MA1125 – Calculus
2018 exam solutions

1a. For which values of a, b is the function f continuous at $x = 3$? Explain.

$$f(x) = \begin{cases} ax^2 + bx + 3 & \text{if } x < 3 \\ 2a + b + 1 & \text{if } x = 3 \\ 3x^2 + 2x + b & \text{if } x > 3 \end{cases}.$$

Since f is a polynomial on the intervals $(-\infty, 3)$ and $(3, +\infty)$, one easily finds that

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (ax^2 + bx + 3) = 9a + 3b + 3,$$
$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (3x^2 + 2x + b) = 27 + 6 + b.$$

In particular, the function f is continuous at the given point if and only if

$$9a + 3b + 3 = 33 + b = 2a + b + 1.$$

Solving this system of equations, we obtain a unique solution which is given by

$$2a + 1 = 33 \implies a = 16 \implies 144 + 3b + 3 = 33 + b \implies b = -57.$$

In other words, f is continuous at the given point if and only if $a = 16$ and $b = -57$.

1b. Use the ε-δ definition of limits to compute the limit

$$L = \lim_{x \to 2} (4x^2 - 5x + 1).$$

Let $f(x) = 4x^2 - 5x + 1$ for convenience. Then $f(2) = 7$ and one has

$$|f(x) - f(2)| = |4x^2 - 5x - 6| = |x - 2| \cdot |4x + 3|.$$

The factor $|x - 2|$ is related to our usual assumption that $0 \neq |x - 2| < \delta$. To estimate the remaining factor $|4x + 3|$, we assume that $\delta \leq 1$ for simplicity and we note that

$$|x - 2| < \delta \leq 1 \implies -1 < x - 2 < 1$$
$$\implies 1 < x < 3 \implies 7 < 4x + 3 < 15.$$

Combining the estimates $|x - 2| < \delta$ and $|4x + 3| < 15$, one may then conclude that

$$|f(x) - f(2)| = |x - 2| \cdot |4x + 3| < 15\delta \leq \varepsilon,$$

as long as $\delta \leq \varepsilon/15$ and $\delta \leq 1$. An appropriate choice of δ is thus $\delta = \min(\varepsilon/15, 1)$.

1c. Compute each of the following limits.

$$\lim_{x\to 2}\frac{2x^3-5x-6}{3x^3-4x^2-8}, \qquad \lim_{x\to+\infty}\frac{\ln x}{x}, \qquad \lim_{x\to 0}(3x+e^x)^{1/x}.$$

The first limit has the form 0/0, so one may use L'Hôpital's rule to find that

$$L_1=\lim_{x\to 2}\frac{6x^2-5}{9x^2-8x}=\frac{24-5}{36-16}=\frac{19}{20}.$$

The second limit has the form ∞/∞ and one may apply L'Hôpital's rule to get

$$L_2=\lim_{x\to+\infty}\frac{1/x}{1}=\lim_{x\to+\infty}\frac{1}{x}=0.$$

The third limit involves a non-constant exponent which can be eliminated by writing

$$\ln L_3=\ln\lim_{x\to 0}(3x+e^x)^{1/x}=\lim_{x\to 0}\ln(3x+e^x)^{1/x}=\lim_{x\to 0}\frac{\ln(3x+e^x)}{x}.$$

This gives a limit of the form 0/0, so one may use L'Hôpital's rule to find that

$$\ln L_3=\lim_{x\to 0}\frac{(3x+e^x)^{-1}\cdot(3+e^x)}{1}=\frac{3+1}{0+1}=4.$$

Since $\ln L_3=4$, the original limit L_3 is then equal to $L_3=e^{\ln L_3}=e^4$.

