MAU11201 – Calculus
Homework #1 solutions

1. Find the domain and the range of the function f which is defined by

$$f(x) = \frac{2-3x}{7-2x}.$$

The domain consists of all points $x \neq 7/2$. To find the range, we note that

$$y = \frac{2-3x}{7-2x} \iff 7y - 2xy = 2 - 3x \iff 3x - 2xy = 2 - 7y$$
$$\iff x(3-2y) = 2 - 7y \iff x = \frac{2-7y}{3-2y}.$$

The rightmost formula determines the value of x that satisfies $y = f(x)$. Since the formula makes sense for any number $y \neq 3/2$, the range consists of all numbers $y \neq 3/2$.

2. Show that the function $f\colon (0,1) \to (0,2)$ is bijective in the case that

$$f(x) = \frac{4x}{3-x}.$$

To show that the given function is injective, we note that

$$\frac{4x_1}{3-x_1} = \frac{4x_2}{3-x_2} \implies 12x_1 - 4x_1x_2 = 12x_2 - 4x_1x_2$$
$$\implies 12x_1 = 12x_2 \implies x_1 = x_2.$$

To show that the given function is surjective, we note that

$$y = \frac{4x}{3-x} \iff 3y - xy = 4x \iff 3y = x(y+4) \iff x = \frac{3y}{y+4}.$$

The rightmost equation determines the value of x such that $y = f(x)$ and we need to check that $0 < x < 1$ if and only if $0 < y < 2$. When $0 < y < 2$, we have $x = \frac{3y}{y+4} > 0$ and also

$$1 - x = 1 - \frac{3y}{y+4} = \frac{y+4-3y}{y+4} = \frac{4-2y}{y+4} = \frac{2(2-y)}{y+4} > 0,$$

so $0 < x < 1$. Conversely, suppose that $0 < x < 1$. Then $y = \frac{4x}{3-x} > 0$ and also

$$2 - y = 2 - \frac{4x}{3-x} = \frac{6-2x-4x}{3-x} = \frac{6(1-x)}{3-x} > 0 \implies 0 < y < 2.$$

3. Find the domain and the range of the function f which is defined by

$$f(x) = \sqrt{4 - \sqrt{x}}.$$

The domain consists of all numbers x with $x \geq 0$ and $4 - \sqrt{x} \geq 0$. This gives $\sqrt{x} \leq 4$ and also $x \leq 16$, so the domain is $[0, 16]$. To find the range, we note that

$$y = \sqrt{4 - \sqrt{x}} \quad \Longrightarrow \quad y^2 = 4 - \sqrt{x} \quad \Longrightarrow \quad \sqrt{x} = 4 - y^2 \quad \Longrightarrow \quad x = (4 - y^2)^2.$$

Note that the first equation implies $y \geq 0$, while the third one implies $4 - y^2 \geq 0$. These restrictions should be observed before squaring the equations. The range is thus $[0, 2]$.

4. Express the following polynomials as the product of linear factors.

$$f(x) = 3x^3 + 4x^2 - 5x - 2, \qquad g(x) = x^3 - \frac{7x^2}{6} + \frac{1}{6}.$$

When it comes to $f(x)$, the possible rational roots are $\pm 1, \pm 2, \pm 1/3, \pm 2/3$. Checking these possibilities, one finds that $x = 1$, $x = -2$ and $x = -1/3$ are all roots. According to the factor theorem, each of $x - 1$, $x + 2$ and $x + 1/3$ is thus a factor and one has

$$f(x) = 3(x - 1)(x + 2)(x + 1/3) = (x - 1)(x + 2)(3x + 1).$$

When it comes to $g(x)$, let us first clear denominators and write

$$6g(x) = 6x^3 - 7x^2 + 1.$$

The only possible rational roots are $\pm 1, \pm 1/2, \pm 1/3, \pm 1/6$. Checking these possibilities, one finds that $x = 1$, $x = 1/2$ and $x = -1/3$ are all roots. It easily follows that

$$6g(x) = 6(x - 1)(x - 1/2)(x + 1/3) \quad \Longrightarrow \quad g(x) = (x - 1)(x - 1/2)(x + 1/3).$$

5. Determine all angles $0 \leq \theta \leq 2\pi$ such that $4\sin^2\theta + 4\sin\theta = 3$.

Letting $x = \sin\theta$ for convenience, one finds that $4x^2 + 4x - 3 = 0$ and

$$x = \frac{-4 \pm \sqrt{16 + 4 \cdot 12}}{8} = \frac{-4 \pm 8}{8} \quad \Longrightarrow \quad x = \frac{1}{2}, -\frac{3}{2}.$$

Since $x = \sin\theta$ must lie between -1 and 1, the only relevant solution is $x = \sin\theta = \frac{1}{2}$. In view of the graph of the sine function, there should be two angles $0 \leq \theta \leq 2\pi$ that satisfy this condition. The first one is $\theta_1 = \frac{\pi}{6}$ and the second one is $\theta_2 = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

MAU11201 – Calculus
Homework #2 solutions

1. Determine the inverse function f^{-1} in each of the following cases.

$$f(x) = \frac{1}{3}\log_2(2x-6) - 1, \qquad f(x) = \frac{7\cdot 5^x - 3}{4\cdot 5^x + 2}.$$

When it comes to the first case, one can easily check that

$$3(y+1) = \log_2(2x-6) \quad\Longleftrightarrow\quad 2^{3y+3} = 2x-6 \quad\Longleftrightarrow\quad 2^{3y+2} = x-3,$$

so the inverse function is defined by $f^{-1}(y) = 2^{3y+2}+3$. When it comes to the second case,

$$y = \frac{7\cdot 5^x - 3}{4\cdot 5^x + 2} \quad\Longleftrightarrow\quad 4y\cdot 5^x + 2y = 7\cdot 5^x - 3 \quad\Longleftrightarrow\quad 2y+3 = (7-4y)\cdot 5^x$$

and this gives $5^x = \frac{2y+3}{7-4y}$, so the inverse function is defined by $f^{-1}(y) = \log_5 \frac{2y+3}{7-4y}$.

2. Simplify each of the following expressions.

$$\cos\left(\sin^{-1} x\right), \qquad \cos\left(\tan^{-1} x\right), \qquad \log_3(54) - 3\log_3(18) + \log_3(36).$$

To simplify the first expression, let $\theta = \sin^{-1} x$ and note that $\sin\theta = x$. When $x \geq 0$, the angle θ arises in a right triangle with an opposite side of length x and a hypotenuse of length 1. It follows by Pythagoras' theorem that the adjacent side has length $\sqrt{1-x^2}$, so the definition of cosine gives

$$\cos(\sin^{-1} x) = \cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \sqrt{1-x^2}.$$

When $x \leq 0$, the last equation holds with $-x$ instead of x. This changes the term $\sin^{-1} x$ by a minus sign, but the cosine remains unchanged, so the equation is still valid.

