MAU11201 – Calculus
Tutorial problems #1

1. Find the domain and the range of the function f which is defined by

$$f(x) = \frac{4-3x}{6-5x}.$$

2. Find the domain and the range of the function f which is defined by

$$f(x) = \sqrt{x - x^2}.$$

3. Show that the function $f\colon (0,1) \to (0,\infty)$ is bijective in the case that

$$f(x) = \frac{1}{x} - 1.$$

4. Express the following polynomials as the product of linear factors.

$$f(x) = 2x^3 - 7x^2 + 9, \qquad g(x) = x^3 - \frac{3x}{4} - \frac{1}{4}.$$

5. Use the addition formulas for sine and cosine to prove the identity

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha \cdot \tan\beta}.$$

6. Show that the function $f\colon (0,\infty) \to \mathbb{R}$ is injective in the case that

$$f(x) = \frac{2x-1}{3x+2}.$$

7. Find the roots of the polynomial $f(x) = x^3 + x^2 - 5x - 2$.

8. Determine the range of the quadratic $f(x) = ax^2 + bx + c$ in the case that $a > 0$.

9. Relate the sines and the cosines of two angles θ_1, θ_2 whose sum is equal to 2π.

10. Determine all angles $0 \leq \theta \leq 2\pi$ such that $2\cos^2\theta + 7\cos\theta = 4$.

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Tutorial problems #2

1. Determine the inverse function f^{-1} in each of the following cases.

$$f(x) = \log_3(2x-5) - 1, \qquad f(x) = \frac{2\cdot 5^x + 7}{3\cdot 5^x - 4}.$$

2. Simplify each of the following expressions.

$$\sec\left(\tan^{-1} x\right), \qquad \cos\left(\sin^{-1} x\right), \qquad \log_2 18 - 2\log_2 3.$$

3. Use the ε-δ definition of limits to compute $\lim_{x\to 3} f(x)$ in the case that

$$f(x) = \left\{ \begin{array}{ll} 3x-7 & \text{if } x \leq 3 \\ 8-2x & \text{if } x > 3 \end{array} \right\}.$$

4. Compute each of the following limits.

$$L = \lim_{x\to 2} \frac{x^3 - 2x^2 + 5x - 1}{x-3}, \qquad M = \lim_{x\to 2} \frac{x^3 - 3x^2 + 4x - 4}{x-2}.$$

5. Use the ε-δ definition of limits to compute $\lim_{x\to 3}(3x^2 - 7x + 2)$.

6. For which value of a does the limit $\lim_{x\to 2} f(x)$ exist? Explain.

$$f(x) = \left\{ \begin{array}{ll} 2x^2 - ax + 3 & \text{if } x \leq 2 \\ 4x^3 + 3x - a & \text{if } x > 2 \end{array} \right\}.$$

7. Determine the inverse function f^{-1} in the case that $f\colon [2,\infty) \to [1,\infty)$ is defined by

$$f(x) = 2x^2 - 8x + 9.$$

8. Compute each of the following limits.

$$L = \lim_{x\to 3} \frac{x^3 - 5x^2 + 7x - 3}{x-3}, \qquad M = \lim_{x\to 3} \frac{2x^3 - 9x^2 + 27}{(x-3)^2}.$$

9. Use the ε-δ definition of limits to compute $\lim_{x\to 2} \frac{1}{x}$.

10. Use the ε-δ definition of limits to compute $\lim_{x\to 2}(4x^2 - 5x + 1)$.

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Tutorial problems #3

1. Show that there exists a real number $0 < x < \pi/2$ that satisfies the equation

$$x^3 \cos x + x^2 \sin x = 2.$$

2. For which values of a, b is the function f continuous at the point $x = 3$? Explain.

$$f(x) = \left\{ \begin{array}{cl} 2x^2 + ax + b & \text{if } x < 3 \\ 2a + b + 1 & \text{if } x = 3 \\ 5x^2 - bx + 2a & \text{if } x > 3 \end{array} \right\}.$$

3. Show that $f(x) = x^3 - 3x^2 + 1$ has three roots in the interval $(-1, 3)$. Hint: you need only consider the values that are attained by f at the integers $-1 \leq x \leq 3$.

4. Compute each of the following limits.

$$L = \lim_{x\to+\infty} \frac{2x^4 - 7x + 3}{3x^4 - 5x^2 + 1}, \qquad M = \lim_{x\to 2^-} \frac{2x^2 + 3x - 4}{3x^3 - 7x^2 + 4x - 4}.$$

5. Use the definition of the derivative to compute $f'(x_0)$ in each of the following cases.

$$f(x) = 3x^2, \qquad f(x) = 2/x, \qquad f(x) = (2x + 3)^2.$$

6. Show that there exists a real number $0 < x < \pi/2$ that satisfies the equation

$$x^2 + x - 1 = \sin x.$$

7. Show that $f(x) = 3x^3 - 5x + 1$ has three roots in the interval $(-2, 2)$. Hint: you need only consider the values that are attained by f at the integers $-2 \leq x \leq 2$.