2a. Show that $f(x)$ has a unique root in the interval $(1,2)$ when

$$f(x)=x^4-2x^3+x^2-1.$$

Being a polynomial, f is continuous on the interval $[1,2]$ and we also have

$$f(1)=1-2+1-1=-1, \qquad f(2)=16-16+4-1=3.$$

Since $f(1)$ and $f(2)$ have opposite signs, f must have a root that lies in $(1,2)$. To show it is unique, suppose that f has two roots in $(1,2)$. Then f' must have a root in this interval by Rolle's theorem. On the other hand, it is easy to check that

$$f'(x)=4x^3-6x^2+2x=2x(2x^2-3x+1)=2x(x-1)(2x-1).$$

Since f' has no roots in $(1,2)$, we conclude that f has exactly one root in $(1,2)$.

2b. Find the intervals on which $f(x)$ is increasing/decreasing and the intervals on which $f(x)$ is concave up/down. Find the limits of $f(x)$ at infinity and then use this information to draw a rough sketch of the graph of f.

$$f(x) = \frac{(x-1)^2}{x^2+1}.$$

To say that $f(x)$ is increasing is to say that $f'(x) > 0$. Using the quotient rule, we get

$$f'(x) = \frac{2(x-1)(x^2+1) - 2x(x-1)^2}{(x^2+1)^2} = \frac{2(x-1)(x+1)}{(x^2+1)^2}.$$

Since the denominator is always positive, $f(x)$ is increasing if and only if

$$(x-1)(x+1) > 0 \quad \Longleftrightarrow \quad x \in (-\infty, -1) \cup (1, +\infty).$$

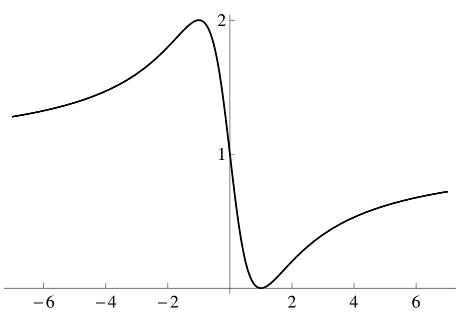
To say that $f(x)$ is concave up is to say that $f''(x) > 0$. In this case, one has

$$\begin{aligned} f''(x) &= \frac{4x(x^2+1)^2 - 2(x^2+1)\cdot 2x(2x^2-2)}{(x^2+1)^4} \\ &= \frac{4x(x^2+1) - 4x(2x^2-2)}{(x^2+1)^3} = \frac{4x(3-x^2)}{(x^2+1)^3}. \end{aligned}$$

To determine the sign of $f''(x)$, we factor this expression and we construct the table below. According to the table, $f(x)$ is concave up if and only if $x \in (-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$. As for the limits of the function at infinity, these are easily found to be

$$\lim_{x\to\pm\infty} \frac{(x-1)^2}{x^2+1} = \lim_{x\to\pm\infty} \frac{x^2-2x+1}{x^2+1} = \lim_{x\to\pm\infty} \frac{x^2}{x^2} = 1.$$

Once we now combine all this information, we obtain the graph which is depicted below.



	$-\sqrt{3}$	0	$\sqrt{3}$	
$4x$	$-$	$-$	$+$	$+$
$\sqrt{3}-x$	$+$	$+$	$+$	$-$
$\sqrt{3}+x$	$-$	$+$	$+$	$+$
$f''(x)$	$+$	$-$	$+$	$-$

Figure 1: The graph of $f(x) = \frac{(x-1)^2}{x^2+1}$.

3a. Find the linear approximation to the function f at the point x_0.

$$f(x) = \frac{e^{3x-3} \cdot (3x^2+5x-2)^2}{\sqrt{\cos(\pi x)+5}}, \qquad x_0 = 1.$$

First, we use logarithmic differentiation to determine $f'(x)$. Since

$$\begin{aligned} \ln|f(x)| &= \ln e^{3x-3} + \ln|3x^2+5x-2|^2 - \ln\sqrt{\cos(\pi x)+5} \\ &= 3x - 3 + 2\ln|3x^2+5x-2| - \frac{1}{2}\ln(\cos(\pi x)+5), \end{aligned}$$

one may differentiate both sides and use the chain rule to find that

$$\frac{f'(x)}{f(x)} = 3 + \frac{2(6x+5)}{3x^2+5x-2} + \frac{\pi\sin(\pi x)}{2(\cos(\pi x)+5)} \quad\Longrightarrow\quad \frac{f'(1)}{f(1)} = 3 + \frac{2\cdot 11}{6}.$$