To simplify the second expression, let $\theta = \tan^{-1} x$ and note that $\tan\theta = x$. When $x \geq 0$, the angle θ arises in a right triangle with an opposite side of length x and an adjacent side of length 1. It follows by Pythagoras' theorem that the hypotenuse has length $\sqrt{1+x^2}$, so the definition of cosine gives

$$\cos(\tan^{-1} x) = \cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{1}{\sqrt{1+x^2}}.$$

When $x \leq 0$, the last equation holds with $-x$ instead of x. This changes the term $\tan^{-1} x$ by a minus sign, but the cosine remains unchanged, so the equation is still valid.

As for the third expression, one may use the properties of logarithms to get

$$\log_3(54) - 3\log_3(18) + \log_3(36) = \log_3 \frac{54\cdot 36}{18^3} = \log_3 \frac{3\cdot 2}{18} = \log_3 3^{-1} = -1.$$

3. Use the ε-δ definition of limits to compute $\lim_{x\to 3} f(x)$ in the case that

$$f(x) = \left\{ \begin{array}{ll} 3x-4 & \text{if } x \leq 3 \\ 4x-7 & \text{if } x > 3 \end{array} \right\}.$$

In this case, x is approaching 3 and $f(x)$ is either $3x-4$ or $4x-7$. We thus expect the limit to be $L = 5$. To prove this formally, we let $\varepsilon > 0$ and estimate the expression

$$|f(x) - 5| = \left\{ \begin{array}{ll} |3x-9| & \text{if } x \leq 3 \\ |4x-12| & \text{if } x > 3 \end{array} \right\} = \left\{ \begin{array}{ll} 3|x-3| & \text{if } x \leq 3 \\ 4|x-3| & \text{if } x > 3 \end{array} \right\}.$$

If we assume that $0 \neq |x-3| < \delta$, then we may use the last equation to get

$$|f(x) - 5| \leq 4|x-3| < 4\delta.$$

Since our goal is to show that $|f(x) - 5| < \varepsilon$, an appropriate choice of δ is thus $\delta = \varepsilon/4$.

4. Compute each of the following limits.

$$L = \lim_{x\to 1} \frac{3x^3 - 7x^2 + 6x - 2}{x-1}, \qquad M = \lim_{x\to 2} \frac{2x^3 - 7x^2 + 4x + 4}{(x-2)^2}.$$

When it comes to the first limit, division of polynomials gives

$$L = \lim_{x\to 1} \frac{3x^3 - 7x^2 + 6x - 2}{x-1} = \lim_{x\to 1} (3x^2 - 4x + 2) = 3 - 4 + 2 = 1.$$

When it comes to the second limit, division of polynomials gives

$$M = \lim_{x\to 2} \frac{2x^3 - 7x^2 + 4x + 4}{x^2 - 4x + 4} = \lim_{x\to 2} (2x + 1) = 4 + 1 = 5.$$

5. Use the ε-δ definition of limits to compute $\lim_{x\to 2} (3x^2 - 4x + 7)$.

Let $f(x) = 3x^2 - 4x + 7$ for convenience. Then $f(2) = 11$ and one has

$$|f(x) - f(2)| = |3x^2 - 4x - 4| = |x-2| \cdot |3x+2|.$$

The factor $|x-2|$ is related to our usual assumption that $0 \neq |x-2| < \delta$. To estimate the remaining factor $|3x+2|$, we assume that $\delta \leq 1$ for simplicity and we note that

$$\begin{aligned} |x-2| < \delta \leq 1 \quad &\Longrightarrow \quad -1 < x - 2 < 1 \\ &\Longrightarrow \quad 1 < x < 3 \quad \Longrightarrow \quad 5 < 3x + 2 < 11. \end{aligned}$$

Combining the estimates $|x-2| < \delta$ and $|3x+2| < 11$, one may then conclude that

$$|f(x) - f(2)| = |x-2| \cdot |3x+2| < 11\delta \leq \varepsilon,$$

as long as $\delta \leq \varepsilon/11$ and $\delta \leq 1$. An appropriate choice of δ is thus $\delta = \min(\varepsilon/11, 1)$.

MAU11201 – Calculus
Homework #3 solutions

1. Show that there exists a real number $0 < x < \pi/2$ that satisfies the equation

$$x \sin x + x \cos x = 1.$$

Consider the function f which is defined as the difference of the two sides, namely

$$f(x) = x \sin x + x \cos x - 1.$$

Being a composition of continuous functions, f is then continuous and we also have

$$f(0) = -1 < 0, \qquad f(\pi/2) = \frac{\pi}{2} - 1 = \frac{\pi - 2}{2} > 0.$$

In view of Bolzano's theorem, this already implies that f has a root $0 < x < \pi/2$.

2. For which values of a, b is the function f continuous at the point $x = 3$? Explain.

$$f(x) = \left\{ \begin{array}{cc} 4x^2 + ax + b & \text{if } x < 3 \\ a + b - 2 & \text{if } x = 3 \\ 2x^3 - bx + a & \text{if } x > 3 \end{array} \right\}.$$

Since f is a polynomial on the intervals $(-\infty, 3)$ and $(3, +\infty)$, one easily finds that

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (4x^2 + ax + b) = 36 + 3a + b,$$
$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (2x^3 - bx + a) = 54 - 3b + a.$$

In particular, the function f is continuous at the given point if and only if

$$36 + 3a + b = 54 - 3b + a = a + b - 2.$$

Solving this system of equations, we obtain a unique solution which is given by

$$54 - 3b = b - 2 \implies 4b = 56 \implies b = 14 \implies a = -19.$$

In other words, f is continuous at the given point if and only if $a = -19$ and $b = 14$.

3. Show that $f(x) = 2x^5 - 3x^3 - 5x + 1$ has three roots in the interval $(-2, 2)$. Hint: you need only consider the values that are attained by f at the points ± 2, ± 1 and 0.

Being a polynomial, the given function is continuous and one can easily check that

$$f(-2) = -29, \qquad f(-1) = 7, \qquad f(0) = 1, \qquad f(1) = -5, \qquad f(2) = 31.$$

Since the values $f(-2)$ and $f(-1)$ have opposite signs, f has a root that lies in $(-2, -1)$. The same argument yields a second root in $(0, 1)$ and also a third root in $(1, 2)$.