8. Compute each of the following limits.

$$L = \lim_{x\to-\infty} \frac{6x^3 - 5x^2 + 7}{5x^4 - 3x + 1}, \qquad M = \lim_{x\to 2^+} \frac{x^3 + x^2 - 5x - 2}{x^3 - 5x^2 + 8x - 4}.$$

9. Use the Squeeze Theorem to show that $\lim_{x\to 0} x^2 \sin(1/x) = 0$.

10. Suppose that f is continuous with $f(0) < 1$. Show that there exists some $\delta > 0$ such that $f(x) < 1$ for all $-\delta < x < \delta$. Hint: use the ε-δ definition for some suitable ε.

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Tutorial problems #4

1. Compute the derivative $y' = \frac{dy}{dx}$ in each of the following cases.

$$y = \ln(\sec x) + e^{\tan x}, \qquad y = \sin(\sec^2(4x)).$$

2. Compute the derivative $y' = \frac{dy}{dx}$ in the case that $x^2 \sin y = y^2 e^x$.

3. Compute the derivative $y' = \frac{dy}{dx}$ in each of the following cases.

$$y = x^2 \cdot \tan^{-1}(2x), \qquad y = (x \cdot \sin x)^x.$$

4. Compute the derivative $f'(x_0)$ in the case that

$$f(x) = \frac{(x^3 + 5x^2 + 2)^3 \cdot e^{\sin x}}{\sqrt{x^2 + 4x + 1}}, \qquad x_0 = 0.$$

5. Compute the derivative $y' = \frac{dy}{dx}$ in the case that

$$y = \sin^{-1} u, \qquad u = \ln(2z^2 + 3z + 1), \qquad z = \frac{3x - 1}{2x + 5}.$$

6. Compute the derivative $y' = \frac{dy}{dx}$ in each of the following cases.

$$y = (e^{2x} + x^3)^4, \qquad y = \tan(x \sin x).$$

7. Compute the derivative $y' = \frac{dy}{dx}$ in the case that $x^2 + y^2 = \sin(xy)$.

8. Compute the derivative $f'(x_0)$ in the case that

$$f(x) = \frac{(x^2 + 3x + 1)^4 \cdot \sqrt{2x + \cos x}}{(e^x + x)^3}, \qquad x_0 = 0.$$

9. Compute the derivative $y' = \frac{dy}{dx}$ in the case that

$$y = \frac{2u - 1}{3u + 1}, \qquad u = \sin(e^z), \qquad z = \tan^{-1}(x^2).$$

10. Compute the derivative $f'(1)$ in the case that $x^2 f(x) + x f(x)^3 = 2$ for all x.

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Tutorial problems #5

1. Show that the polynomial $f(x) = x^3 - 5x^2 - 8x + 1$ has exactly one root in $(0, 1)$.

2. Let $b > 1$ be a given constant. Use the mean value theorem to show that

$$1 - \frac{1}{b} < \ln b < b - 1.$$

3. Compute each of the following limits.

$$L_1 = \lim_{x\to 2} \frac{2x^3 - 5x^2 + 5x - 6}{3x^3 - 5x^2 - 4}, \qquad L_2 = \lim_{x\to\infty} \frac{\ln x}{x^2}, \qquad L_3 = \lim_{x\to 0} (x + \cos x)^{1/x}.$$

4. For which values of x is $f(x) = (\ln x)^2$ increasing? For which values is it concave up?

5. Find the intervals on which f is increasing/decreasing and the intervals on which f is concave up/down. Use this information to sketch the graph of f.

$$f(x) = \frac{x^2}{x^2 + 3}.$$

6. Show that the polynomial $f(x) = x^3 + x^2 - 5x + 1$ has exactly two roots in $(0, 2)$.

7. Use the mean value theorem for the case $f(x) = \sqrt{x + 4}$ to show that

$$2 + \frac{1}{2} < \sqrt{7} < 2 + \frac{3}{4}.$$

8. Compute each of the following limits.

$$L_1 = \lim_{x\to 2} \frac{x^3 - 5x^2 + 8x - 4}{x^3 - 3x^2 + 4}, \qquad L_2 = \lim_{x\to 1} \frac{\ln x}{x^4 - 1}, \qquad L_3 = \lim_{x\to 0^+} \frac{\ln(\sin x)}{\ln(\tan x)}.$$

9. For which values of x is $f(x) = e^{-2x^2}$ increasing? For which values is it concave up?

10. Show that there exists a unique number $1 < x < \pi$ such that $x^3 = 3\sin x + 1$.

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Tutorial problems #6

1. Let $a_1, a_2, \ldots, a_n$ be some given constants and let f be the function defined by

$$f(x) = (x - a_1)^2 + (x - a_2)^2 + \ldots + (x - a_n)^2.$$

Show that $f(x)$ becomes minimum when x is equal to $\overline{x} = (a_1 + a_2 + \ldots + a_n)/n$.

