MA1125 – Calculus
Homework #1 solutions

1. Find the domain and the range of the function f which is defined by

$$f(x) = \frac{3-2x}{5-3x}.$$

The domain consists of all points $x \neq 5/3$. To find the range, we note that

$$y = \frac{3-2x}{5-3x} \iff 5y - 3xy = 3 - 2x \iff 2x - 3xy = 3 - 5y$$
$$\iff x(2-3y) = 3 - 5y \iff x = \frac{3-5y}{2-3y}.$$

The rightmost formula determines the value of x that satisfies $y = f(x)$. Since the formula makes sense for any number $y \neq 2/3$, the range consists of all numbers $y \neq 2/3$.

2. Find the domain and the range of the function f which is defined by

$$f(x) = \frac{\sqrt{2x-1}}{x}.$$

When it comes to the domain, we need to have $x \neq 0$ and $2x-1 \geq 0$. This gives $x \geq 1/2$ and the domain is $[1/2, +\infty)$. Since x is non-negative, the same is true for $y = f(x)$ and

$$y^2 = \frac{2x-1}{x^2} \iff y^2x^2 = 2x - 1 \iff y^2x^2 - 2x + 1 = 0.$$

If it happens that $y = 0$, then $x = 1/2$. If it happens that $y \neq 0$, then the last equation is quadratic in x and one may use the quadratic formula to conclude that

$$x = \frac{2 \pm \sqrt{4-4y^2}}{2y^2} = \frac{1 \pm \sqrt{1-y^2}}{y^2}.$$

This leads to the restriction $1 - y^2 \geq 0$, which gives $y^2 \leq 1$ and thus $-1 \leq y \leq 1$. Since y is also non-negative, however, the range of the given function is merely $[0, 1]$.

3. Show that the function $f\colon (0,1) \to (1,\infty)$ is bijective in the case that

$$f(x) = \frac{1+x}{1-x}.$$

To show that the given function is injective, we note that

$$\begin{aligned}\frac{1+x_1}{1-x_1} = \frac{1+x_2}{1-x_2} \quad &\implies \quad 1 - x_2 + x_1 - x_1x_2 = 1 - x_1 + x_2 - x_1x_2 \\ &\implies \quad 2x_1 = 2x_2 \quad \implies \quad x_1 = x_2.\end{aligned}$$

To show that the given function is surjective, we note that

$$y = \frac{1+x}{1-x} \quad \iff \quad y - xy = 1 + x \quad \iff \quad y - 1 = xy + x \quad \iff \quad x = \frac{y-1}{y+1}.$$

The rightmost formula determines the value of x such that $y = f(x)$ and we need to check that $0 < x < 1$ if and only if $y > 1$. When $y > 1$, we have $y + 1 > y - 1 > 0$, so $0 < x < 1$. When $0 < x < 1$, we have $0 < 1 - x < 1 + x$ and this gives $y > 1$, as needed.

4. Express the following polynomials as the product of linear factors.

$$f(x) = 3x^3 - 2x^2 - 7x - 2, \qquad g(x) = x^3 + x^2 - \frac{7x}{4} + \frac{1}{2}.$$

When it comes to $f(x)$, the possible rational roots are $\pm 1, \pm 2, \pm 1/3, \pm 2/3$. Checking these possibilities, one finds that $x = -1$, $x = 2$ and $x = -1/3$ are all roots. According to the factor theorem, each of $x + 1$, $x - 2$ and $x + 1/3$ is thus a factor and one has

$$f(x) = 3(x+1)(x-2)(x+1/3) = (x+1)(x-2)(3x+1).$$

When it comes to $g(x)$, let us first clear denominators and write

$$4g(x) = 4x^3 + 4x^2 - 7x + 2.$$

The only possible rational roots are $\pm 1, \pm 2, \pm 1/2, \pm 1/4$. Checking these possibilities, one finds that only $x = -2$ and $x = 1/2$ are roots. This gives two of the factors and then the third can be found using division of polynomials. More precisely, one has

$$4g(x) = (x+2)(4x^2 - 4x + 1) = (x+2)(2x-1)^2 \quad \implies \quad g(x) = \frac{1}{4}(x+2)(2x-1)^2.$$

5. Determine all angles $0 \leq \theta \leq 2\pi$ such that $2\sin^2\theta + 9\sin\theta = 5$.

Letting $x = \sin\theta$ for convenience, one finds that $2x^2 + 9x - 5 = 0$ and

$$x = \frac{-9 \pm \sqrt{81 + 4 \cdot 10}}{4} = \frac{-9 \pm 11}{4} \quad \implies \quad x = \frac{1}{2}, -5.$$

Since $x = \sin\theta$ must lie between -1 and 1, the only relevant solution is $x = \sin\theta = \frac{1}{2}$. In view of the graph of the sine function, there should be two angles $0 \leq \theta \leq 2\pi$ that satisfy this condition. The first one is $\theta_1 = \frac{\pi}{6}$ and the second one is $\theta_2 = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

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Homework #2 solutions

1. Determine the inverse function f^{-1} in each of the following cases.

$$f(x) = 3 - \log_2(2x-4), \qquad f(x) = \frac{2\cdot 7^x + 3}{5\cdot 7^x + 4}.$$

When it comes to the first case, one can easily check that

$$3 - y = \log_2(2x-4) \quad\Longleftrightarrow\quad 2^{3-y} = 2x - 4 \quad\Longleftrightarrow\quad 2^{2-y} = x - 2,$$

so the inverse function is defined by $f^{-1}(y) = 2^{2-y} + 2$. When it comes to the second case,

$$y = \frac{2\cdot 7^x + 3}{5\cdot 7^x + 4} \quad\Longleftrightarrow\quad 5y\cdot 7^x + 4y = 2\cdot 7^x + 3 \quad\Longleftrightarrow\quad 7^x(5y-2) = 3 - 4y$$

and this gives $7^x = \frac{3-4y}{5y-2}$, so the inverse function is defined by $f^{-1}(y) = \log_7 \frac{3-4y}{5y-2}$.

2. Simplify each of the following expressions.

$$\cos\left(\tan^{-1} x\right), \qquad \sin\left(\cos^{-1} x\right), \qquad \log_2 \frac{4^x + 8^x}{2^x + 4^x}.$$

To simplify the first expression, let $\theta = \tan^{-1} x$ and note that $\tan\theta = x$. When $x \geq 0$, the angle θ arises in a right triangle with an opposite side of length x and an adjacent side of length 1. It follows by Pythagoras' theorem that the hypotenuse has length $\sqrt{1+x^2}$, so the definition of cosine gives

$$\cos(\tan^{-1} x) = \cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{1}{\sqrt{1+x^2}}.$$

When $x \leq 0$, the last equation holds with $-x$ instead of x. This changes the term $\tan^{-1} x$ by a minus sign, but the cosine remains unchanged, so the equation is still valid.