Since $f(1) = \frac{6^2}{\sqrt{4}} = 18$, one has $f'(1) = 54 + 66 = 120$ and the linear approximation is

$$L(x) = f'(1)(x-1) + f(1) = 120(x-1) + 18 = 120x - 102.$$

3b. A rectangle is inscribed in an equilateral triangle of side $a > 0$. How large can the area of the rectangle be, if one of its sides lies along the base of the triangle?

Let x, y be the two sides of the rectangle and assume that x lies along the base of the triangle. Then one can relate the two sides x, y by noting that

$$\tan 60^\circ = \frac{y}{(a-x)/2} \quad\Longrightarrow\quad \sqrt{3} = \frac{2y}{a-x} \quad\Longrightarrow\quad y = \frac{\sqrt{3}}{2}(a-x).$$

We need to maximise the area A of the rectangle and this is given by

$$A(x) = xy = \frac{\sqrt{3}}{2}x(a-x) = \frac{\sqrt{3}}{2}(ax - x^2), \qquad 0 \leq x \leq a.$$

Since $A'(x) = \frac{\sqrt{3}}{2}(a-2x)$, the only points at which the maximum value may occur are the points $x = 0$, $x = a$ and $x = \frac{a}{2}$. Since $A(0) = A(a) = 0$, the maximum is $A(\frac{a}{2}) = \frac{a^2\sqrt{3}}{8}$.

4a. Compute the volume of a sphere of radius $r > 0$.

One may obtain such a sphere by rotating the upper semicircle $f(x) = \sqrt{r^2 - x^2}$ around the x-axis. Thus, the volume of the sphere is given by the integral

$$\int_{-r}^{r} \pi f(x)^2\,dx = \int_{-r}^{r} \pi(r^2 - x^2)\,dx = \pi\left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r} = \pi\left(\frac{2r^3}{3} + \frac{2r^3}{3}\right) = \frac{4\pi r^3}{3}.$$

4b. Compute the indefinite integral

$$\int \frac{9x^2+6x-7}{x^3-x}\,dx.$$

Since the denominator can be factored, one may use partial fractions to write

$$\frac{9x^2+6x-7}{x^3-x} = \frac{9x^2+6x-7}{x(x^2-1)} = \frac{9x^2+6x-7}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

for some constants A, B, C that need to be determined. Clearing denominators gives

$$9x^2+6x-7 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

and one may look at some suitable choices of x to find that

$$x = -1, 0, 1 \quad \Longrightarrow \quad -4 = 2C, \qquad -7 = -A, \qquad 8 = 2B.$$

Solving these equations, we now get $C = -2$, $A = 7$ and $B = 4$, which finally gives

$$\begin{aligned}\int \frac{9x^2+6x-7}{x^3-x}\,dx &= \int \frac{7}{x}\,dx + \int \frac{4}{x-1}\,dx - \int \frac{2}{x+1}\,dx \\ &= 7\ln|x| + 4\ln|x-1| - 2\ln|x+1| + K.\end{aligned}$$

5a. Compute the length of the graph of f in the case that

$$f(x) = \ln(\cos x), \qquad 0 \leq x \leq \pi/6.$$

The length of the graph is given by the integral of $\sqrt{1+f'(x)^2}$. In this case,

$$f'(x) = \frac{1}{\cos x} \cdot (\cos x)' = -\frac{\sin x}{\cos x} = -\tan x$$

so the expression $1 + f'(x)^2$ can be written in the form

$$1 + f'(x)^2 = 1 + \tan^2 x = \sec^2 x.$$

Taking the square root of both sides, we conclude that the length of the graph is

$$\int_0^{\pi/6} \sqrt{1+f'(x)^2}\,dx = \int_0^{\pi/6} \sec x\,dx = \Big[\ln|\sec x + \tan x|\Big]_0^{\pi/6} = \frac{\ln 3}{2}.$$