4. Compute each of the following limits.

$$L = \lim_{x\to+\infty} \frac{2x^4 - 4x^2 + 5}{3x^4 - 7x + 2}, \qquad M = \lim_{x\to 3^-} \frac{x^3 - 5x + 4}{x^3 - 8x - 3}.$$

Since the first limit involves infinite values of x, it should be clear that

$$L = \lim_{x\to+\infty} \frac{2x^4 - 4x^2 + 5}{3x^4 - 7x + 2} = \lim_{x\to+\infty} \frac{2x^4}{3x^4} = \frac{2}{3}.$$

For the second limit, the denominator becomes zero when $x = 3$, while the numerator is nonzero at that point. Thus, one needs to factor the denominator and this gives

$$M = \lim_{x\to 3^-} \frac{x^3 - 5x + 4}{(x-3)(x^2 + 3x + 1)} = \lim_{x\to 3^-} \frac{16}{19(x-3)} = -\infty.$$

5. Use the definition of the derivative to compute $f'(x_0)$ in each of the following cases.

$$f(x) = (3x+1)^2, \qquad f(x) = (x^2 - 1)^2.$$

The derivative of the first function is given by the limit

$$\begin{aligned} f'(x_0) &= \lim_{x\to x_0} \frac{(3x+1)^2 - (3x_0+1)^2}{x - x_0} = \lim_{x\to x_0} \frac{(3x - 3x_0)(3x + 3x_0 + 2)}{x - x_0} \\ &= \lim_{x\to x_0} 3(3x + 3x_0 + 2) = 3(6x_0 + 2) = 6(3x_0 + 1). \end{aligned}$$

The derivative of the second function is given by the limit

$$\begin{aligned} f'(x_0) &= \lim_{x\to x_0} \frac{(x^2-1)^2 - (x_0^2-1)^2}{x - x_0} = \lim_{x\to x_0} \frac{(x^2 - x_0^2)(x^2 + x_0^2 - 2)}{x - x_0} \\ &= \lim_{x\to x_0} (x + x_0)(x^2 + x_0^2 - 2) = 2x_0(2x_0^2 - 2) = 4x_0(x_0^2 - 1). \end{aligned}$$

MAU11201 – Calculus
Homework #4 solutions

1. Compute the derivative $y' = \frac{dy}{dx}$ in each of the following cases.

$$y = \ln(\tan x) + 2(\sec x)^5, \qquad y = \tan^{-1}(\sin(2x)).$$

When it comes to the first function, one may use the chain rule to get

$$\begin{aligned} y' &= \frac{1}{\tan x} \cdot (\tan x)' + 10(\sec x)^4 \cdot (\sec x)' \\ &= \frac{1}{\tan x} \cdot \sec^2 x + 10\sec^4 x \cdot \sec x \tan x = \frac{\sec^2 x}{\tan x} + 10\sec^5 x \cdot \tan x. \end{aligned}$$

When it comes to the second function, one similarly finds that

$$y' = \frac{1}{\sin^2(2x) + 1} \cdot \sin(2x)' = \frac{2\cos(2x)}{\sin^2(2x) + 1}.$$

2. Compute the derivative $y' = \frac{dy}{dx}$ in the case that $y^2 \cos x + x^3 e^y = x^2 y^3$.

Differentiating both sides of the given equation, one finds that

$$2yy' \cos x - y^2 \sin x + 3x^2 e^y + x^3 e^y y' = 2xy^3 + 3x^2 y^2 y'.$$

We now collect the terms that contain y' on the left hand side and we get

$$(2y \cos x + x^3 e^y - 3x^2 y^2) y' = 2xy^3 + y^2 \sin x - 3x^2 e^y.$$

Solving this equation for y', one may thus conclude that

$$y' = \frac{2xy^3 + y^2 \sin x - 3x^2 e^y}{2y \cos x + x^3 e^y - 3x^2 y^2}.$$

3. Compute the derivative $f'(x_0)$ in the case that

$$f(x) = \frac{(x^3+2)^3 \cdot e^{4x} \cdot \cos(5\tan x)}{\sqrt{x^3+1}}, \qquad x_0 = 0.$$

First, we use logarithmic differentiation to determine $f'(x)$. In this case, we have

$$\begin{aligned}\ln|f(x)| &= \ln|x^3+2|^3 + \ln e^{4x} + \ln|\cos(5\tan x)| + \ln|x^3+1|^{-1/2} \\ &= 3\ln|x^3+2| + 4x + \ln|\cos(5\tan x)| - \frac{1}{2}\ln|x^3+1|.\end{aligned}$$

Differentiating both sides of this equation, one easily finds that

$$\frac{f'(x)}{f(x)} = \frac{3\cdot 3x^2}{x^3+2} + 4 - \frac{\sin(5\tan x)\cdot 5\sec^2 x}{\cos(5\tan x)} - \frac{3x^2}{2(x^3+1)}.$$

To compute the derivative $f'(0)$, one may then substitute $x=0$ to conclude that

$$\frac{f'(0)}{f(0)} = 0+4-0-0 = 4 \quad\Longrightarrow\quad f'(0) = 4f(0) = 32.$$

4. Show that the derivative of the inverse tangent function is given by

$$\left(\tan^{-1} x\right)' = \frac{1}{1+x^2}.$$

Using Theorem 3.19 with $f(x)=\tan x$ and $g(x)=\tan^{-1} x$, one finds that

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\sec^2(g(x))} = \cos^2 g(x) = \cos^2(\tan^{-1} x).$$

Let $\theta = \tan^{-1} x$ for simplicity and note that $\tan\theta = x$. When $x\geq 0$, the angle θ arises in a right triangle with an opposite side of length x and an adjacent side of length 1. It follows by Pythagoras' theorem that the hypotenuse has length $\sqrt{1+x^2}$, so

$$g'(x) = \cos^2(\tan^{-1} x) = \cos^2\theta = \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \frac{1}{1+x^2}.$$

When $x\leq 0$, the last equation holds with $-x$ instead of x. This changes the term $\tan^{-1} x$ by a minus sign, but the cosine remains unchanged, so the equation is still valid.

5. Compute the derivative $f'(2)$ in the case that $x^2e^{f(x)} + 3xe^{2f(x)} = 2$ for all x.

Let us write $x^2e^y + 3xe^{2y} = 2$ for simplicity. Differentiating both sides, we get

$$2xe^y + x^2e^yy' + 3e^{2y} + 3xe^{2y}\cdot 2y' = 0.$$

We now collect the terms that contain y' on the left hand side and we get

$$(x^2e^y + 6xe^{2y})y' = -2xe^y - 3e^{2y} \quad\Longrightarrow\quad y' = -\frac{2xe^y+3e^{2y}}{x^2e^y+6xe^{2y}} = -\frac{2x+3e^y}{x^2+6xe^y}.$$

To determine the value of y that corresponds to $x = 2$, we note that

$$x^2 e^y + 3xe^{2y} = 2 \quad \Longrightarrow \quad 4e^y + 6e^{2y} = 2 \quad \Longrightarrow \quad 3e^{2y} + 2e^y - 1 = 0.$$

Let $z = e^y$ for convenience. Then $3z^2 + 2z - 1 = 0$ and the quadratic formula gives

$$z = \frac{-2 \pm \sqrt{4 + 4 \cdot 3}}{6} = \frac{-2 \pm 4}{6} = \frac{1}{3}, -1.$$