To simplify the second expression, one may use a similar approach or simply note that

$$\theta = \cos^{-1} x \quad\Longrightarrow\quad \cos\theta = x \quad\Longrightarrow\quad \sin^2\theta = 1 - \cos^2\theta = 1 - x^2.$$

Since $\theta = \cos^{-1} x$ lies between 0 and π by definition, $\sin\theta$ is non-negative and

$$\sin^2\theta = 1 - x^2 \quad\Longrightarrow\quad \sin\theta = \sqrt{1-x^2}.$$

As for the third expression, one may simplify the given fraction to conclude that

$$\log_2 \frac{4^x + 8^x}{2^x + 4^x} = \log_2 \frac{4^x(1+2^x)}{2^x(1+2^x)} = \log_2 2^x = x.$$

3. Use the ε-δ definition of limits to compute $\lim_{x\to 2} f(x)$ in the case that

$$f(x) = \left\{ \begin{array}{ll} 2x-5 & \text{if } x \leq 2 \\ 5-3x & \text{if } x > 2 \end{array} \right\}.$$

In this case, x is approaching 2 and $f(x)$ is either $2x-5$ or $5-3x$. We thus expect the limit to be $L = -1$. To prove this formally, we let $\varepsilon > 0$ and estimate the expression

$$|f(x)+1| = \left\{ \begin{array}{ll} |2x-4| & \text{if } x \leq 2 \\ |6-3x| & \text{if } x > 2 \end{array} \right\} = \left\{ \begin{array}{ll} 2|x-2| & \text{if } x \leq 2 \\ 3|x-2| & \text{if } x > 2 \end{array} \right\}.$$

If we assume that $0 \neq |x-2| < \delta$, then we may use the last equation to get

$$|f(x)+1| \leq 3|x-2| < 3\delta.$$

Since our goal is to show that $|f(x)+1| < \varepsilon$, an appropriate choice of δ is thus $\delta = \varepsilon/3$.

4. Compute each of the following limits.

$$L = \lim_{x\to 1} \frac{x^3-4x^2+4x-1}{x-1}, \qquad M = \lim_{x\to 1} \frac{3x^3-7x^2+5x-1}{(x-1)^2}.$$

When it comes to the first limit, division of polynomials gives

$$L = \lim_{x\to 1} \frac{(x-1)(x^2-3x+1)}{x-1} = \lim_{x\to 1} (x^2-3x+1) = 1-3+1 = -1.$$

When it comes to the second limit, division of polynomials gives

$$M = \lim_{x\to 1} \frac{(x^2-2x+1)(3x-1)}{x^2-2x+1} = \lim_{x\to 1} (3x-1) = 3-1 = 2.$$

5. Use the ε-δ definition of limits to compute $\lim_{x\to 3} (5x^2-6x+3)$.

Let $f(x) = 5x^2-6x+3$ for convenience. Then $f(3) = 30$ and one has

$$|f(x)-f(3)| = |5x^2-6x-27| = |x-3| \cdot |5x+9|.$$

The factor $|x-3|$ is related to our usual assumption that $0 \neq |x-3| < \delta$. To estimate the remaining factor $|5x+9|$, we assume that $\delta \leq 1$ for simplicity and we note that

$$\begin{aligned} |x-3| < \delta \leq 1 \quad &\Longrightarrow \quad -1 < x-3 < 1 \\ &\Longrightarrow \quad 2 < x < 4 \quad \Longrightarrow \quad 19 < 5x+9 < 29. \end{aligned}$$

Combining the estimates $|x-3| < \delta$ and $|5x+9| < 29$, one may then conclude that

$$|f(x)-f(3)| = |x-3| \cdot |5x+9| < 29\delta \leq \varepsilon,$$

as long as $\delta \leq \varepsilon/29$ and $\delta \leq 1$. An appropriate choice of δ is thus $\delta = \min(\varepsilon/29, 1)$.

MA1125 – Calculus
Homework #3 solutions

1. Show that there exists a real number $0 < x < \pi$ that satisfies the equation

$$x^2 = \frac{x^2+1}{2+\sin x} + 4.$$

Consider the function f which is defined as the difference of the two sides, namely

$$f(x) = \frac{x^2+1}{2+\sin x} + 4 - x^2.$$

Being a composition of continuous functions, f is then continuous and we also have

$$f(0) = \frac{1}{2} + 4 > 0, \qquad f(\pi) = \frac{\pi^2+1}{2} + 4 - \pi^2 = \frac{9-\pi^2}{2} < 0.$$

In view of Bolzano's theorem, this already implies that f has a root $0 < x < \pi$.

2. For which values of a, b is the function f continuous at the point $x = 2$? Explain.

$$f(x) = \left\{ \begin{array}{cl} 2x^3 - ax^2 + bx & \text{if } x < 2 \\ a^2 + b & \text{if } x = 2 \\ 2x^2 + bx - a & \text{if } x > 2 \end{array} \right\}.$$

Since f is a polynomial on the intervals $(-\infty, 2)$ and $(2, +\infty)$, it should be clear that

$$\begin{aligned} \lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} (2x^3 - ax^2 + bx) = 16 - 4a + 2b, \\ \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} (2x^2 + bx - a) = 8 + 2b - a. \end{aligned}$$

In particular, the function f is continuous at the given point if and only if

$$16 - 4a + 2b = 8 + 2b - a = a^2 + b.$$

Solving this system of equations, one obtains a unique solution which is given by

$$16 - 4a = 8 - a \quad\Longrightarrow\quad 3a = 8 \quad\Longrightarrow\quad a = \frac{8}{3} \quad\Longrightarrow\quad b = a^2 + a - 8 = \frac{16}{9}.$$

In other words, f is continuous at the given point if and only if $a = 8/3$ and $b = 16/9$.

3. Show that $f(x) = x^5 - x^2 - 3x + 1$ has three roots in the interval $(-2, 2)$. Hint: you need only consider the values that are attained by f at the points ± 2, ± 1 and 0.

Being a polynomial, the given function is continuous and one can easily check that

$$f(-2) = -29, \qquad f(-1) = 2, \qquad f(0) = 1, \qquad f(1) = -2, \qquad f(2) = 23.$$

Since the values $f(-2)$ and $f(-1)$ have opposite signs, f has a root that lies in $(-2, -1)$. The same argument yields a second root in $(0, 1)$ and also a third root in $(1, 2)$.