5b. Compute each of the following indefinite integrals.

$$\int \sin^2 x \cdot \cos^3 x \, dx, \qquad \int x^3 e^{-x^2} \, dx.$$

For the first integral, we use the substitution $u = \sin x$. Since $du = \cos x \, dx$, we get

$$\begin{aligned} \int \sin^2 x \cdot \cos^3 x \, dx &= \int \sin^2 x \cdot (1 - \sin^2 x) \cdot \cos x \, dx = \int u^2 (1 - u^2) \, du \\ &= \int (u^2 - u^4) \, du = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C. \end{aligned}$$

For the second integral, we use the substitution $u = -x^2$. Since $du = -2x \, dx$, we get

$$\int x^3 e^{-x^2} \, dx = \int x^2 e^{-x^2} \cdot x \, dx = \frac{1}{2} \int u e^u \, du.$$

Next, we integrate by parts with $dv = e^u \, du$. This gives $v = e^u$ and so

$$\begin{aligned} \int x^3 e^{-x^2} \, dx &= \frac{1}{2} u e^u - \frac{1}{2} \int e^u \, du = \frac{1}{2} u e^u - \frac{1}{2} e^u + C \\ &= -\frac{1}{2} x^2 e^{-x^2} - \frac{1}{2} e^{-x^2} + C. \end{aligned}$$

6a. Test each of the following series for convergence.

$$\sum_{n=1}^\infty \frac{n\sqrt{n}}{n^2+1}, \qquad \sum_{n=1}^\infty \frac{2n+1}{n!}, \qquad \sum_{n=1}^\infty \sin \frac{1}{n}.$$

When it comes to the first series, we use the limit comparison test with

$$a_n = \frac{n\sqrt{n}}{n^2+1}, \qquad b_n = \frac{n\sqrt{n}}{n^2} = \frac{1}{\sqrt{n}}.$$

To show that the limit comparison test is applicable in this case, we note that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1.$$

Since $\sum_{n=1}^\infty b_n$ is a divergent p-series, we conclude that $\sum_{n=1}^\infty a_n$ diverges as well. When it comes to the second series, we use the ratio test and we compute

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2n+3}{2n+1} \cdot \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{2n+3}{(2n+1)(n+1)} = 0.$$

Since this limit is smaller than 1, the second series converges. For the last series, we let

$$a_n = \sin\frac{1}{n}, \qquad b_n = \frac{1}{n}.$$

To see that the limit comparison test is applicable, we note that L'Hôpital's rule gives

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sin(1/n)}{1/n} = \lim_{n\to\infty}\frac{\cos(1/n)\cdot(1/n)'}{(1/n)'} = \cos 0 = 1.$$

Since $\sum_{n=1}^{\infty} b_n$ is a divergent p-series, we conclude that $\sum_{n=1}^{\infty} a_n$ diverges as well.

6b. Find the radius of convergence of the power series

$$f(x) = \sum_{n=0}^{\infty}\frac{(n!)^2}{(2n)!}\cdot x^n.$$

One uses the ratio test to find the radius of convergence. In this case, we have

$$\begin{aligned} L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{(n+1)!}{n!}\cdot\frac{(n+1)!}{n!}\cdot\frac{(2n)!}{(2n+2)!}\cdot\frac{|x|^{n+1}}{|x|^n} \\ &= \lim_{n\to\infty}\frac{(n+1)^2\cdot|x|}{(2n+1)(2n+2)} = \lim_{n\to\infty}\frac{n^2|x|}{4n^2} = \frac{|x|}{4}. \end{aligned}$$

Thus, the series converges when $|x| < 4$ and diverges when $|x| > 4$, so the radius is $R = 4$.