2. Find the global minimum and the global maximum values that are attained by

$$f(x) = 3x^4 - 16x^3 + 18x^2 - 1, \qquad 0 \leq x \leq 2.$$

3. Find the linear approximation to the function f at the point x_0 in the case that

$$f(x) = \frac{(x^2+1)^4 \cdot e^{x^2-1}}{\sqrt{3x+1}}, \qquad x_0 = 1.$$

4. The top of a 5m ladder is sliding down a wall at the rate of 0.25 m/sec. How fast is the base sliding away from the wall when the top lies 3 metres above the ground?

5. Let $n > 0$ be a given constant. Show that $x^n \ln x \geq -\frac{1}{ne}$ for all $x > 0$.

6. Find the global minimum and the global maximum values that are attained by

$$f(x) = x^2 \cdot e^{4-2x}, \qquad -1 \leq x \leq 2.$$

7. Find the point on the graph of $y = 2\sqrt{x}$ which lies closest to the point $(2, 1)$.

8. If a right triangle has a hypotenuse of length $a > 0$, how large can its perimeter be?

9. Two cars are driving in opposite directions along two parallel roads which are 300m apart. If one is driving at 50 m/sec and the other is driving at 30 m/sec, how fast is the distance between them changing 5 seconds after they pass one another?

10. Show that $f(x) = x^4 + 5x - 1$ has a unique root in $(0, 1)$ and use Newton's method with initial guess $x_1 = 0$ to approximate this root within two decimal places.

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Tutorial problems #7

1. Find the area of the region enclosed by the graphs of $f(x) = 3x^2$ and $g(x) = x + 4$.

2. Compute the volume of the solid that is obtained when the graph of $f(x) = x^2 + 3$ is rotated around the x-axis over the interval $[0, 2]$.

3. Compute the length of the graph of $f(x) = \frac{1}{3}(x^2 + 2)^{3/2}$ over the interval $[1, 3]$.

4. Find both the mass and the centre of mass for a thin rod whose density is given by

$$\delta(x) = x^2 + 4x + 1, \qquad 0 \leq x \leq 2.$$

5. A chain that is 4m long has a uniform density of 3kg/m. If the chain is hanging from the top of a tall building, then how much work is needed to pull it up to the top?

6. Find the area of the region enclosed by the graphs of $f(x)$ and $g(x)$ in the case that

$$f(x) = \sin x, \qquad g(x) = \cos x, \qquad 0 \leq x \leq \pi/2.$$

7. The graph of $f(x) = 2e^{6x}$ is rotated around the x-axis over the interval $[0, a]$. If the volume of the resulting solid is equal to π, then what is the value of a?

8. Compute the length of the graph of $f(x) = x^{3/2} - \frac{1}{3}x^{1/2}$ over the interval $[0, 2]$.

9. Show that the function f is integrable on $[0, 1]$ for any given constants a, b when

$$f(x) = \left\{ \begin{array}{ll} a & \text{if } x \neq 0 \\ b & \text{if } x = 0 \end{array} \right\}.$$

10. Compute each of the following improper integrals.

$$I_1 = \int_2^\infty \frac{dx}{(x-1)^5}, \qquad I_2 = \int_2^3 \frac{dx}{\sqrt[4]{x-2}}, \qquad I_3 = \int_0^\infty \frac{dx}{x^2+1}.$$

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Tutorial problems #8

1. Compute each of the following indefinite integrals.

$$\int \frac{x^2}{x^3+1}\,dx, \qquad \int \frac{x^2}{x+1}\,dx.$$

2. Compute each of the following indefinite integrals.

$$\int \sin^2 x \cdot \cos^3 x\,dx, \qquad \int \sec^5 x \cdot \tan x\,dx.$$

3. Find the volume of the solid that is obtained by rotating the graph of $f(x) = \tan x$ around the x-axis over the interval $[0, \pi/4]$.