Since $z = e^y$ must be positive, the only acceptable solution is $z = e^y = 1/3$ and so

$$y' = -\frac{2x + 3e^y}{x^2 + 6xe^y} = -\frac{2x + 1}{x^2 + 2x} = -\frac{5}{8}.$$

MAU11201 – Calculus
Homework #5 solutions

1. Show that $f(x) = 2x^3 - 3x^2 - 4x + 1$ has exactly one root in $(0, 1)$.

Being a polynomial, f is continuous on the interval $[0, 1]$ and we also have

$$f(0) = 1, \qquad f(1) = 2 - 3 - 4 + 1 = -4.$$

Since $f(0)$ and $f(1)$ have opposite signs, f must have a root that lies in $(0, 1)$. To show it is unique, suppose that f has two roots in $(0, 1)$. Then f' must have a root in this interval by Rolle's theorem. On the other hand, it is easy to check that

$$f'(x) = 0 \quad \Longrightarrow \quad 6x^2 - 6x - 4 = 0 \quad \Longrightarrow \quad x = \frac{3 \pm \sqrt{33}}{6}.$$

Since f' has no roots in $(0, 1)$, we conclude that f has exactly one root in $(0, 1)$.

2. Compute each of the following limits.

$$L_1 = \lim_{x\to 2} \frac{3x^2 - 5x - 2}{2x^2 - 7x + 6}, \qquad L_2 = \lim_{x\to\infty} \frac{(\ln x)^2}{x}, \qquad L_3 = \lim_{x\to 0^+} (e^{3x} + \sin x)^{2/x}.$$

The first limit has the form $0/0$, so one may use L'Hôpital's rule to find that

$$L_1 = \lim_{x\to 2} \frac{6x - 5}{4x - 7} = \frac{12 - 5}{8 - 7} = 7.$$

The second limit has the form ∞/∞ and one may apply L'Hôpital's rule to get

$$L_2 = \lim_{x\to\infty} \frac{2(\ln x) \cdot 1/x}{1} = \lim_{x\to\infty} \frac{2\ln x}{x}.$$

This is still a limit of the form ∞/∞ and another application of L'Hôpital's rule gives

$$L_2 = \lim_{x\to\infty} \frac{2/x}{1} = \lim_{x\to\infty} \frac{2}{x} = 0.$$

The third limit involves a non-constant exponent which can be eliminated by writing

$$\ln L_3 = \ln \lim_{x\to 0^+} (e^{3x} + \sin x)^{2/x} = \lim_{x\to 0^+} \ln(e^{3x} + \sin x)^{2/x} = \lim_{x\to 0^+} \frac{2\ln(e^{3x} + \sin x)}{x}.$$

This gives a limit of the form $0/0$, so one may use L'Hôpital's rule to find that

$$\ln L_3 = \lim_{x\to 0^+} \frac{2(e^{3x} + \sin x)^{-1} \cdot (3e^{3x} + \cos x)}{1} = \frac{2(3 + 1)}{1 + 0} = 8.$$

Since $\ln L_3 = 8$, the original limit L_3 is then equal to $L_3 = e^{\ln L_3} = e^8$.

3. On which intervals is f increasing? On which intervals is it concave up?

$$f(x) = \ln(4x^2 + 1).$$

To say that $f(x)$ is increasing is to say that $f'(x) > 0$. Let us then compute

$$f'(x) = \frac{1}{4x^2+1} \cdot (4x^2+1)' = \frac{8x}{4x^2+1}.$$

Since the denominator is always positive, $f(x)$ is increasing if and only if $x > 0$. Next, we look at concavity. To say that $f(x)$ is concave up is to say that $f''(x) > 0$. In this case,

$$f''(x) = \frac{8(4x^2+1) - 8x \cdot 8x}{(4x^2+1)^2} = \frac{8(4x^2+1-8x^2)}{(4x^2+1)^2} = \frac{8(1+2x)(1-2x)}{(4x^2+1)^2}.$$

To determine the sign of this expression, one needs to find the sign of each of the factors. According to the table below, $f(x)$ is concave up if and only if $x \in (-1/2, 1/2)$.

		$-1/2$	$1/2$
$8(1+2x)$	$-$	$+$	$+$
$1-2x$	$+$	$+$	$-$
$f''(x)$	$-$	$+$	$-$

4. Find the intervals on which f is increasing/decreasing and the intervals on which f is concave up/down. Use this information to sketch the graph of f.

$$f(x) = \frac{x}{x^2+1}.$$

To say that $f(x)$ is increasing is to say that $f'(x) > 0$. Let us then compute

$$f'(x) = \frac{x^2+1-2x \cdot x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}.$$

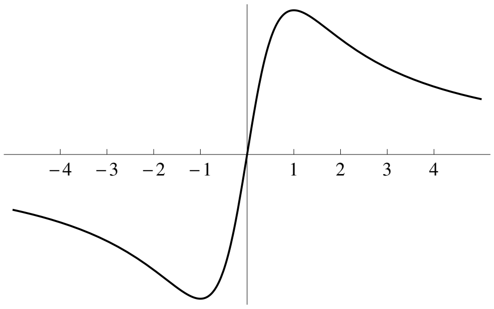
Since the denominator is always positive, $f(x)$ is increasing if and only if

$$1 - x^2 > 0 \quad \Longleftrightarrow \quad x^2 < 1 \quad \Longleftrightarrow \quad -1 < x < 1.$$

To say that $f(x)$ is concave up is to say that $f''(x) > 0$. In this case, we have

$$\begin{aligned} f''(x) &= \frac{-2x(x^2+1)^2 - 2(x^2+1) \cdot 2x(1-x^2)}{(x^2+1)^4} \\ &= \frac{-2x(x^2+1) - 4x(1-x^2)}{(x^2+1)^3} = -\frac{2x(x^2+1+2-2x^2)}{(x^2+1)^3} \\ &= -\frac{2x(3-x^2)}{(x^2+1)^3} = \frac{2x(x-\sqrt{3})(x+\sqrt{3})}{(x^2+1)^3}. \end{aligned}$$

To determine the sign of this expression, one needs to find the sign of each of the factors. According to the table below, $f(x)$ is concave up if and only if $x \in (-\sqrt{3}, 0) \cup (\sqrt{3}, +\infty)$.



		$-\sqrt{3}$		0		$\sqrt{3}$	
$2x$	$-$		$-$		$+$		$+$
$x-\sqrt{3}$	$-$		$-$		$-$		$+$
$x+\sqrt{3}$	$-$		$+$		$+$		$+$
$f''(x)$	$-$		$+$		$-$		$+$

Figure 1: The graph of $f(x) = \dfrac{x}{x^2+1}$.

5. Show that the cubic polynomial $f(x) = x^3 + ax^2 + bx + c$ has a unique real root for any given constants a, b, c such that $a^2 < 3b$.