4. Compute each of the following limits.

$$L = \lim_{x \to +\infty} \frac{3x^3 - 2x + 4}{5x^3 - x^2 + 7}, \qquad M = \lim_{x \to 2^-} \frac{x^3 + 5x^2 - 4}{3x^3 - 16x + 8}.$$

Since the first limit involves infinite values of x, it should be clear that

$$L = \lim_{x \to +\infty} \frac{3x^3 - 2x + 4}{5x^3 - x^2 + 7} = \lim_{x \to +\infty} \frac{3x^3}{5x^3} = \frac{3}{5}.$$

For the second limit, the denominator becomes zero when $x = 2$, while the numerator is nonzero at that point. Thus, one needs to factor the denominator and this gives

$$M = \lim_{x \to 2^-} \frac{x^3 + 5x^2 - 4}{(x-2)(3x^2 + 6x - 4)} = \lim_{x \to 2^-} \frac{24}{20(x-2)} = -\infty.$$

5. Use the definition of the derivative to compute $f'(x_0)$ in each of the following cases.

$$f(x) = x^3, \qquad f(x) = 1/x^2, \qquad f(x) = (3x+4)^2.$$

The derivative of the first function is given by the limit

$$f'(x_0) = \lim_{x \to x_0} \frac{x^3 - x_0^3}{x - x_0} = \lim_{x \to x_0} \frac{(x - x_0)(x^2 + xx_0 + x_0^2)}{x - x_0} = x_0^2 + x_0^2 + x_0^2 = 3x_0^2.$$

To compute the derivative of the second function, we begin by writing

$$f(x) - f(x_0) = \frac{1}{x^2} - \frac{1}{x_0^2} = \frac{(x_0 - x)(x_0 + x)}{x^2 x_0^2}.$$

Once we now divide this expression by $x - x_0$, we may also conclude that

$$f'(x_0) = \lim_{x \to x_0} \frac{-(x_0 + x)}{x^2 x_0^2} = -\frac{2x_0}{x_0^4} = -\frac{2}{x_0^3}.$$

Finally, the derivative of the third function is given by the limit

$$f'(x_0) = \lim_{x \to x_0} \frac{(3x+4)^2 - (3x_0+4)^2}{x - x_0} = \lim_{x \to x_0} \frac{(3x + 3x_0 + 8)(3x - 3x_0)}{x - x_0} = 6(3x_0 + 4).$$

MA1125 – Calculus
Homework #4 solutions

1. Compute the derivative $y' = \frac{dy}{dx}$ in each of the following cases.

$$y = \tan(e^x) + e^{\sec x}, \qquad y = \cos(\sin^2(\ln x)).$$

When it comes to the first function, one may use the chain rule to get

$$y' = \sec^2(e^x) \cdot (e^x)' + e^{\sec x} \cdot (\sec x)' = e^x \sec^2(e^x) + e^{\sec x} \sec x \tan x.$$

When it comes to the second function, one similarly finds that

$$\begin{aligned} y' &= -\sin(\sin^2(\ln x)) \cdot [\sin^2(\ln x)]' \\ &= -\sin(\sin^2(\ln x)) \cdot 2\sin(\ln x) \cdot [\sin(\ln x)]' \\ &= -\sin(\sin^2(\ln x)) \cdot 2\sin(\ln x) \cdot \cos(\ln x) \cdot 1/x. \end{aligned}$$

2. Compute the derivative $y' = \frac{dy}{dx}$ in the case that $y^2 \sin x + x^2 \sin y = x^2 y$.

Differentiating both sides of the given equation, one finds that

$$2yy' \sin x + y^2 \cos x + 2x \sin y + x^2 y' \cos y = 2xy + x^2 y'.$$

We now collect the terms that contain y' on the left hand side and we get

$$(2y \sin x + x^2 \cos y - x^2) y' = 2xy - y^2 \cos x - 2x \sin y.$$

Solving this equation for y', one may thus conclude that

$$y' = \frac{2xy - y^2 \cos x - 2x \sin y}{2y \sin x + x^2 \cos y - x^2}.$$

3. Compute the derivative $y' = \frac{dy}{dx}$ in each of the following cases.

$$y = e^{\sin x} \cdot \cos(e^x), \qquad y = (x \cdot \tan x)^{\ln x}.$$

When it comes to the first function, we use the product rule and the chain rule to get

$$y' = e^{\sin x} \cos x \cdot \cos(e^x) - e^{\sin x} \cdot e^x \sin(e^x).$$

When it comes to the second function, logarithmic differentiation gives

$$\begin{aligned} \ln y = \ln x \cdot \ln(x \tan x) \quad &\Longrightarrow \quad \frac{y'}{y} = \frac{1}{x} \cdot \ln(x \tan x) + \frac{\ln x}{x \tan x} \cdot (x \tan x)' \\ &\Longrightarrow \quad \frac{y'}{y} = \frac{\ln(x \tan x)}{x} + \frac{\ln x}{x \tan x} \cdot (\tan x + x \sec^2 x) \\ &\Longrightarrow \quad y' = (x \cdot \tan x)^{\ln x} \cdot \left(\frac{\ln(x \tan x)}{x} + \frac{\ln x}{x} + \frac{\ln x}{\sin x \cos x} \right). \end{aligned}$$

4. Compute the derivative $f'(x_0)$ in the case that

$$f(x) = \frac{(x^2+3)^2 \cdot x^{\ln x} \cdot e^{4-4x}}{\sqrt{e^{2x-2}+3}}, \qquad x_0 = 1.$$

First, we use logarithmic differentiation to determine $f'(x)$. In this case, we have

$$\begin{aligned} \ln f(x) &= \ln(x^2+3)^2 + \ln x^{\ln x} + \ln e^{4-4x} - \ln(e^{2x-2}+3)^{1/2} \\ &= 2\ln(x^2+3) + (\ln x)^2 + 4 - 4x - \frac{1}{2}\ln(e^{2x-2}+3). \end{aligned}$$

Differentiating both sides of this equation, one easily finds that

$$\frac{f'(x)}{f(x)} = \frac{2 \cdot 2x}{x^2+3} + \frac{2\ln x}{x} - 4 - \frac{2e^{2x-2}}{2(e^{2x-2}+3)}.$$

To compute the derivative $f'(1)$, one may then substitute $x=1$ to conclude that

$$\frac{f'(1)}{f(1)} = \frac{4}{4} + 2\ln 1 - 4 - \frac{2}{8} = -\frac{13}{4} \quad \Longrightarrow \quad f'(1) = -\frac{13}{4} \cdot 8 = -26.$$

