3} + C.$$

For the second integral, we let $u = x+1$. This gives $du = dx$, so it easily follows that

$$\begin{aligned}\int \frac{x^2}{x+1}\,dx &= \int \frac{(u-1)^2}{u}\,du = \int \frac{u^2-2u+1}{u}\,du = \int \left(u - 2 + \frac{1}{u}\right)du \\ &= \frac{u^2}{2} - 2u + \ln|u| + C = \frac{(x+1)^2}{2} - 2(x+1) + \ln|x+1| + C.\end{aligned}$$

2. Compute each of the following indefinite integrals.

$$\int \sin^2 x \cdot \cos^3 x\,dx, \qquad \int \sec^5 x \cdot \tan x\,dx.$$

For the first integral, we use the substitution $u = \sin x$. Since $du = \cos x\,dx$, we get

$$\begin{aligned}\int \sin^2 x\cdot\cos^3 x\,dx &= \int \sin^2 x\cdot\cos^2 x\cdot\cos x\,dx = \int u^2(1-u^2)\,du \\ &= \int (u^2-u^4)\,du = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.\end{aligned}$$

For the second integral, we let $u = \sec x$. This gives $du = \sec x \tan x\,dx$ and so

$$\int \sec^5 x\cdot\tan x\,dx = \int u^4\,du = \frac{u^5}{5} + C = \frac{\sec^5 x}{5} + C.$$

3. Find the volume of the solid that is obtained by rotating the graph of $f(x) = \tan x$ around the x-axis over the interval $[0, \pi/4]$.

The volume of the solid is the integral of $\pi f(x)^2$ and this is given by

$$\text{Volume} = \pi\int_0^{\pi/4} \tan^2 x\,dx = \pi\int_0^{\pi/4} (\sec^2 x - 1)\,dx = \pi\Big[\tan x - x\Big]_0^{\pi/4} = \pi - \frac{\pi^2}{4}.$$

4. Compute each of the following indefinite integrals.

$$\int \frac{x^3 - x}{x^2 + 5}\, dx, \qquad \int \frac{x^2 + 5}{x^3 - x}\, dx.$$

When it comes to the first integral, one may use division of polynomials to write

$$\int \frac{x^3 - x}{x^2 + 5}\, dx = \int \left(x - \frac{6x}{x^2 + 5} \right) dx.$$

To integrate the fraction, we let $u = x^2 + 5$. Since $du = 2x\, dx$, we find that

$$\begin{aligned} \int \frac{x^3 - x}{x^2 + 5}\, dx &= \frac{x^2}{2} - \int \frac{6x\, dx}{x^2 + 5} = \frac{x^2}{2} - \int \frac{3\, du}{u} \\ &= \frac{x^2}{2} - 3 \ln u + C = \frac{x^2}{2} - 3 \ln(x^2 + 5) + C. \end{aligned}$$

When it comes to the second integral, one may use partial fractions to write

$$\frac{x^2 + 5}{x^3 - x} = \frac{x^2 + 5}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{x + 1}$$

for some constants A, B and C. Clearing denominators gives rise to the identity

$$x^2 + 5 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

and this should be valid for all x. Let us then look at some special values of x to get

$$x = -1, 0, 1 \quad \Longrightarrow \quad 6 = 2C, \qquad 5 = -A, \qquad 6 = 2B.$$

This gives $A = -5$ and $B = C = 3$, so the second integral can be expressed in the form

$$\begin{aligned} \int \frac{x^2 + 5}{x^3 - x}\, dx &= \int \left(-\frac{5}{x} + \frac{3}{x - 1} + \frac{3}{x + 1} \right) dx \\ &= -5 \ln |x| + 3 \ln |x - 1| + 3 \ln |x + 1| + K. \end{aligned}$$

5. Compute each of the following indefinite integrals.

$$\int \sin^{-1} x\, dx, \qquad \int e^{\sqrt{x}}\, dx.$$

For the first integral, let $u = \sin^{-1} x$ and $dv = dx$. Then $du = \frac{dx}{\sqrt{1-x^2}}$ and $v = x$, so

$$\int \sin^{-1} x\, dx = uv - \int v\, du = x \sin^{-1} x - \int \frac{x\, dx}{\sqrt{1 - x^2}}.$$

To compute the rightmost integral, we let $w = 1 - x^2$. This gives $dw = -2x\,dx$ and

$$\int \sin^{-1} x\,dx = x\sin^{-1} x + \frac{1}{2}\int \frac{dw}{\sqrt{w}} = x\sin^{-1} x + \frac{1}{2}\int w^{-1/2}\,dw$$
$$= x\sin^{-1} x + w^{1/2} + C = x\sin^{-1} x + \sqrt{1-x^2} + C.$$