Since f is a polynomial, it is certainly continuous and we also have

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x^3 = -\infty, \qquad \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x^3 = +\infty.$$

In view of the intermediate value theorem, f must then attain all values, so it must have a real root. Suppose that f has two roots $x_1 < x_2$. Then f' must have a root in (x_1, x_2) by Rolle's theorem. On the other hand, it is easy to check that

$$f'(x) = 3x^2 + 2ax + b$$

is a quadratic whose discriminant $\Delta = (2a)^2 - 4\cdot 3b = 4(a^2 - 3b)$ is negative. Thus, f' does not have any real roots and this means that f has a unique real root.

MAU11201 – Calculus
Homework #6 solutions

1. Find the global minimum and the global maximum values that are attained by

$$f(x) = 4x^3 + x^2 - 2x - 1, \qquad 0 \leq x \leq 1.$$

The derivative of the given function can be expressed in the form

$$f'(x) = 12x^2 + 2x - 2 = 2(6x^2 + x - 1) = 2(3x - 1)(2x + 1).$$

Thus, the only points at which the minimum/maximum value may occur are the points

$$x = 0, \qquad x = 1, \qquad x = 1/3, \qquad x = -1/2.$$

We exclude the rightmost point, as it does not lie in the given interval, and we compute

$$f(0) = -1, \qquad f(1) = 4 + 1 - 2 - 1 = 2, \qquad f(1/3) = \frac{4}{27} + \frac{1}{9} - \frac{2}{3} - 1 = -\frac{38}{27}.$$

This means that the minimum is $f(1/3) = -38/27$ and the maximum is $f(1) = 2$.

2. Find the linear approximation to the function f at the point x_0 in the case that

$$f(x) = \frac{3x^4 - 4x + 2}{x^2 + 3x + 1}, \qquad x_0 = 0.$$

To find the derivative of $f(x)$ at the given point, we use the quotient rule to get

$$f'(x) = \frac{(12x^3 - 4)(x^2 + 3x + 1) - (2x + 3)(3x^4 - 4x + 2)}{(x^2 + 3x + 1)^2} \quad \Longrightarrow \quad f'(0) = \frac{-4 - 6}{1^2} = -10.$$

Since $f(0) = 2$, the linear approximation is thus $L(x) = -10x + 2$.

3. Show that $f(x) = x^3 - 4x^2 + 1$ has exactly two roots in $(-1, 1)$ and use Newton's method with $x_1 = \pm 1$ to approximate these roots within two decimal places.

To prove existence using Bolzano's theorem, we note that f is continuous with

$$f(-1) = -1 - 4 + 1 < 0, \qquad f(0) = 1 > 0, \qquad f(1) = 1 - 4 + 1 < 0.$$

In view of Bolzano's theorem, f must then have a root in $(-1, 0)$ and another root in $(0, 1)$, so it has two roots in $(-1, 1)$. Suppose that it has three roots in $(-1, 1)$. Then f' must have two roots in this interval by Rolle's theorem. On the other hand,

$$f'(x) = 3x^2 - 8x = x(3x - 8)$$

has only one root in $(-1, 1)$. This implies that f can only have two roots in $(-1, 1)$.

To use Newton's method to approximate the roots, we repeatedly apply the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 4x_n^2 + 1}{3x_n^2 - 8x_n}.$$

Starting with the initial guess $x_1 = -1$, one obtains the approximations

$$x_2 = -0.6364, \qquad x_3 = -0.4972, \qquad x_4 = -0.4735, \qquad x_5 = -0.4728.$$

Starting with the initial guess $x_1 = 1$, one obtains the approximations

$$x_2 = 0.6, \qquad x_3 = 0.5398, \qquad x_4 = 0.5374, \qquad x_5 = 0.5374.$$

This suggests that the two roots are roughly -0.47 and 0.53 within two decimal places.

4. A rectangle is inscribed in an equilateral triangle of side length $a > 0$ with one of its sides along the base of the triangle. How large can the area of the rectangle be?

Let x, y be the two sides of the rectangle and assume that x lies along the base of the triangle. Then one can relate the two sides x, y by noting that

$$\tan 60^\circ = \frac{y}{(a-x)/2} \implies \sqrt{3} = \frac{2y}{a-x} \implies y = \frac{\sqrt{3}}{2}(a-x).$$

We need to maximise the area A of the rectangle and this is given by

$$A(x) = xy = \frac{\sqrt{3}}{2}x(a-x) = \frac{\sqrt{3}}{2}(ax - x^2), \qquad 0 \leq x \leq a.$$

Since $A'(x) = \frac{\sqrt{3}}{2}(a - 2x)$, the only points at which the maximum value may occur are the points $x = 0$, $x = a$ and $x = \frac{a}{2}$. Since $A(0) = A(a) = 0$, the maximum is $A(\frac{a}{2}) = \frac{a^2\sqrt{3}}{8}$.

5. A ladder 5m long is resting against a vertical wall. The bottom of the ladder slides away from the wall at the rate of 0.2m/s. How fast is the angle θ between the ladder and the wall changing when the bottom of the ladder lies 3m away from the wall?

Let x be the horizontal distance between the base of the ladder and the wall, and let y be the vertical distance between the top of the ladder and the floor. We must then have

$$x(t)^2 + y(t)^2 = 5^2 \implies 2x(t)x'(t) + 2y(t)y'(t) = 0.$$

At the given moment, $x'(t) = 0.2 = 1/5$ and also $x(t) = 3$, so it easily follows that

$$y'(t) = -\frac{x(t)x'(t)}{y(t)} = -\frac{x(t)x'(t)}{\sqrt{5^2 - x(t)^2}} = -\frac{3/5}{\sqrt{5^2 - 3^2}} = -\frac{3}{20}.$$

We now need to determine θ'. Using the chain rule along with the quotient rule, we get

$$\tan\theta = \frac{x}{y} \quad\Longrightarrow\quad \sec^2\theta\cdot\theta' = \frac{x'y - y'x}{y^2} \quad\Longrightarrow\quad \theta' = \frac{x'y - y'x}{y^2}\cdot\cos^2\theta.$$

Since $\cos\theta = y/5$ and the other variables are already known, we may conclude that

$$\theta' = \frac{x'y - y'x}{y^2}\cdot\cos^2\theta = \frac{4(1/5) - 3(-3/20)}{4^2}\cdot\left(\frac{4}{5}\right)^2 = \frac{1}{20}.$$

MAU11201 – Calculus
Homework #7 solutions

1. Find the area of the region enclosed by the graphs of $f(x) = 3x^2$ and $g(x) = x + 2$.

The graph of the parabola $f(x) = 3x^2$ meets the graph of the line $g(x) = x + 2$ when

$$3x^2 = x + 2 \iff 3x^2 - x - 2 = 0 \iff (3x + 2)(x - 1) = 0.$$

Since the line lies above the parabola at the points $-2/3 \leq x \leq 1$, the area is then

$$\int_{-2/3}^{1} [g(x) - f(x)]\, dx = \int_{-2/3}^{1} \left[x + 2 - 3x^2\right] dx = \left[\frac{x^2}{2} + 2x - x^3\right]_{-2/3}^{1} = \frac{125}{54}.$$

2. Compute the volume of a sphere of radius $r > 0$. Hint: one may obtain such a sphere by rotating the upper semicircle $f(x) = \sqrt{r^2 - x^2}$ around the x-axis.