5. Compute the derivative $y' = \frac{dy}{dx}$ in the case that

$$y = \tan^{-1} u, \qquad u = \sqrt{2z^3+1}, \qquad z = \frac{x^2-3}{x^2+1}.$$

Differentiating the given equations, one easily finds that

$$\frac{dy}{du} = \frac{1}{u^2+1}, \qquad \frac{du}{dz} = \frac{6z^2}{2\sqrt{2z^3+1}}, \qquad \frac{dz}{dx} = \frac{2x(x^2+1) - 2x(x^2-3)}{(x^2+1)^2} = \frac{8x}{(x^2+1)^2}.$$

According to the chain rule, the derivative $\frac{dy}{dx}$ is the product of these factors, namely

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dz}\frac{dz}{dx} = \frac{1}{u^2+1} \cdot \frac{3z^2}{\sqrt{2z^3+1}} \cdot \frac{8x}{(x^2+1)^2}.$$

MA1125 – Calculus
Homework #5 solutions

1. Show that the polynomial $f(x) = x^3 - 4x^2 - 3x + 1$ has exactly one root in $(0, 2)$.

Being a polynomial, f is continuous on the interval $[0, 2]$ and we also have

$$f(0) = 1, \qquad f(2) = 8 - 16 - 6 + 1 = -13.$$

Since $f(0)$ and $f(2)$ have opposite signs, f must have a root that lies in $(0, 2)$. To show it is unique, suppose that f has two roots in $(0, 2)$. Then f' must have a root in this interval by Rolle's theorem. On the other hand, it is easy to check that

$$f'(x) = 3x^2 - 8x - 3 = (3x + 1)(x - 3).$$

Since f' has no roots in $(0, 2)$, we conclude that f has exactly one root in $(0, 2)$.

2. Suppose that $0 < a < b$. Use the mean value theorem to show that

$$1 - \frac{a}{b} < \ln b - \ln a < \frac{b}{a} - 1.$$

Since $f(x) = \ln x$ is differentiable with $f'(x) = 1/x$, the mean value theorem gives

$$\frac{\ln b - \ln a}{b - a} = f'(c) = \frac{1}{c}$$

for some point $a < c < b$. Using this fact to estimate the right hand side, one finds that

$$\frac{1}{b} < \frac{1}{c} < \frac{1}{a} \quad \Longrightarrow \quad \frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a} \quad \Longrightarrow \quad 1 - \frac{a}{b} < \ln b - \ln a < \frac{b}{a} - 1.$$

3. Compute each of the following limits.

$$L_1 = \lim_{x \to 3} \frac{2x^3 - 8x^2 + 7x - 3}{3x^3 - 8x^2 - x - 6}, \qquad L_2 = \lim_{x \to \infty} \frac{x^2}{e^x}, \qquad L_3 = \lim_{x \to 0} (e^x + x)^{1/x}.$$

The first limit has the form $0/0$, so one may use L'Hôpital's rule to find that

$$L_1 = \lim_{x \to 3} \frac{6x^2 - 16x + 7}{9x^2 - 16x - 1} = \frac{54 - 48 + 7}{81 - 48 - 1} = \frac{13}{32}.$$

The second limit has the form ∞/∞ and one may apply L'Hôpital's rule twice to get

$$L_2 = \lim_{x \to \infty} \frac{2x}{e^x} = \lim_{x \to \infty} \frac{2}{e^x} = 0.$$

The third limit involves a non-constant exponent which can be eliminated by writing

$$\ln L_3 = \ln \lim_{x\to 0} (e^x + x)^{1/x} = \lim_{x\to 0} \ln(e^x + x)^{1/x} = \lim_{x\to 0} \frac{\ln(e^x + x)}{x}.$$

This gives a limit of the form 0/0, so one may use L'Hôpital's rule to find that

$$\ln L_3 = \lim_{x\to 0} \frac{e^x + 1}{e^x + x} = \frac{1+1}{1+0} = 2.$$

Since $\ln L_3 = 2$, the original limit L_3 is then equal to $L_3 = e^{\ln L_3} = e^2$.

4. On which intervals is f increasing? On which intervals is it concave up?

$$f(x) = \frac{x}{x^2+3}.$$

To say that $f(x)$ is increasing is to say that $f'(x) > 0$. Let us then compute

$$f'(x) = \frac{x^2 + 3 - 2x \cdot x}{(x^2+3)^2} = \frac{3 - x^2}{(x^2+3)^2}.$$

Since the denominator is always positive, $f(x)$ is increasing if and only if

$$3 - x^2 > 0 \quad \Longleftrightarrow \quad x^2 < 3 \quad \Longleftrightarrow \quad -\sqrt{3} < x < \sqrt{3}.$$

To say that $f(x)$ is concave up is to say that $f''(x) > 0$. In this case, we have

$$\begin{aligned} f''(x) &= \frac{-2x(x^2+3)^2 - 2(x^2+3) \cdot 2x \cdot (3-x^2)}{(x^2+3)^4} \\ &= \frac{-2x(x^2+3) - 4x(3-x^2)}{(x^2+3)^3} \\ &= -\frac{2x(x^2+3+6-2x^2)}{(x^2+3)^3} = -\frac{2x(3-x)(3+x)}{(x^2+3)^3}. \end{aligned}$$

To determine the sign of this expression, one needs to find the sign of each of the factors. According to the table below, $f(x)$ is concave up if and only if $x \in (-3, 0) \cup (3, +\infty)$.