Finally, we integrate $e^{\sqrt{x}}$. If we let $u = \sqrt{x}$, then $x = u^2$ and $dx = 2u\,du$, so

$$\int e^{\sqrt{x}}\,dx = 2\int ue^u\,du.$$

Once we now integrate by parts with $dv = 2e^u\,du$, we get $v = 2e^u$ and also

$$\int e^{\sqrt{x}}\,dx = 2ue^u - 2\int e^u\,du = 2ue^u - 2e^u + C = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C.$$

6. Find the area of the region enclosed by the graphs of $f(x) = e^{2x}$ and $g(x) = 4e^x - 3$.

Letting $z = e^x$ for simplicity, we get $f(x) = z^2$ and $g(x) = 4z - 3$. It easily follows that

$$f(x) \leq g(x) \quad\iff\quad z^2 \leq 4z - 3 \quad\iff\quad (z-3)(z-1) \leq 0 \quad\iff\quad 1 \leq z \leq 3.$$

In other words, $f(x) \leq g(x)$ if and only if $0 \leq x \leq \ln 3$, so the area of the region is

$$\text{Area} = \int_0^{\ln 3} [g(x) - f(x)]\,dx = \int_0^{\ln 3} (4e^x - 3 - e^{2x})\,dx$$
$$= \left[4e^x - 3x - \frac{1}{2}e^{2x}\right]_0^{\ln 3} = 4 - 3\ln 3.$$

7. Compute each of the following indefinite integrals.

$$\int \frac{dx}{(1+x)\sqrt{x}}, \qquad \int x(\ln x)^2\,dx.$$

For the first integral, we let $u = \sqrt{x}$. This gives $x = u^2$ and $dx = 2u\,du$, so

$$\int \frac{dx}{(1+x)\sqrt{x}} = \int \frac{2u\,du}{(1+u^2)u} = \int \frac{2\,du}{1+u^2} = 2\tan^{-1} u + C = 2\tan^{-1}\sqrt{x} + C.$$

For the second integral, we let $u = (\ln x)^2$ and $dv = x\,dx$. Then $du = \frac{2\ln x}{x}\,dx$ and $v = \frac{x^2}{2}$, so

$$\int x(\ln x)^2\,dx = \frac{x^2}{2}(\ln x)^2 - \int \frac{2\ln x}{x}\cdot\frac{x^2}{2}\,dx = \frac{x^2}{2}(\ln x)^2 - \int x(\ln x)\,dx.$$

Next, we take $u = \ln x$ and $dv = x\,dx$. Since $du = \frac{dx}{x}$ and $v = \frac{x^2}{2}$, we conclude that

$$\int x(\ln x)^2\,dx = \frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x + \int \frac{x}{2}\,dx = \frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}\ln x + \frac{x^2}{4} + C.$$

8. Compute each of the following indefinite integrals.

$$\int \frac{2\,dx}{(x^2+1)^2}, \qquad \int x^2\sqrt{1-x^2}\,dx.$$

For the first integral, let $x = \tan\theta$ for some angle $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and note that

$$x^2+1 = \tan^2\theta + 1 = \sec^2\theta, \qquad dx = \sec^2\theta\,d\theta.$$

The given integral can thus be expressed in the form

$$\int \frac{2\,dx}{(x^2+1)^2} = \int \frac{2\sec^2\theta\,d\theta}{\sec^4\theta} = \int 2\cos^2\theta\,d\theta.$$

Using the half-angle formula for cosine, one may now simplify to arrive at

$$\int \frac{2\,dx}{(x^2+1)^2} = \int (1+\cos(2\theta))\,d\theta = \left(\theta + \frac{\sin(2\theta)}{2}\right) = (\theta + \sin\theta\cos\theta)\,.$$

We need to express this equation in terms of $x = \tan\theta$. When $x \geq 0$, the angle θ appears in a right triangle with an opposite side of length x and an adjacent side of length 1. This makes the hypotenuse of length $\sqrt{x^2+1}$, so one finds that

$$\int \frac{2\,dx}{(x^2+1)^2} = \tan^{-1}x + \frac{x}{\sqrt{x^2+1}}\cdot\frac{1}{\sqrt{x^2+1}} = \tan^{-1}x + \frac{x}{x^2+1}.$$

When $x = \tan\theta \leq 0$, the expression $\theta + \sin\theta\cos\theta$ changes by a minus sign and the same is true for the right hand side of the last equation. Thus, the equation remains valid.