4. Compute each of the following indefinite integrals.

$$\int \frac{x^3-x}{x^2+5}\,dx, \qquad \int \frac{x^2+5}{x^3-x}\,dx.$$

5. Compute each of the following indefinite integrals.

$$\int \sin^{-1} x\,dx, \qquad \int e^{\sqrt{x}}\,dx.$$

6. Find the area of the region enclosed by the graphs of $f(x) = e^{2x}$ and $g(x) = 4e^x - 3$.

7. Compute each of the following indefinite integrals.

$$\int \frac{dx}{(1+x)\sqrt{x}}, \qquad \int x(\ln x)^2\,dx.$$

8. Compute each of the following indefinite integrals.

$$\int \frac{2\,dx}{(x^2+1)^2}, \qquad \int x^2\sqrt{1-x^2}\,dx.$$

9. Let $a > 0$ be given. Use integration by parts to find a reduction formula for

$$I_n = \int \frac{dx}{(x^2+a^2)^n}.$$

10. Use integration by parts to compute the indefinite integral

$$\int \sin(\ln x)\,dx.$$

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Tutorial problems #9

1. Compute each of the following indefinite integrals.

$$\int e^{2x}\cos(e^x)\,dx, \qquad \int \frac{\sin^3 x}{\cos^6 x}\,dx.$$

2. Compute each of the following indefinite integrals.

$$\int \frac{\sqrt{x}}{x+1}\,dx, \qquad \int \frac{\sqrt{x}}{x-1}\,dx.$$

3. Show that each of the following sequences converges.

$$a_n = \sqrt{\frac{n^2+1}{n^3+2}}, \qquad b_n = \frac{\sin n}{n^2}, \qquad c_n = n^{1/n}.$$

4. Define a sequence $\{a_n\}$ by setting $a_1 = 1$ and $a_{n+1} = \sqrt{6+a_n}$ for each $n \geq 1$. Show that $1 \leq a_n \leq a_{n+1} \leq 3$ for each $n \geq 1$, use this fact to conclude that the sequence converges and then find its limit.

5. Use the formula for a geometric series to compute each of the following sums.

$$\sum_{n=0}^{\infty} \frac{2^n}{7^n}, \qquad \sum_{n=1}^{\infty} \frac{3^{n+2}}{2^{3n+1}}, \qquad \sum_{n=2}^{\infty} \frac{3^{n+1}}{4^{n+2}}.$$

6. Compute each of the following indefinite integrals.

$$\int \frac{2x+3}{x^2-4x+3}\,dx, \qquad \int \frac{2x+3}{x^2-4x+5}\,dx.$$

7. Compute each of the following indefinite integrals.

$$\int \sqrt{1-x^2}\,dx, \qquad \int \frac{\sqrt{1-x}}{\sqrt{1+x}}\,dx.$$

8. Define a sequence $\{a_n\}$ by setting $a_1 = 1$ and $a_{n+1} = 3 + \sqrt{a_n}$ for each $n \geq 1$. Show that $1 \leq a_n \leq a_{n+1} \leq 9$ for each $n \geq 1$, use this fact to conclude that the sequence converges and then find its limit.

9. An ant starts out at the origin in the xy-plane and walks 1 unit south, then 1/2 units east, then 1/4 units north, then 1/8 units west, then 1/16 units south, and so on. If it continues like that indefinitely, which point in the xy-plane will it eventually reach?

10. Suppose the series $\sum_{n=1}^{\infty} a_n$ converges. Show that the series $\sum_{n=1}^{\infty} \frac{1}{1+a_n}$ diverges.

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Tutorial problems #10

1. Test each of the following series for convergence.

$$\sum_{n=1}^{\infty} \frac{2+\sin n}{n}, \qquad \sum_{n=1}^{\infty} \frac{2+\sin n}{n^2}.$$

2. Test each of the following series for convergence.

$$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}, \qquad \sum_{n=1}^{\infty} \frac{ne^{1/n}}{n^3+1}.$$

3. Test each of the following series for convergence.

$$\sum_{n=1}^{\infty} \frac{\ln n}{n}, \qquad \sum_{n=1}^{\infty} \frac{\ln n}{n!}.$$

4. Find the radius of convergence for each of the following power series.

$$\sum_{n=0}^{\infty} \frac{nx^n}{3^n}, \qquad \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \cdot x^n.$$

5. Assuming that $|x| < 1$, use the formula for a geometric series to show that

$$\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

6. Find the radius of convergence for each of the following power series.

$$\sum_{n=0}^{\infty} \frac{nx^{2n}}{4^n}, \qquad \sum_{n=0}^{\infty} \frac{3^n x^n}{2n+1}.$$

7. Use differentiation to show that the following power series is equal to $\ln(1+x)$.

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}, \qquad |x| < 1.$$

8. Use differentiation to show that the following power series is equal to $\tan^{-1} x$.

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad |x| < 1.$$

9. Let $a \in \mathbb{R}$ be a given number. Find the radius of convergence for the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{a(a-1)(a-2)\cdots(a-n+1)}{n!} \cdot x^n.$$

10. Show that $\sum_{n=0}^{\infty} a_n y^n$ converges absolutely, if $\sum_{n=0}^{\infty} a_n x^n$ converges and $|y| < |x|$.