		-3	0	3
$-2x$	$+$	$+$	$-$	$-$
$3-x$	$+$	$+$	$+$	$-$
$3+x$	$-$	$+$	$+$	$+$
$f''(x)$	$-$	$+$	$-$	$+$

5. Find the intervals on which f is increasing/decreasing and the intervals on which f is concave up/down. Use this information to sketch the graph of f.

$$f(x) = \frac{(x-1)^2}{x^2+1}.$$

To say that $f(x)$ is increasing is to say that $f'(x) > 0$. Let us then compute

$$f'(x) = \frac{2(x-1)(x^2+1) - 2x \cdot (x-1)^2}{(x^2+1)^2} = \frac{2(x-1)(x+1)}{(x^2+1)^2} = \frac{2(x^2-1)}{(x^2+1)^2}.$$

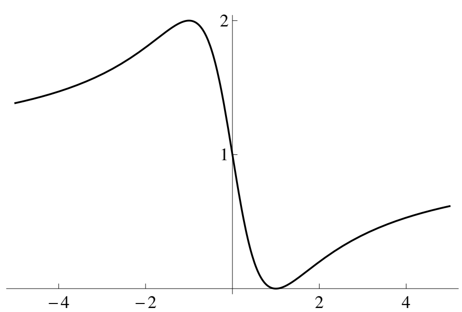
Since the denominator is always positive, $f(x)$ is increasing if and only if

$$x^2 - 1 > 0 \iff x^2 > 1 \iff x \in (-\infty, -1) \cup (1, +\infty).$$

To say that $f(x)$ is concave up is to say that $f''(x) > 0$. In this case, we have

$$\begin{aligned} f''(x) &= \frac{4x \cdot (x^2+1)^2 - 2(x^2+1) \cdot 2x \cdot 2(x^2-1)}{(x^2+1)^4} \\ &= \frac{4x(x^2+1) - 8x(x^2-1)}{(x^2+1)^3} = \frac{4x(3-x^2)}{(x^2+1)^3}. \end{aligned}$$

To determine the sign of this expression, one needs to find the sign of each of the factors. According to the table below, $f(x)$ is concave up if and only if $x \in (-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$.



	$-\sqrt{3}$		0		$\sqrt{3}$	
$4x$	$-$		$-$	$+$		$+$
$3-x^2$	$-$		$+$	$+$		$-$
$f''(x)$	$+$		$-$	$+$		$-$

Figure 1: The graph of $f(x) = \dfrac{(x-1)^2}{x^2+1}$.

MA1125 – Calculus
Homework #6 solutions

1. Find the global minimum and the global maximum values that are attained by

$$f(x) = x^3 - 6x^2 + 9x - 5, \qquad 0 \leq x \leq 2.$$

The derivative of the given function can be expressed in the form

$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3).$$

Thus, the only points at which the minimum/maximum value may occur are the points

$$x = 0, \qquad x = 2, \qquad x = 1, \qquad x = 3.$$

We exclude the rightmost point, as it does not lie in the given interval, and we compute

$$f(0) = -5, \qquad f(2) = 8 - 24 + 18 - 5 = -3, \qquad f(1) = 1 - 6 + 9 - 5 = -1.$$

This means that the minimum value is $f(0) = -5$ and the maximum value is $f(1) = -1$.

2. If a right triangle has a hypotenuse of length $a > 0$, how large can its area be?

Let us denote by x, y the other sides of the triangle. Then $x^2 + y^2 = a^2$ and the area is

$$A(x) = \frac{1}{2}\,xy = \frac{1}{2}\,x\sqrt{a^2 - x^2}, \qquad 0 \leq x \leq a.$$

The value of x that maximises this expression is the value of x that maximises its square

$$f(x) = A(x)^2 = \frac{1}{4}\,x^2(a^2 - x^2) = \frac{1}{4}\,(a^2x^2 - x^4).$$

Let us then worry about $f(x)$, instead. The derivative of this function is given by

$$f'(x) = \frac{1}{4}\,(2a^2x - 4x^3) = \frac{x}{2}\,(a^2 - 2x^2).$$

Thus, the only points at which the maximum value may occur are the points

$$x = 0, \qquad x = a, \qquad x = \frac{a}{\sqrt{2}}.$$

Since $f(0) = f(a) = 0$, the maximum value is $f(a/\sqrt{2})$ and the largest possible area is

$$A(a/\sqrt{2}) = \frac{a}{2\sqrt{2}} \cdot \sqrt{a^2 - \frac{a^2}{2}} = \frac{a}{2\sqrt{2}} \cdot \frac{a}{\sqrt{2}} = \frac{a^2}{4}.$$

3. A balloon is rising vertically at the rate of 1 m/sec. When it reaches 48m above the ground, a bicycle passes under it moving at 3 m/sec along a flat, straight road. How fast is the distance between the bicycle and the balloon increasing 16 seconds later?

Let x be the horizontal distance between the balloon and the bicycle, and let y be the height of the balloon. Then x, y are the sides of a right triangle whose hypotenuse is the distance z between the balloon and the bicycle. It follows by Pythagoras' theorem that

$$x(t)^2 + y(t)^2 = z(t)^2 \quad \Longrightarrow \quad 2x(t)x'(t) + 2y(t)y'(t) = 2z(t)z'(t).$$

At the given moment, $x'(t) = 3$ and $y'(t) = 1$, while $x(t) = 16 \cdot 3$ and $y(t) = 48 + 16$, so

$$z'(t) = \frac{x(t)x'(t) + y(t)y'(t)}{\sqrt{x(t)^2 + y(t)^2}} = \frac{48 \cdot 3 + 64}{\sqrt{48^2 + 64^2}} = \frac{208}{80} = \frac{13}{5}.$$

4. Find the linear approximation to the function f at the point x_0 in the case that

$$f(x) = \frac{x^3 - 2x + 4}{x^2 + 2}, \qquad x_0 = 0.$$

To find the derivative of $f(x)$ at the given point, we use the quotient rule to get

$$f'(x) = \frac{(3x^2 - 2) \cdot (x^2 + 2) - 2x \cdot (x^3 - 2x + 4)}{(x^2 + 2)^2} \quad \Longrightarrow \quad f'(0) = -\frac{4}{2^2} = -1.$$

Since $f(0) = 4/2 = 2$, the linear approximation is thus $L(x) = -(x - 0) + 2 = 2 - x$.

5. Show that $f(x) = x^3 - 4x + 1$ has two roots in $(0, 2)$ and use Newton's method with initial guesses $x_1 = 0, 2$ to approximate these roots within two decimal places.

To prove existence using Bolzano's theorem, we note that f is continuous with

$$f(0) = 1, \qquad f(1) = 1 - 4 + 1 = -2, \qquad f(2) = 8 - 8 + 1 = 1.$$

In view of Bolzano's theorem, f must then have a root in $(0, 1)$ and another root in $(1, 2)$, so it has two roots in $(0, 2)$. Suppose that it has three roots in $(0, 2)$. Then f' must have two roots in this interval by Rolle's theorem. On the other hand, $f'(x) = 3x^2 - 4$ has only one root in $(0, 2)$. This implies that f can only have two roots in $(0, 2)$.

To use Newton's method to approximate the roots, we repeatedly apply the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 4x_n + 1}{3x_n^2 - 4}.$$

Starting with the initial guess $x_1 = 0$, one obtains the approximations

$$x_1 = 0, \qquad x_2 = 0.25, \qquad x_3 = 0.2540983607, \qquad x_4 = 0.2541016884.$$

Starting with the initial guess $x_1 = 2$, one obtains the approximations

$$x_1 = 2, \qquad x_2 = 1.875, \qquad x_3 = 1.860978520, \qquad x_4 = 1.860805879.$$

This suggests that the two roots are roughly 0.25 and 1.86 within two decimal places.