Finally, we look at the integral of $x^2\sqrt{1-x^2}$. Taking $x = \sin\theta$, we get

$$\begin{aligned}\int x^2\sqrt{1-x^2}\,dx &= \int \sin^2\theta\cos^2\theta\,d\theta = \int \frac{1-\cos(2\theta)}{2}\cdot\frac{1+\cos(2\theta)}{2}\,d\theta \\ &= \frac{1}{4}\int (1-\cos^2(2\theta))\,d\theta = \frac{1}{4}\int\left(1 - \frac{1+\cos(4\theta)}{2}\right)d\theta \\ &= \frac{1}{8}\int (1-\cos(4\theta))\;d\theta = \frac{1}{8}\left(\theta - \frac{1}{4}\sin(4\theta)\right).\end{aligned}$$

It remains to simplify the right hand side. The addition formulas for sine and cosine give

$$\sin(4\theta) = 2\sin(2\theta)\cos(2\theta) = 4\sin\theta\cos\theta\cdot(\cos^2\theta - \sin^2\theta).$$

Since $\sin\theta = x$, one has $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$ and so

$$\sin(4\theta) = 4x\sqrt{1-x^2}\cdot(1-x^2-x^2) = 4x(1-x^2)^{3/2} - 4x^3\sqrt{1-x^2}.$$

Once we now combine the above computations, we may finally conclude that

$$\int x^2\sqrt{1-x^2}\,dx = \frac{1}{8}\sin^{-1}x - \frac{x}{8}(1-x^2)^{3/2} + \frac{x^3}{8}\sqrt{1-x^2} + C.$$

9. Let $a > 0$ be given. Use integration by parts to find a reduction formula for

$$I_n = \int \frac{dx}{(x^2+a^2)^n}.$$

If we let $u = (x^2+a^2)^{-n}$ and $dv = dx$, then $du = -2nx(x^2+a^2)^{-n-1}\,dx$ and $v = x$, so

$$\begin{aligned} I_n &= x(x^2+a^2)^{-n} + 2n\int x^2(x^2+a^2)^{-n-1}\,dx \\ &= \frac{x}{(x^2+a^2)^n} + 2n\int \frac{x^2+a^2-a^2}{(x^2+a^2)^{n+1}}\,dx \\ &= \frac{x}{(x^2+a^2)^n} + 2nI_n - 2na^2 I_{n+1}. \end{aligned}$$

Rearranging terms, one may thus express the integral I_{n+1} in terms of I_n to find that

$$I_{n+1} = \frac{2n-1}{2na^2}\cdot I_n + \frac{x}{2na^2(x^2+a^2)^n}.$$

10. Use integration by parts to compute the indefinite integral

$$\int \sin(\ln x)\,dx.$$

Letting $u = \sin(\ln x)$ and $dv = dx$, we get $du = \cos(\ln x)\cdot\frac{dx}{x}$ and $v = x$, so

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - \int \cos(\ln x)\,dx.$$

Letting $u = \cos(\ln x)$ and $dv = dx$, we similarly get $du = -\sin(\ln x)\cdot\frac{dx}{x}$ and $v = x$, so

$$\int \cos(\ln x)\,dx = x\cos(\ln x) + \int \sin(\ln x)\,dx.$$

Once we now combine the last two equations, we get an identity of the form

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\,dx.$$

Moving the rightmost integral to the left hand side, we may thus conclude that

$$\int \sin(\ln x)\,dx = \frac{x}{2}\sin(\ln x) - \frac{x}{2}\cos(\ln x) + C.$$

MAU11201 – Calculus
Tutorial solutions #9

1. Compute each of the following indefinite integrals.

$$\int e^{2x}\cos(e^x)\,dx, \qquad \int \frac{\sin^3 x}{\cos^6 x}\,dx.$$

For the first integral, we let $u = e^x$. Since $du = e^x\,dx$, one finds that

$$\int e^{2x}\cos(e^x)\,dx = \int e^x\cos(e^x)\cdot e^x\,dx = \int u\cos u\,du.$$

Next, we integrate by parts with $dv = \cos u\,du$. This gives $v = \sin u$ and so

$$\begin{aligned}\int e^{2x}\cos(e^x)\,dx = u\sin u - \int \sin u\,du &= u\sin u + \cos u + C\\ &= e^x\sin(e^x) + \cos(e^x) + C.\end{aligned}$$