The volume of the sphere is the integral of $\pi f(x)^2$ and this is given by

$$\int_{-r}^{r} \pi(r^2 - x^2)\, dx = \pi \left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r} = \pi \left(\frac{2r^3}{3} + \frac{2r^3}{3}\right) = \frac{4\pi r^3}{3}.$$

3. Compute the length of the graph of $f(x) = \frac{x^4}{16} + \frac{1}{2x^2}$ over the interval $[1, 3]$.

The length of the graph is given by the integral of $\sqrt{1 + f'(x)^2}$. In this case,

$$f'(x) = \frac{4x^3}{16} - \frac{2}{2x^3} = \frac{x^3}{4} - \frac{1}{x^3} \implies f'(x)^2 = \frac{x^6}{16} + \frac{1}{x^6} - \frac{1}{2}$$

so the expression $1 + f'(x)^2$ can be written in the form

$$1 + f'(x)^2 = \frac{x^6}{16} + \frac{1}{x^6} + \frac{1}{2} = \left(\frac{x^3}{4} + \frac{1}{x^3}\right)^2.$$

Taking the square root of both sides, we conclude that the length of the graph is

$$\int_1^3 \sqrt{1 + f'(x)^2}\, dx = \int_1^3 \left(\frac{x^3}{4} + \frac{1}{x^3}\right) dx = \left[\frac{x^4}{16} - \frac{1}{2x^2}\right]_1^3 = \frac{49}{9}.$$

4. Find both the mass and the centre of mass for a thin rod whose density is given by

$$\delta(x) = x^2 + 2x + 3, \qquad 1 \leq x \leq 2.$$

The mass of the rod is merely the integral of its density function, namely

$$M = \int_1^2 \delta(x)\,dx = \int_1^2 (x^2 + 2x + 3)\,dx = \left[\frac{x^3}{3} + x^2 + 3x\right]_1^2 = \frac{25}{3}.$$

The centre of mass is given by a similar formula and one finds that

$$\overline{x} = \frac{1}{M}\int_1^2 x\delta(x)\,dx = \frac{3}{25}\int_1^2 (x^3 + 2x^2 + 3x)\,dx = \frac{3}{25}\left[\frac{x^4}{4} + \frac{2x^3}{3} + \frac{3x^2}{2}\right]_1^2 = \frac{31}{20}.$$

5. Use the definition of integrals and Riemann sums to compute the value of the limit

$$\lim_{n\to\infty}\left(\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \ldots + \frac{n}{n^2+n^2}\right).$$

First of all, we note that the given sum can be expressed in the form

$$\sum_{k=1}^n \frac{n}{n^2+k^2} = \sum_{k=1}^n \frac{n/n^2}{1+(k/n)^2} = \sum_{k=1}^n \frac{1}{n}\cdot f(k/n),$$

where $f(x) = \frac{1}{1+x^2}$. It is thus a Riemann sum for the function f on $[0,1]$. If we divide this interval into n equal parts and choose $x_k^* = k/n$ for each k, then we get

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^n f(x_k^*)\Delta x = \lim_{n\to\infty}\sum_{k=1}^n \frac{1}{n}\cdot f(k/n).$$

Once we now combine the last two equations, we may finally conclude that

$$\lim_{n\to\infty}\sum_{k=1}^n \frac{n}{n^2+k^2} = \lim_{n\to\infty}\sum_{k=1}^n \frac{1}{n}\cdot f(k/n) = \int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1 = \frac{\pi}{4}.$$

MAU11201 – Calculus
Homework #8 solutions

1. Compute each of the following indefinite integrals.

$$\int \cos\sqrt{x}\,dx, \qquad \int x^2\cdot\sqrt{x+1}\,dx.$$

For the first integral, we let $u=\sqrt{x}$. This gives $x=u^2$ and $dx=2u\,du$, so

$$\int \cos\sqrt{x}\,dx = \int 2u\cos u\,du.$$

We now integrate by parts using $dv=2\cos u\,du$. Since $v=2\sin u$, we find that

$$\begin{aligned}\int \cos\sqrt{x}\,dx = 2u\sin u - \int 2\sin u\,du &= 2u\sin u + 2\cos u + C\\ &= 2\sqrt{x}\cdot\sin\sqrt{x} + 2\cos\sqrt{x} + C.\end{aligned}$$

For the second integral, we let $u=x+1$. Then $du=dx$ and $x=u-1$, so

$$\begin{aligned}\int x^2\cdot\sqrt{x+1}\,dx &= \int (u-1)^2\sqrt{u}\,du = \int (u^2-2u+1)\sqrt{u}\,du\\ &= \int (u^{5/2}-2u^{3/2}+u^{1/2})\,du = \frac{2}{7}\,u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C\\ &= \frac{2}{7}\,(x+1)^{7/2} - \frac{4}{5}(x+1)^{5/2} + \frac{2}{3}(x+1)^{3/2} + C.\end{aligned}$$

2. Compute each of the following indefinite integrals.

$$\int \sin^3 x\cdot\cos^2 x\,dx, \qquad \int \tan^4 x\cdot\sec^6 x\,dx.$$

For the first integral, we use the substitution $u=\cos x$. Since $du=-\sin x\,dx$, we get

$$\begin{aligned}\int \sin^3 x\cdot\cos^2 x\,dx &= \int \cos^2 x\cdot(1-\cos^2 x)\cdot\sin x\,dx = -\int u^2(1-u^2)\,du\\ &= \int (u^4-u^2)\,du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.\end{aligned}$$

For the second integral, we use the substitution $u=\tan x$. Since $du=\sec^2 x\,dx$, we get

$$\begin{aligned}\int \tan^4 x\cdot\sec^6 x\,dx &= \int \tan^4 x\cdot(1+\tan^2 x)^2\cdot\sec^2 x\,dx = \int u^4(1+u^2)^2\,du\\ &= \int (u^4+2u^6+u^8)\,du = \frac{1}{5}u^5 + \frac{2}{7}u^7 + \frac{1}{9}u^9 + C\\ &= \frac{\tan^5 x}{5} + \frac{2\tan^7 x}{7} + \frac{\tan^9 x}{9} + C.\end{aligned}$$