MA1125 – Calculus
Homework #7 solutions

1. Find the area of the region enclosed by the graphs of $f(x) = 2x^2$ and $g(x) = x + 6$.

The graph of the parabola $f(x) = 2x^2$ meets the graph of the line $g(x) = x + 6$ when

$$2x^2 = x + 6 \iff 2x^2 - x - 6 = 0 \iff (2x + 3)(x - 2) = 0.$$

Since the line lies above the parabola at the points $-3/2 \leq x \leq 2$, the area is then

$$\int_{-3/2}^{2} [g(x) - f(x)]\, dx = \int_{-3/2}^{2} \left[x + 6 - 2x^2\right] dx = \left[\frac{x^2}{2} + 6x - \frac{2x^3}{3}\right]_{-3/2}^{2} = \frac{343}{24}.$$

2. Compute the volume of a sphere of radius $r > 0$. Hint: one may obtain such a sphere by rotating the upper semicircle $f(x) = \sqrt{r^2 - x^2}$ around the x-axis.

The volume of the sphere is the integral of $\pi f(x)^2$ and this is given by

$$\int_{-r}^{r} \pi(r^2 - x^2)\, dx = \pi \left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r} = \pi \left(\frac{2r^3}{3} + \frac{2r^3}{3}\right) = \frac{4\pi r^3}{3}.$$

3. Compute the length of the graph of $f(x) = \frac{1}{3}x^{3/2}$ over the interval $[0, 5]$.

The length of the graph is given by the integral of $\sqrt{1 + f'(x)^2}$ and one has

$$f'(x) = \frac{1}{3} \cdot \frac{3}{2} \cdot x^{1/2} = \frac{\sqrt{x}}{2} \implies 1 + f'(x)^2 = 1 + \frac{x}{4} = \frac{x + 4}{4}.$$

Taking the square root of both sides, we conclude that the length of the graph is

$$\int_0^5 \sqrt{1 + f'(x)^2}\, dx = \int_0^5 \frac{(x + 4)^{1/2}}{2}\, dx = \left[\frac{(x + 4)^{3/2}}{3}\right]_0^5 = \frac{3^3 - 2^3}{3} = \frac{19}{3}.$$

4. Find both the mass and the centre of mass for a thin rod whose density is given by

$$\delta(x) = x^3 + 2x^2 + 5x, \qquad 0 \le x \le 2.$$

The mass of the rod is merely the integral of its density function, namely

$$M = \int_0^2 \delta(x)\,dx = \int_0^2 (x^3 + 2x^2 + 5x)\,dx = \left[\frac{x^4}{4} + \frac{2x^3}{3} + \frac{5x^2}{2}\right]_0^2 = \frac{58}{3}.$$

The centre of mass is given by a similar formula and one finds that

$$\overline{x} = \frac{1}{M}\int_0^2 x\delta(x)\,dx = \frac{3}{58}\int_0^2 (x^4 + 2x^3 + 5x^2)\,dx = \frac{3}{58}\left[\frac{x^5}{5} + \frac{x^4}{2} + \frac{5x^3}{3}\right]_0^2 = \frac{208}{145}.$$

5. A cylindrical tank of radius 2m and height 3m is full with water of density 1000kg/m^3. How much work is needed to pump out the water through a hole in the top?

Consider a cross section of the tank which has arbitrarily small height dx and lies x metres from the top. The volume of this cylindrical cross section is

$$V = \pi \cdot \text{radius}^2 \cdot \text{height} = 4\pi\,dx.$$

Its mass is volume times density, namely $m = 4000\pi\,dx$, and the force needed to pump out this part is mass times acceleration, namely mg. The overall amount of work is thus

$$\text{Work} = \int mg \cdot x = \int_0^3 4{,}000\pi g \cdot x\,dx = 4{,}000\pi g \left[\frac{x^2}{2}\right]_0^3 = 18{,}000\pi g.$$

MA1125 – Calculus
Homework #8 solutions

1. Compute each of the following indefinite integrals.

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx, \qquad \int x\sqrt{1-x}\,dx.$$

For the first integral, we let $u=\sqrt{x}$. This gives $x=u^2$ and $dx=2u\,du$, so

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int \frac{\sin u}{u}\cdot 2u\,du = \int 2\sin u\,du = -2\cos u + C = -2\cos\sqrt{x} + C.$$

For the second integral, we let $u=1-x$. This gives $x=1-u$ and $dx=-du$, so

$$\begin{aligned}\int x\sqrt{1-x}\,dx &= -\int (1-u)\sqrt{u}\,du = \int (u^{3/2}-u^{1/2})\,du \\ &= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C = \frac{2}{5}(1-x)^{5/2} - \frac{2}{3}(1-x)^{3/2} + C.\end{aligned}$$

2. Compute each of the following indefinite integrals.

$$\int \sin^3 x\cdot\cos^4 x\,dx, \qquad \int \tan^2 x\cdot\sec^6 x\,dx.$$

For the first integral, we use the substitution $u=\cos x$. Since $du=-\sin x\,dx$, we get

$$\begin{aligned}\int \sin^3 x\cdot\cos^4 x\,dx &= \int \sin^2 x\cdot\cos^4 x\cdot\sin x\,dx = -\int (1-u^2)\cdot u^4\,du \\ &= \int (u^6-u^4)\,du = \frac{1}{7}u^7 - \frac{1}{5}u^5 + C = \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C.\end{aligned}$$

For the second integral, we use the substitution $u=\tan x$. Since $du=\sec^2 x\,dx$, we get

$$\begin{aligned}\int \tan^2 x\cdot\sec^6 x\,dx &= \int \tan^2 x\cdot\sec^4 x\cdot\sec^2 x\,dx = \int u^2(1+u^2)^2\,du \\ &= \int (u^2+u^6+2u^4)\,du = \frac{1}{3}u^3 + \frac{1}{7}u^7 + \frac{2}{5}u^5 + C \\ &= \frac{\tan^3 x}{3} + \frac{\tan^7 x}{7} + \frac{2\tan^5 x}{5} + C.\end{aligned}$$