For the second integral, it is better to simplify the given expression and write

$$\int \frac{\sin^3 x}{\cos^6 x}\,dx = \int \frac{\tan^3 x}{\cos^3 x}\,dx = \int \sec^3 x\cdot\tan^3 x\,dx.$$

To compute this integral, we let $u = \sec x$. Then $du = \sec x\tan x\,dx$ and we get

$$\begin{aligned}\int \frac{\sin^3 x}{\cos^6 x}\,dx &= \int \sec^2 x\cdot\tan^2 x\cdot\sec x\tan x\,dx = \int u^2(u^2-1)\,du\\ &= \int (u^4-u^2)\,du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C.\end{aligned}$$

2. Compute each of the following indefinite integrals.

$$\int \frac{\sqrt{x}}{x+1}\,dx, \qquad \int \frac{\sqrt{x}}{x-1}\,dx.$$

In each case, we let $u = \sqrt{x}$ to simplify. Since $x = u^2$, we have $dx = 2u\,du$ and

$$\int \frac{\sqrt{x}}{x+1}\,dx = \int \frac{u}{u^2+1}\cdot 2u\,du = \int \frac{2u^2}{u^2+1}\,du.$$

This is a rational function that can be simplified using division of polynomials, so

$$\begin{aligned}\int \frac{\sqrt{x}}{x+1}\,dx &= \int \frac{2(u^2+1)-2}{u^2+1}\,du = \int \left(2 - \frac{2}{u^2+1}\right)du\\ &= 2u - 2\tan^{-1}u + C = 2\sqrt{x} - 2\tan^{-1}\sqrt{x} + C.\end{aligned}$$

For the second integral, we proceed in a similar fashion to find that

$$\int \frac{\sqrt{x}}{x-1}\,dx = \int \frac{2u^2}{u^2-1}\,du = \int \left(2 + \frac{2}{u^2-1}\right)du.$$

In this case, however, one needs to use partial fractions to write

$$\frac{2}{u^2-1} = \frac{2}{(u+1)(u-1)} = \frac{A}{u+1} + \frac{B}{u-1}$$

for some constants A, B that need to be determined. Clearing denominators gives

$$2 = A(u-1) + B(u+1),$$

so we may take $u = \pm 1$ to find that $2B = 2 = -2A$. It easily follows that

$$\begin{aligned}\int \frac{\sqrt{x}}{x-1}\,dx = \int \left(2 + \frac{1}{u-1} - \frac{1}{u+1}\right)du &= 2u + \ln|u-1| - \ln|u+1| + C \\ &= 2\sqrt{x} + \ln|\sqrt{x}-1| - \ln(\sqrt{x}+1) + C.\end{aligned}$$

3. Show that each of the following sequences converges.

$$a_n = \sqrt{\frac{n^2+1}{n^3+2}}, \qquad b_n = \frac{\sin n}{n^2}, \qquad c_n = n^{1/n}.$$

Since the limit of a square root is the square root of the limit, it should be clear that

$$\lim_{n\to\infty} \frac{n^2+1}{n^3+2} = \lim_{n\to\infty} \frac{n^2}{n^3} = \lim_{n\to\infty} \frac{1}{n} = 0 \quad\Longrightarrow\quad \lim_{n\to\infty} a_n = \sqrt{0} = 0.$$

The limit of the second sequence is also zero because $-1/n^2 \leq b_n \leq 1/n^2$ for each $n \geq 1$. This means that b_n lies between two sequences that converge to zero. Finally, one has

$$c_n = n^{1/n} \quad\Longrightarrow\quad \ln c_n = \ln n^{1/n} = \frac{\ln n}{n}.$$

Since $\ln n \to \infty$ as $n \to \infty$, one may use L'Hôpital's rule to conclude that

$$\lim_{n\to\infty} \ln c_n = \lim_{n\to\infty} \frac{1/n}{1} = 0 \quad\Longrightarrow\quad \lim_{n\to\infty} c_n = e^0 = 1.$$

4. Define a sequence $\{a_n\}$ by setting $a_1 = 1$ and $a_{n+1} = \sqrt{6 + a_n}$ for each $n \geq 1$. Show that $1 \leq a_n \leq a_{n+1} \leq 3$ for each $n \geq 1$, use this fact to conclude that the sequence converges and then find its limit.