3. Compute each of the following indefinite integrals.

$$\int \frac{x^2}{\sqrt{9-x}}\,dx, \qquad \int \frac{x^2}{\sqrt{9-x^2}}\,dx.$$

For the first integral, we let $u = 9 - x$. This gives $x = 9 - u$ and $dx = -du$, so

$$\begin{aligned}\int \frac{x^2}{\sqrt{9-x}}\,dx &= -\int \frac{(9-u)^2}{\sqrt{u}}\,du = \int \frac{18u - u^2 - 81}{u^{1/2}}\,du \\ &= \int \left(18u^{1/2} - u^{3/2} - 81u^{-1/2}\right)\,du = 12u^{3/2} - \frac{2}{5}u^{5/2} - 162u^{1/2} + C \\ &= 12(9-x)^{3/2} - \frac{2}{5}(9-x)^{5/2} - 162(9-x)^{1/2} + C.\end{aligned}$$

For the second integral, let $x = 3\sin\theta$ for some angle $-\pi/2 \leq \theta \leq \pi/2$. Then

$$\begin{aligned}\int \frac{x^2}{\sqrt{9-x^2}}\,dx &= \int \frac{9\sin^2\theta}{3\cos\theta} \cdot 3\cos\theta\,d\theta = \int 9\sin^2\theta\,d\theta = \frac{9}{2}\int (1 - \cos(2\theta))\,d\theta \\ &= \frac{9\theta}{2} - \frac{9\sin(2\theta)}{4} + C = \frac{9\theta}{2} - \frac{9\sin\theta\cos\theta}{2} + C.\end{aligned}$$

Since $\sin\theta = x/3$ by above, we also have $\cos\theta = \sqrt{1 - x^2/9}$ and this finally gives

$$\begin{aligned}\int \frac{x^2}{\sqrt{9-x^2}}\,dx &= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{9}{2}\frac{x}{3}\sqrt{1 - \frac{x^2}{9}} + C \\ &= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C.\end{aligned}$$

4. Compute each of the following indefinite integrals.

$$\int \frac{2x+1}{x^2-3x+2}\,dx, \qquad \int \frac{2+e^x}{3-e^x}\,dx.$$

When it comes to the first integral, one may use partial fractions to write

$$\frac{2x+1}{x^2-3x+2} = \frac{2x+1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

for some constants A and B. Clearing denominators gives rise to the identity

$$2x + 1 = A(x-2) + B(x-1)$$

and this should be valid for all x. Let us then look at some special values of x to get

$$x = 1, 2 \quad\Longrightarrow\quad 3 = -A, \qquad 5 = B.$$

This gives $A = -3$ and $B = 5$, so it easily follows that

$$\int \frac{2x+1}{x^2-3x+2}\,dx = \int \left(-\frac{3}{x-1} + \frac{5}{x-2}\right) dx = -3\ln|x-1| + 5\ln|x-2| + K.$$

When it comes to the second integral, we let $u = e^x$. This gives $du = e^x\,dx$ and so

$$\int \frac{2+e^x}{3-e^x}\,dx = \int \frac{2+e^x}{e^x(3-e^x)}\,e^x\,dx = \int \frac{2+u}{u(3-u)}\,du.$$

Proceeding as before, we use partial fractions to obtain a decomposition of the form

$$\frac{2+u}{u(3-u)} = \frac{A}{u} + \frac{B}{3-u} \quad\Longrightarrow\quad 2+u = A(3-u) + Bu.$$

Taking $u = 0$ gives $2 = 3A$ and taking $u = 3$ gives $5 = 3B$, so it easily follows that

$$\begin{aligned}\int \frac{2+u}{u(3-u)}\,du &= \int \left(\frac{2/3}{u} + \frac{5/3}{3-u}\right) du \\ &= \frac{2}{3}\ln|u| - \frac{5}{3}\ln|3-u| + K = \frac{2x}{3} - \frac{5}{3}\ln|3-e^x| + K.\end{aligned}$$

5. Find the volume of the solid that is obtained by rotating the graph of $f(x) = \sin x$ around the x-axis over the interval $[0, \pi]$.

The volume of the solid is the integral of $\pi f(x)^2$ and this is given by

$$\text{Volume} = \pi \int_0^\pi \sin^2 x\,dx = \frac{\pi}{2}\int_0^\pi (1-\cos(2x))\,dx = \frac{\pi}{2}\left[x - \frac{\sin(2x)}{2}\right]_0^\pi = \frac{\pi^2}{2}.$$

MAU11201 – Calculus
Homework #9 solutions

1. Compute each of the following indefinite integrals.

$$\int \frac{x^2-2x-3}{x^3-x^2}\,dx, \qquad \int \frac{x^3-x^2}{x^2-2x-3}\,dx.$$

When it comes to the first integral, one may use partial fractions to write

$$\frac{x^2-2x-3}{x^3-x^2} = \frac{x^2-2x-3}{x^2(x-1)} = \frac{Ax+B}{x^2} + \frac{C}{x-1}$$

for some constants A, B and C. Clearing denominators gives rise to the identity

$$x^2-2x-3 = (Ax+B)(x-1) + Cx^2$$

and this should be valid for all x. Let us then look at some special values of x to get

$$x=0,1,2 \quad\Longrightarrow\quad -3=-B, \qquad -4=C, \qquad -3=2A+B+4C.$$

This gives $B=3$, $C=-4$ and $2A=-3-3+16=10$, so it easily follows that

$$\begin{aligned}\int \frac{x^2-2x-3}{x^3-x^2}\,dx &= \int \left(\frac{5}{x}+\frac{3}{x^2}-\frac{4}{x-1}\right)dx \\ &= 5\ln|x| - \frac{3}{x} - 4\ln|x-1| + K.\end{aligned}$$

When it comes to the second integral, one may use division of polynomials to write

$$\frac{x^3-x^2}{x^2-2x-3} = x+1+\frac{5x+3}{x^2-2x-3} = x+1+\frac{5x+3}{(x+1)(x-3)}.$$

The rightmost rational function is now proper and it can be decomposed as

$$\frac{5x+3}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}.$$

Clearing denominators, we get $5x+3 = A(x-3)+B(x+1)$ and this implies

$$x=-1,3 \quad\Longrightarrow\quad -2=-4A, \qquad 18=4B \quad\Longrightarrow\quad A=1/2, \qquad B=9/2.$$

Once we now combine our computations above, we may finally conclude that

$$\begin{aligned}\int \frac{x^3-x^2}{x^2-2x-3}\,dx &= \int \left(x+1+\frac{1/2}{x+1}+\frac{9/2}{x-3}\right)dx \\ &= \frac{x^2}{2} + x + \frac{1}{2}\ln|x+1| + \frac{9}{2}\ln|x-3| + K.\end{aligned}$$