3. Compute each of the following indefinite integrals.

$$\int x^3(\ln x)^2\,dx, \qquad \int x^3\sqrt{4-x^2}\,dx.$$

For the first integral, let $u = (\ln x)^2$ and $dv = x^3\,dx$. Then $du = \frac{2\ln x}{x}\,dx$ and $v = \frac{x^4}{4}$, so

$$\int x^3(\ln x)^2\,dx = \frac{x^4}{4}(\ln x)^2 - \int \frac{2\ln x}{x}\cdot\frac{x^4}{4}\,dx = \frac{x^4}{4}(\ln x)^2 - \frac{1}{2}\int x^3(\ln x)\,dx.$$

Next, we take $u = \ln x$ and $dv = x^3\,dx$. Since $du = \frac{1}{x}\,dx$ and $v = \frac{x^4}{4}$, we conclude that

$$\int x^3(\ln x)^2\,dx = \frac{x^4}{4}(\ln x)^2 - \frac{x^4}{8}\ln x + \int \frac{x^3}{8}\,dx = \frac{x^4}{4}(\ln x)^2 - \frac{x^4}{8}\ln x + \frac{x^4}{32} + C.$$

For the second integral, let $x = 2\sin\theta$ for some angle $-\pi/2 \le \theta \le \pi/2$. Then

$$\int x^3\sqrt{4-x^2}\,dx = \int 8\sin^3\theta\cdot\sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,d\theta = 32\int \sin^3\theta\cdot\cos^2\theta\,d\theta.$$

This can be further simplified by letting $u = \cos\theta$, in which case $du = -\sin\theta\,d\theta$ and

$$\begin{aligned}\int x^3\sqrt{4-x^2}\,dx &= -32\int (1-u^2)\cdot u^2\,du = 32\int (u^4-u^2)\,du\\ &= 32\left(\frac{1}{5}u^5 - \frac{1}{3}u^3\right) = 32\left(\frac{\cos^5\theta}{5} - \frac{\cos^3\theta}{3}\right) + C.\end{aligned}$$

Since $4\cos^2\theta = 4 - 4\sin^2\theta = 4 - x^2$, we also have $\cos\theta = \frac{1}{2}(4-x^2)^{1/2}$ and so

$$\int x^3\sqrt{4-x^2}\,dx = \frac{(4-x^2)^{5/2}}{5} - \frac{4(4-x^2)^{3/2}}{3} + C.$$

4. Find the area of the region enclosed by the graphs of $f(x) = e^{2x}$ and $g(x) = 3e^x - 2$.

Letting $z = e^x$ for simplicity, we get $f(x) = z^2$ and $g(x) = 3z - 2$. It easily follows that

$$f(x) \le g(x) \iff z^2 \le 3z - 2 \iff (z-2)(z-1) \le 0 \iff 1 \le z \le 2.$$

In other words, $f(x) \le g(x)$ if and only if $0 \le x \le \ln 2$, so the area of the region is

$$\begin{aligned}\text{Area} &= \int_0^{\ln 2} [g(x) - f(x)]\,dx = \int_0^{\ln 2} (3e^x - 2 - e^{2x})\,dx\\ &= \left[3e^x - 2x - \frac{1}{2}e^{2x}\right]_0^{\ln 2} = \frac{3}{2} - 2\ln 2.\end{aligned}$$

5. Find the volume of the solid that is obtained by rotating the graph of $f(x) = xe^x$ around the x-axis over the interval $[0, 1]$.

The volume of the solid is the integral of $\pi f(x)^2$ and this is given by

$$\text{Volume} = \pi \int_0^1 x^2 e^{2x}\, dx.$$

To simplify this expression, let $u = x^2$ and $dv = e^{2x}\, dx$. Then $du = 2x\, dx$ and $v = \frac{1}{2}e^{2x}$, so

$$\int_0^1 x^2 e^{2x}\, dx = \left[\frac{x^2}{2}e^{2x}\right]_0^1 - \int_0^1 xe^{2x}\, dx.$$

Once again, we take $u = x$ and $dv = e^{2x}\, dx$. Then $du = dx$ and $v = \frac{1}{2}e^{2x}$, so

$$\begin{aligned}\int_0^1 x^2 e^{2x}\, dx &= \left[\frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x}\right]_0^1 + \frac{1}{2}\int_0^1 e^{2x}\, dx \\ &= \left[\frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{1}{4}e^{2x}\right]_0^1.\end{aligned}$$

The volume of the solid is given by π times the last integral, so it is given by

$$\text{Volume} = \pi\left[\frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{1}{4}e^{2x}\right]_0^1 = \frac{\pi(e^2 - 1)}{4}.$$

MA1125 – Calculus
Homework #9 solutions

1. Compute each of the following indefinite integrals.

$$\int \frac{x^3 - 3x + 2}{x+3}\, dx, \qquad \int \frac{x+3}{x^3 - 3x + 2}\, dx.$$

When it comes to the first integral, one may use division of polynomials to find that

$$\begin{aligned} \int \frac{x^3 - 3x + 2}{x+3}\, dx &= \int \left(x^2 - 3x + 6 - \frac{16}{x+3} \right) dx \\ &= \frac{x^3}{3} - \frac{3x^2}{2} + 6x - 16 \ln|x+3| + C. \end{aligned}$$

When it comes to the second integral, one may use partial fractions to write

$$\frac{x+3}{x^3 - 3x + 2} = \frac{x+3}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

for some constants A, B and C. Clearing denominators gives rise to the identity

$$x + 3 = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

and this should be valid for all x. Let us then look at some special values of x to get

$$x = -2, 1, 0 \quad \Longrightarrow \quad 1 = 9C, \qquad 4 = 3B, \qquad 3 = -2A + 2B + C.$$

This gives $C = 1/9$, $B = 4/3$ and $2A = 2B + C - 3 = 8/3 + 1/9 - 3 = -2/9$, so

$$\begin{aligned} \int \frac{x+3}{x^3 - 3x + 2}\, dx &= \int \left(-\frac{1/9}{x-1} + \frac{4/3}{(x-1)^2} + \frac{1/9}{x+2} \right) dx \\ &= -\frac{1}{9} \ln|x-1| - \frac{4}{3}(x-1)^{-1} + \frac{1}{9} \ln|x+2| + C. \end{aligned}$$

2. Compute each of the following indefinite integrals.

$$\int \frac{x+3}{x + \sqrt{x}}\, dx, \qquad \int \frac{e^x + 3}{e^x + 1}\, dx.$$

In the first case, we let $u = \sqrt{x}$ to simplify. Since $x = u^2$, we have $dx = 2u\, du$ and

$$\int \frac{x+3}{x + \sqrt{x}}\, dx = \int \frac{u^2 + 3}{u^2 + u} \cdot 2u\, du = 2 \int \frac{u^2 + 3}{u + 1}\, du.$$