Since the first two terms are $a_1 = 1$ and $a_2 = \sqrt{7}$, the statement

$$1 \leq a_n \leq a_{n+1} \leq 3$$

does hold when $n = 1$. Suppose that it holds for some n, in which case

$$\begin{aligned} 7 \leq 6 + a_n \leq 6 + a_{n+1} \leq 9 \quad &\Longrightarrow \quad \sqrt{7} \leq a_{n+1} \leq a_{n+2} \leq 3 \\ &\Longrightarrow \quad 1 \leq a_{n+1} \leq a_{n+2} \leq 3. \end{aligned}$$

In particular, the statement holds for $n + 1$ as well, so it actually holds for all $n \in \mathbb{N}$. This shows that the given sequence is monotonic and bounded, hence also convergent; denote its limit by L. Using the definition of the sequence, we then find that

$$a_{n+1} = \sqrt{6 + a_n} \quad \Longrightarrow \quad \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{6 + a_n} \quad \Longrightarrow \quad L = \sqrt{6 + L}.$$

This leads to the quadratic equation $L^2 = 6 + L$ which implies that $L = -2, 3$. Since the terms of the sequence satisfy $1 \leq a_n \leq 3$, however, the limit must be $L = 3$.

5. Use the formula for a geometric series to compute each of the following sums.

$$\sum_{n=0}^{\infty} \frac{2^n}{7^n}, \qquad \sum_{n=1}^{\infty} \frac{3^{n+2}}{2^{3n+1}}, \qquad \sum_{n=2}^{\infty} \frac{3^{n+1}}{4^{n+2}}.$$

The first sum is the sum of a geometric series with $x = 2/7$ and one easily finds that

$$\sum_{n=0}^{\infty} \frac{2^n}{7^n} = \sum_{n=0}^{\infty} \left(\frac{2}{7}\right)^n = \frac{1}{1 - 2/7} = \frac{7}{5}.$$

The second sum is the sum of a geometric series with $x = 3/8$ and we similarly get

$$\sum_{n=1}^{\infty} \frac{3^{n+2}}{2^{3n+1}} = \frac{3^2}{2} \sum_{n=1}^{\infty} \left(\frac{3}{8}\right)^n = \frac{9}{2} \cdot \frac{3/8}{1 - 3/8} = \frac{27}{10}.$$

To compute the third sum, we shift the index of summation to conclude that

$$\sum_{n=2}^{\infty} \frac{3^{n+1}}{4^{n+2}} = \sum_{n=1}^{\infty} \frac{3^{n+1+1}}{4^{n+1+2}} = \frac{9}{64} \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n = \frac{9}{64} \cdot \frac{3/4}{1 - 3/4} = \frac{27}{64}.$$

6. Compute each of the following indefinite integrals.

$$\int \frac{2x+3}{x^2-4x+3}\,dx, \qquad \int \frac{2x+3}{x^2-4x+5}\,dx.$$

When it comes to the first integral, one may use partial fractions to write

$$\frac{2x+3}{x^2-4x+3} = \frac{2x+3}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3}$$

for some constants A and B. Clearing denominators gives rise to the identity

$$2x+3 = A(x-3) + B(x-1)$$

and this should be valid for all x. Let us then look at some special values of x to get

$$x = 1, 3 \quad \Longrightarrow \quad 5 = -2A, \qquad 9 = 2B.$$

This gives $A = -5/2$ and $B = 9/2$, so it easily follows that

$$\int \frac{2x+3}{x^2-4x+3}\,dx = \int \left(-\frac{5/2}{x-1} + \frac{9/2}{x-3} \right) dx = -\frac{5}{2}\ln|x-1| + \frac{9}{2}\ln|x-3| + C.$$

When it comes to the second integral, one may complete the square to write

$$\int \frac{2x+3}{x^2-4x+5}\,dx = \int \frac{2x+3}{x^2-4x+4+1}\,dx = \int \frac{2x+3}{(x-2)^2+1}\,dx.$$

Using the substitution $u = x-2$, we now get $du = dx$ and also $x = u+2$, so

$$\begin{aligned}\int \frac{2x+3}{x^2-4x+5}\,dx &= \int \frac{2u+7}{u^2+1}\,du = \int \frac{2u}{u^2+1}\,du + \int \frac{7}{u^2+1}\,du \\ &= \ln(u^2+1) + 7\tan^{-1} u + C \\ &= \ln(x^2-4x+5) + 7\tan^{-1}(x-2) + C.\end{aligned}$$