2. Compute each of the following indefinite integrals.

$$\int \frac{2+\sqrt{x}}{x+\sqrt{x}}\,dx, \qquad \int \ln(x^2+x)\,dx.$$

For the first integral, we let $u=\sqrt{x}$ to simplify. Since $x=u^2$ and $dx=2u\,du$, we get

$$\begin{aligned}\int \frac{2+\sqrt{x}}{x+\sqrt{x}}\,dx &= \int \frac{2+u}{u^2+u}\cdot 2u\,du = \int \frac{4+2u}{u+1}\,du = \int \left(2+\frac{2}{u+1}\right)du \\ &= 2u+2\ln|u+1|+C = 2\sqrt{x}+2\ln(\sqrt{x}+1)+C.\end{aligned}$$

For the second integral, we let $u=\ln(x^2+x)$ and $dv=dx$. Then $du=\frac{2x+1}{x^2+x}\,dx$, so

$$\begin{aligned}\int \ln(x^2+x)\,dx &= x\ln(x^2+x) - \int \frac{2x+1}{x^2+x}\cdot x\,dx \\ &= x\ln(x^2+x) - \int \frac{2x+1}{x+1}\,dx = x\ln(x^2+x) - \int \left(2-\frac{1}{x+1}\right)dx \\ &= x\ln(x^2+x) - 2x + \ln|x+1| + C.\end{aligned}$$

3. Use integration by parts and induction to show that

$$\int_0^{\pi/2} \sin^{2n+1} x\,dx = \frac{(2^n\cdot n!)^2}{(2n+1)!} \quad \text{for each integer } n\geq 0.$$

We integrate by parts with $u=\sin^{2n}x$ and $dv=\sin x\,dx$. Since $v=-\cos x$, we get

$$\begin{aligned}\int \sin^{2n+1}x\,dx &= -\sin^{2n}x\cos x + \int 2n\sin^{2n-1}x\cdot\cos^2 x\,dx \\ &= -\sin^{2n}x\cos x + \int 2n\sin^{2n-1}x\cdot(1-\sin^2 x)\,dx \\ &= -\sin^{2n}x\cos x + 2n\int \sin^{2n-1}x\,dx - 2n\int \sin^{2n+1}x\,dx.\end{aligned}$$

Next, we rearrange terms and we evaluate the integral over $[0,\pi/2]$ to find that

$$\int_0^{\pi/2}\sin^{2n+1}x\,dx = -\left[\frac{\sin^{2n}x\cos x}{2n+1}\right]_0^{\pi/2} + \frac{2n}{2n+1}\int_0^{\pi/2}\sin^{2n-1}x\,dx.$$

Since $\sin 0=0$ and $\cos(\pi/2)=0$, this leads to an identity of the form

$$I_n = \int_0^{\pi/2}\sin^{2n+1}x\,dx = \frac{2n}{2n+1}\int_0^{\pi/2}\sin^{2n-1}x\,dx = \frac{2n}{2n+1}\cdot I_{n-1}.$$

We now use this identity to establish the given formula. When $n = 0$, we have

$$I_0 = \int_0^{\pi/2} \sin x \, dx = \Big[-\cos x \Big]_0^{\pi/2} = 1 = \frac{(2^0 \cdot 0!)^2}{1!}$$

and the formula holds. If we assume that it holds for some n, then we also have

$$\begin{aligned} I_{n+1} = \frac{2n+2}{2n+3} \cdot I_n &= \frac{2n+2}{2n+3} \cdot \frac{(2^n \cdot n!)^2}{(2n+1)!} \\ &= \frac{2^2(n+1)^2}{(2n+2)(2n+3)} \cdot \frac{(2^n \cdot n!)^2}{(2n+1)!} = \frac{(2^{n+1} \cdot (n+1)!)^2}{(2n+3)!}. \end{aligned}$$

In particular, the formula holds for $n + 1$ as well, so it holds for all $n \geq 0$ by induction.

4. Show that each of the following sequences converges.

$$a_n = \sqrt{\frac{4n^2+5}{9n^2+7}}, \qquad b_n = \frac{(-1)^n}{n^2+1}, \qquad c_n = n \tan \frac{2}{n}.$$

Since the limit of a square root is the square root of the limit, it should be clear that

$$\lim_{n\to\infty} \frac{4n^2+5}{9n^2+7} = \lim_{n\to\infty} \frac{4n^2}{9n^2} = \frac{4}{9} \quad \Longrightarrow \quad \lim_{n\to\infty} a_n = \sqrt{\frac{4}{9}} = \frac{2}{3}.$$

The limit of the second sequence is zero because $-\frac{1}{n^2+1} \leq b_n \leq \frac{1}{n^2+1}$ for each $n \geq 1$. This means that b_n lies between two sequences that converge to zero. Finally, one has

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} n \tan \frac{2}{n} = \lim_{n\to\infty} \frac{\tan(2/n)}{1/n}.$$

This is a limit of the form $0/0$, so one may use L'Hôpital's rule to conclude that

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{\sec^2(2/n) \cdot (2/n)'}{(1/n)'} = \lim_{n\to\infty} 2\sec^2(2/n) = 2\sec^2 0 = 2.$$

5. Define a sequence $\{a_n\}$ by setting $a_1 = 1$ and $a_{n+1} = 2\sqrt{4 + a_n}$ for each $n \geq 1$. Show that $1 \leq a_n \leq a_{n+1} \leq 8$ for each $n \geq 1$, use this fact to conclude that the sequence converges and then find its limit.

Since the first two terms are $a_1 = 1$ and $a_2 = 2\sqrt{5}$, the statement

$$1 \leq a_n \leq a_{n+1} \leq 8$$

does hold when $n = 1$. Suppose that it holds for some n, in which case

$$\begin{aligned} 5 \leq 4 + a_n \leq 4 + a_{n+1} \leq 12 \quad &\Longrightarrow \quad 2\sqrt{5} \leq 2\sqrt{4 + a_n} \leq 2\sqrt{4 + a_{n+1}} \leq 2\sqrt{12} \\ &\Longrightarrow \quad 2\sqrt{5} \leq a_{n+1} \leq a_{n+2} \leq 2\sqrt{12} \\ &\Longrightarrow \quad 1 \leq a_{n+1} \leq a_{n+2} \leq 8. \end{aligned}$$

In particular, the statement holds for $n + 1$ as well, so it actually holds for all $n \in \mathbb{N}$. This shows that the given sequence is monotonic and bounded, hence also convergent; denote its limit by L. Using the definition of the sequence, we then find that

$$a_{n+1} = 2\sqrt{4 + a_n} \quad \Longrightarrow \quad \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} 2\sqrt{4 + a_n} \quad \Longrightarrow \quad L = 2\sqrt{4 + L}.$$

This gives the quadratic equation $L^2 = 4(L + 4)$ which implies that $L = 2 \pm 2\sqrt{5}$. Since the terms of the sequence satisfy $1 \leq a_n \leq 8$, however, the limit must be $L = 2 + 2\sqrt{5}$.