This is a rational function that can be simplified using division of polynomials, so

$$\int \frac{x+3}{x+\sqrt{x}}\,dx = 2\int \frac{u^2-1+4}{u+1}\,du = 2\int \left(u-1+\frac{4}{u+1}\right) du$$
$$= u^2 - 2u + 8\ln|u+1| + C = x - 2\sqrt{x} + 8\ln(\sqrt{x}+1) + C.$$

For the second integral, we proceed similarly with $u = e^x$. Since $du = e^x\,dx$, we find that

$$\int \frac{e^x+3}{e^x+1}\,dx = \int \frac{e^x+3}{e^x(e^x+1)} \cdot e^x\,dx = \int \frac{u+3}{u(u+1)}\,du.$$

In this case, however, one needs to use partial fractions to write

$$\frac{u+3}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$$

for some constants A, B that need to be determined. Clearing denominators gives

$$u + 3 = A(u+1) + Bu,$$

so one may take $u = -1, 0$ to find that $2 = -B$ and $3 = A$. It easily follows that

$$\int \frac{e^x+3}{e^x+1}\,dx = \int \left(\frac{3}{u} - \frac{2}{u+1}\right) du = 3\ln|u| - 2\ln|u+1| + C$$
$$= 3x - 2\ln(e^x+1) + C.$$

3. Compute each of the following indefinite integrals.

$$\int \frac{\sin^3 x}{\cos^8 x}\,dx, \qquad \int \frac{3x+1}{x^2+2x+5}\,dx.$$

For the first integral, it is better to simplify the given expression and write

$$\int \frac{\sin^3 x}{\cos^8 x}\,dx = \int \frac{\tan^3 x}{\cos^5 x}\,dx = \int \sec^5 x \cdot \tan^3 x\,dx.$$

To compute this integral, we let $u = \sec x$. Then $du = \sec x \tan x\,dx$ and we get

$$\int \frac{\sin^3 x}{\cos^8 x}\,dx = \int \sec^4 x \cdot \tan^2 x \cdot \sec x \tan x\,dx = \int u^4(u^2-1)\,du$$
$$= \int (u^6 - u^4)\,du = \frac{1}{7}u^7 - \frac{1}{5}u^5 + C = \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C.$$

For the second integral, the substitution $u = x^2 + 2x + 5$ gives $du = 2(x+1)\,dx$ and this is helpful whenever the numerator is a multiple of $x + 1$. Let us then split the integral as

$$\int \frac{3x+1}{x^2+2x+5}\,dx = \int \frac{3(x+1)-2}{x^2+2x+5}\,dx$$
$$= 3\int \frac{x+1}{x^2+2x+5}\,dx - 2\int \frac{dx}{x^2+2x+5}.$$

For the first part, one may use the substitution $u = x^2 + 2x + 5$ to find that

$$3\int \frac{x+1}{x^2+2x+5}\,dx = \frac{3}{2}\int \frac{du}{u} = \frac{3}{2}\ln|u| + C = \frac{3}{2}\ln|x^2+2x+5| + C.$$

For the second part, one may complete the square and let $u = (x+1)/2$. This gives

$$2\int \frac{dx}{x^2+2x+5} = 2\int \frac{dx}{(x+1)^2+4} = 4\int \frac{du}{4u^2+4}$$
$$= \tan^{-1} u + C = \tan^{-1}\frac{x+1}{2} + C.$$

As for the original integral, this is merely the difference of the two parts, namely

$$\int \frac{3x+1}{x^2+2x+5}\,dx = \frac{3}{2}\ln|x^2+2x+5| - \tan^{-1}\frac{x+1}{2} + C.$$

4. Show that each of the following sequences converges.

$$a_n = \cos\frac{n^2+2}{n^3+1}, \qquad b_n = \frac{(-1)^n}{n^2}, \qquad c_n = n\sin\frac{1}{n}.$$

Since the limit of a cosine is the cosine of the limit, it should be clear that

$$\lim_{n\to\infty}\frac{n^2+2}{n^3+1} = \lim_{n\to\infty}\frac{n^2}{n^3} = \lim_{n\to\infty}\frac{1}{n} = 0 \quad\Longrightarrow\quad \lim_{n\to\infty} a_n = \cos 0 = 1.$$

The limit of the second sequence is zero because $-1/n^2 \leq b_n \leq 1/n^2$ for each $n \geq 1$. This means that b_n lies between two sequences that converge to zero. Finally, one has

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} n\sin\frac{1}{n} = \lim_{n\to\infty}\frac{\sin(1/n)}{1/n}.$$

This is a limit of the form $0/0$, so one may use L'Hôpital's rule to conclude that

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty}\frac{\cos(1/n)\cdot(1/n)'}{(1/n)'} = \lim_{n\to\infty}\cos(1/n) = \cos 0 = 1.$$

5. Define a sequence $\{a_n\}$ by setting $a_1 = 1$ and $a_{n+1} = \sqrt{2a_n + 1}$ for each $n \geq 1$. Show that $1 \leq a_n \leq a_{n+1} \leq 3$ for each $n \geq 1$, use this fact to conclude that the sequence converges and then find its limit.

Since the first two terms are $a_1 = 1$ and $a_2 = \sqrt{3}$, the statement

$$1 \leq a_n \leq a_{n+1} \leq 3$$

does hold when $n = 1$. Suppose that it holds for some n, in which case

$$\begin{aligned} 2 \leq 2a_n \leq 2a_{n+1} \leq 6 \quad &\implies \quad 3 \leq 2a_n + 1 \leq 2a_{n+1} + 1 \leq 7 \\ &\implies \quad \sqrt{3} \leq a_{n+1} \leq a_{n+2} \leq \sqrt{7} \\ &\implies \quad 1 \leq a_{n+1} \leq a_{n+2} \leq 3. \end{aligned}$$

In particular, the statement holds for $n + 1$ as well, so it actually holds for all $n \in \mathbb{N}$. This shows that the given sequence is monotonic and bounded, hence also convergent; denote its limit by L. Using the definition of the sequence, we then find that

$$a_{n+1} = \sqrt{2a_n + 1} \quad \implies \quad \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2a_n + 1} \quad \implies \quad L = \sqrt{2L + 1}.$$

This gives the quadratic equation $L^2 = 2L + 1$ which implies that $L = 1 \pm \sqrt{2}$. Since the terms of the sequence satisfy $1 \leq a_n \leq 3$, however, the limit must be $L = 1 + \sqrt{2}$.