7. Compute each of the following indefinite integrals.

$$\int \sqrt{1-x^2}\,dx, \qquad \int \frac{\sqrt{1-x}}{\sqrt{1+x}}\,dx.$$

For the first integral, we let $x = \sin\theta$. Since $dx = \cos\theta\,d\theta$, this gives

$$\int \sqrt{1-x^2}\,dx = \int \cos\theta \cdot \cos\theta\,d\theta = \frac{1}{2}\int (1+\cos(2\theta))\,d\theta = \frac{\theta}{2} + \frac{\sin(2\theta)}{4} + C.$$

Since $\sin\theta = x$ and $\cos\theta = \sqrt{1-x^2}$, it easily follows that

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\sin^{-1}x + \frac{\sin\theta\cos\theta}{2} + C = \frac{1}{2}\sin^{-1}x + \frac{x}{2}\sqrt{1-x^2} + C.$$

The second integral seems a bit difficult, but one may express it in the form

$$\int \frac{\sqrt{1-x}}{\sqrt{1+x}}\,dx = \int \frac{\sqrt{1-x}}{\sqrt{1+x}} \cdot \frac{\sqrt{1-x}}{\sqrt{1-x}}\,dx = \int \frac{1-x}{\sqrt{1-x^2}}\,dx.$$

Using this fact along with the substitution $u = 1-x^2$, we conclude that

$$\begin{aligned}\int \frac{\sqrt{1-x}}{\sqrt{1+x}}\,dx &= \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{x\,dx}{\sqrt{1-x^2}} = \sin^{-1}x + \int \frac{du}{2\sqrt{u}} \\ &= \sin^{-1}x + \sqrt{u} + C = \sin^{-1}x + \sqrt{1-x^2} + C.\end{aligned}$$

8. Define a sequence $\{a_n\}$ by setting $a_1 = 1$ and $a_{n+1} = 3 + \sqrt{a_n}$ for each $n \geq 1$. Show that $1 \leq a_n \leq a_{n+1} \leq 9$ for each $n \geq 1$, use this fact to conclude that the sequence converges and then find its limit.

Since the first two terms are $a_1 = 1$ and $a_2 = 3 + 1 = 4$, the statement

$$1 \leq a_n \leq a_{n+1} \leq 9$$

does hold when $n = 1$. Suppose that it holds for some n, in which case

$$\begin{aligned}1 \leq \sqrt{a_n} \leq \sqrt{a_{n+1}} \leq 3 \quad &\implies \quad 4 \leq a_{n+1} \leq a_{n+2} \leq 6 \\ &\implies \quad 1 \leq a_{n+1} \leq a_{n+2} \leq 9.\end{aligned}$$

In particular, the statement holds for $n+1$ as well, so it actually holds for all $n \in \mathbb{N}$. This shows that the given sequence is monotonic and bounded, hence also convergent; denote its limit by L. Using the definition of the sequence, we then find that

$$a_{n+1} = 3 + \sqrt{a_n} \quad \implies \quad \lim_{n\to\infty} a_{n+1} = 3 + \lim_{n\to\infty} \sqrt{a_n} \quad \implies \quad L = 3 + \sqrt{L}.$$

This gives the quadratic equation $(L-3)^2 = L$, which one may easily solve to get

$$L^2 - 6L + 9 = L \quad \implies \quad L^2 - 7L + 9 = 0 \quad \implies \quad L = \frac{7 \pm \sqrt{13}}{2}.$$

Since $L - 3 = \sqrt{L} \geq 0$, however, we also have $L \geq 3$ and the limit is $L = \frac{1}{2}(7 + \sqrt{13})$.

9. An ant starts out at the origin in the xy-plane and walks 1 unit south, then 1/2 units east, then 1/4 units north, then 1/8 units west, then 1/16 units south, and so on. If it continues like that indefinitely, which point in the xy-plane will it eventually reach?

When it comes to the vertical displacement of the ant, this is given by the sum

$$y = -1 + \frac{1}{4} - \frac{1}{16} + \ldots = -\sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n .$$

Using the explicit formula for the sum of a geometric series, one now finds that

$$y = -\frac{1}{1+1/4} = -\frac{4}{5} .$$

When it comes to the horizontal displacement of the ant, one similarly has

$$x = \frac{1}{2} - \frac{1}{8} + \frac{1}{32} - \ldots = \frac{1}{2}\sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n = \frac{1/2}{1+1/4} = \frac{2}{5}.$$

Since the ant starts out at the origin, it will eventually reach the point $(2/5, -4/5)$.

10. Suppose the series $\sum_{n=1}^{\infty} a_n$ converges. Show that the series $\sum_{n=1}^{\infty} \frac{1}{1+a_n}$ diverges.

Since $\sum_{n=1}^{\infty} a_n$ converges, we must have $\lim_{n\to\infty} a_n = 0$ by the nth term test, so

$$\lim_{n\to\infty} \frac{1}{1+a_n} = 1.$$

Using the nth term test once again, we conclude that $\sum_{n=1}^{\infty} \frac{1}{1+a_n}$ diverges.

MAU11201 – Calculus
Tutorial solutions #10

1. Test each of the following series for convergence.

$$\sum_{n=1}^{\infty} \frac{2+\sin n}{n}, \qquad \sum_{n=1}^{\infty} \frac{2+\sin n}{n^2}.$$

When it comes to the first series, we use comparison with $1/n$. Since

$$\sum_{n=1}^{\infty} \frac{2+\sin n}{n} \geq \sum_{n=1}^{\infty} \frac{1}{n},$$

the first series is larger than a divergent p-series, so it diverges. A similar argument gives

$$\sum_{n=1}^{\infty} \frac{2+\sin n}{n^2} \leq \sum_{n=1}^{\infty} \frac{3}{n^2},$$

so the second series is smaller than a convergent p-series and thus converges as well.

2. Test each of the following series for convergence.

$$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}, \qquad \sum_{n=1}^{\infty} \frac{ne^{1/n}}{n^3+1}.$$

When it comes to the first series, we use the limit comparison test with

$$a_n = \frac{e^{1/n}}{n}, \qquad b_n = \frac{1}{n}.$$

To show that the limit comparison test is applicable in this case, we note that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} e^{1/n} = e^0 = 1.$$

Since $\sum_{n=1}^{\infty} b_n$ is a divergent p-series, we conclude that $\sum_{n=1}^{\infty} a_n$ diverges as well. When it comes to the second series, we use the limit comparison test with

$$a_n = \frac{ne^{1/n}}{n^3+1}, \qquad b_n = \frac{1}{n^2}.$$

In this case, the limit comparison test is still applicable, as one can easily check that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3 e^{1/n}}{n^3+1} = \lim_{n\to\infty} e^{1/n} = e^0 = 1.$$

Since $\sum_{n=1}^{\infty} b_n$ is a convergent p-series, we conclude that $\sum_{n=1}^{\infty} a_n$ converges as well.

3. Test each of the following series for convergence.

$$\sum_{n=1}^{\infty} \frac{\ln n}{n}, \qquad \sum_{n=1}^{\infty} \frac{\ln n}{n!}.$$

When it comes to the first series, we use the comparison test. Since

$$\sum_{n=1}^{\infty} \frac{\ln n}{n} = \sum_{n=2}^{\infty} \frac{\ln n}{n} \geq \sum_{n=2}^{\infty} \frac{\ln 2}{n},$$

the first series is larger than a divergent p-series and thus diverges. When it comes to the second series, we use the ratio test together with L'Hôpital's rule. This gives

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \cdot \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1/(n+1)}{1/n} \cdot \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Since $L < 1$, it follows by the ratio test that the second series converges.

4. Find the radius of convergence for each of the following power series.

$$\sum_{n=0}^{\infty} \frac{nx^n}{3^n}, \qquad \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \cdot x^n.$$

In each case, we use the ratio test to find the radius of convergence. In the first case,

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{|x|^{n+1}}{|x|^n} \cdot \frac{3^n}{3^{n+1}} = \lim_{n\to\infty} \frac{n+1}{3n} \cdot |x| = \frac{|x|}{3}.$$

In particular, the series converges when $|x| < 3$ and it diverges when $|x| > 3$, so its radius of convergence is $R = 3$. For the second series, we similarly have

$$\begin{aligned} L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{(n+1)!}{n!} \cdot \frac{(2n)!}{(2n+2)!} \cdot \frac{|x|^{n+1}}{|x|^n} \\ &= \lim_{n\to\infty} \frac{(n+1)^2 \cdot |x|}{(2n+1)(2n+2)} = \lim_{n\to\infty} \frac{n^2|x|}{4n^2} = \frac{|x|}{4}. \end{aligned}$$

Thus, the series converges when $|x| < 4$ and diverges when $|x| > 4$, so the radius is $R = 4$.

5. Assuming that $|x| < 1$, use the formula for a geometric series to show that

$$\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

Since $|x| < 1$ by assumption, one may use the formula for a geometric series to get

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = (1-x)^{-1}.$$

This power series can be differentiated term by term, so it easily follows that

$$\sum_{n=0}^{\infty} nx^{n-1} = (1-x)^{-2} \quad \Longrightarrow \quad \sum_{n=0}^{\infty} nx^n = x(1-x)^{-2} = \frac{x}{(1-x)^2}.$$

6. Find the radius of convergence for each of the following power series.

$$\sum_{n=0}^{\infty} \frac{nx^{2n}}{4^n}, \qquad \sum_{n=0}^{\infty} \frac{3^n x^n}{2n+1}.$$

In each case, we use the ratio test to find the radius of convergence. In the first case,

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{x^{2n+2}}{x^{2n}} \cdot \frac{4^n}{4^{n+1}} = \lim_{n\to\infty} \frac{(n+1)x^2}{4n} = \frac{x^2}{4}.$$

Thus, the series converges when $x^2 < 4$ and diverges when $x^2 > 4$, so the radius is $R = 2$. For the second series, we similarly have

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{3^{n+1}}{3^n} \cdot \frac{|x|^{n+1}}{|x|^n} \cdot \frac{2n+1}{2n+3} = \lim_{n\to\infty} 3|x| \cdot \frac{2n+1}{2n+3} = 3|x|.$$

Thus, the series converges when $|x| < \frac{1}{3}$ and diverges when $|x| > \frac{1}{3}$, so the radius is $R = \frac{1}{3}$.

7. Use differentiation to show that the following power series is equal to $\ln(1+x)$.

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}, \qquad |x| < 1.$$

First of all, we note that this power series converges by the ratio test because

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|x|^{n+2}}{|x|^{n+1}} \cdot \frac{n+1}{n+2} = |x| < 1.$$

Differentiating the series term by term, one obtains the geometric series

$$f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1) x^n}{n+1} = \sum_{n=0}^{\infty} (-x)^n.$$

Since $|x| < 1$ by assumption, this series actually converges and one easily finds that

$$f'(x) = \frac{1}{1-(-x)} = \frac{1}{1+x} \quad \Longrightarrow \quad f(x) = \ln(1+x) + C.$$

Since $f(0) = 0$, however, we must have $0 = \ln 1 + C = C$ and thus $f(x) = \ln(1+x)$.

8. Use differentiation to show that the following power series is equal to $\tan^{-1} x$.

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad |x| < 1.$$

First of all, we note that this power series converges by the ratio test because

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|x|^{2n+3}}{|x|^{2n+1}} \cdot \frac{2n+1}{2n+3} = |x|^2 < 1.$$

Differentiating the series term by term, one obtains the geometric series

$$f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{2n+1} = \sum_{n=0}^{\infty} (-x^2)^n.$$

Since $|x| < 1$ by assumption, this series actually converges and one easily finds that

$$f'(x) = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2} \quad \Longrightarrow \quad f(x) = \tan^{-1} x + C.$$

Since $f(0) = 0$, however, we must have $0 = \tan^{-1} 0 + C = C$ and thus $f(x) = \tan^{-1} x$.

9. Let $a \in \mathbb{R}$ be a given number. Find the radius of convergence for the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{a(a-1)(a-2)\cdots(a-n+1)}{n!} \cdot x^n.$$

To find the radius of convergence, we use the ratio test. In this case, we have

$$\begin{aligned} L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{a(a-1)(a-2)\cdots(a-n)}{a(a-1)(a-2)\cdot(a-n+1)} \cdot \frac{n!}{(n+1)!} \cdot \frac{x^{n+1}}{x^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{a-n}{n+1} \cdot x \right| = |x|. \end{aligned}$$

Thus, the series converges when $|x| < 1$ and diverges when $|x| > 1$, so the radius is $R = 1$.

10. Show that $\sum_{n=0}^{\infty} a_n y^n$ converges absolutely, if $\sum_{n=0}^{\infty} a_n x^n$ converges and $|y| < |x|$.

Since the series $\sum_{n=0}^{\infty} a_n x^n$ converges, its nth term approaches zero as $n \to \infty$ and so

$$|a_n x^n| \leq 1 \quad \text{for large enough } n.$$

Suppose that this is true for all $n \geq N$, for instance. One may then conclude that

$$\sum_{n=N}^{\infty} |a_n y^n| = \sum_{n=N}^{\infty} |a_n x^n| \cdot \left|\frac{y}{x}\right|^n \leq \sum_{n=N}^{\infty} \left|\frac{y}{x}\right|^n.$$

The series on the right hand side is a geometric series with ratio $r < 1$ and thus converges. This implies that $\sum_{n=0}^{\infty} |a_n y^n|$ converges as well, so $\sum_{n=0}^{\infty} a_n y^n$ converges absolutely.