1 Basic set theory

- We use capital letters to denote sets and lowercase letters to denote their elements.
- We write $A \subseteq B$ whenever every element of $A$ is also an element of $B$.
- The union $A \cup B$ of two sets consists of the elements $x$ with $x \in A$ or $x \in B$.
- The intersection $A \cap B$ of two sets consists of the elements $x$ with $x \in A$ and $x \in B$.
- The difference $A - B$ of two sets consists of the elements $x$ with $x \in A$, but $x \notin B$.

### Theorem 1.1 – De Morgan’s laws

The difference of a union/intersection is the intersection/union of the differences, namely

$$A - (B \cup C) = (A - B) \cap (A - C), \quad A - (B \cap C) = (A - B) \cup (A - C).$$

**Proof.** To prove the statement about the difference of a union, one argues that

\[
x \in A - (B \cup C) \iff x \in A, \text{ but } x \notin B \cup C
\]

\[
\iff x \in A, \text{ but } x \notin B \text{ and } x \notin C
\]

\[
\iff x \in A - B \text{ and } x \in A - C
\]

\[
\iff x \in (A - B) \cap (A - C).
\]

Since the difference of an intersection can be treated similarly, we omit the details. \hfill \blacksquare

### Definition 1.2 – Image of a set

Given a function $f: A \to B$ and a set $A_1 \subseteq A$, we define $f(A_1) = \{ f(x) : x \in A_1 \}$.

### Theorem 1.3 – Properties of images

Let $f: A \to B$ be a function and let $A_1, A_2 \subseteq A$ be arbitrary.

(a) If $A_1 \subseteq A_2$, then $f(A_1) \subseteq f(A_2)$.

(b) One has $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

(c) One has $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$ and equality holds when $f$ is injective.

(d) One has $f(A_1 - A_2) \supseteq f(A_1) - f(A_2)$ and equality holds when $f$ is injective.

- Thus, images preserve inclusions and unions, but not intersections and differences.
Proof. To prove the first part, suppose that \( A_1 \subseteq A_2 \). We then have
\[
y \in f(A_1) \implies y = f(x) \text{ for some } x \in A_1
\]
\[
\implies y = f(x) \text{ for some } x \in A_2
\]
\[
\implies y \in f(A_2).
\]
This implies that \( f(A_1) \subseteq f(A_2) \), as needed. To prove the second part, we note that
\[
y \in f(A_1 \cup A_2) \iff y = f(x) \text{ for some } x \in A_1 \cup A_2
\]
\[
\iff y = f(x) \text{ for some } x \in A_1 \text{ or some } x \in A_2
\]
\[
\iff y \in f(A_1) \text{ or } y \in f(A_2)
\]
\[
\iff y \in f(A_1) \cup f(A_2).
\]
Next, we turn to the third part. To prove the inclusion, one argues that
\[
y \in f(A_1 \cap A_2) \implies y = f(x) \text{ for some } x \in A_1 \cap A_2
\]
\[
\implies y = f(x) \text{ with } x \in A_1 \text{ and } x \in A_2
\]
\[
\implies y \in f(A_1) \text{ and } y \in f(A_2)
\]
\[
\implies y \in f(A_1) \cap f(A_2).
\]
This shows that \( f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2) \), as needed. If it happens that \( f \) is injective, then we can also establish the opposite inclusion. In that case, one has
\[
y \in f(A_1) \cap f(A_2) \implies y = f(x_1) \text{ for some } x_1 \in A_1 \text{ and } y = f(x_2) \text{ for some } x_2 \in A_2
\]
\[
\implies y = f(x_1) = f(x_2) \text{ with } x_1 \in A_1 \text{ and } x_2 \in A_2
\]
\[
\implies y = f(x_1) = f(x_2) \text{ with } x_1 = x_2 \in A_1 \cap A_2 \text{ (by injectivity)}
\]
\[
\implies y \in f(A_1 \cap A_2).
\]
This completes the proof of the third part. The proof of the last part is quite similar. \(\blacksquare\)

Example 1.4 Consider the case \( f(x) = x^2 \). If we take \( A_1 = [-1, 0] \) and \( A_2 = [0, 1] \), then
\[
A_1 \cap A_2 = \{0\}, \quad f(A_1 \cap A_2) = \{0\}, \quad f(A_1) = [0, 1] = f(A_2).
\]
In particular, \( f(A_1 \cap A_2) = \{0\} \) is a proper subset of \( f(A_1) \cap f(A_2) = [0, 1] \). Similarly,
\[
A_1 - A_2 = [-1, 0), \quad f(A_1 - A_2) = (0, 1], \quad f(A_1) - f(A_2) = \emptyset
\]
and so \( f(A_1) - f(A_2) \) could be a proper subset of \( f(A_1 - A_2) \) when \( f \) is not injective. \(\square\)

**Definition 1.5 – Inverse image of a set**

Given a function \( f: A \to B \) and a set \( B_1 \subseteq B \), we define its inverse image by
\[
f^{-1}(B_1) = \{x \in A : f(x) \in B_1\}.
\]
This set is defined for any function \( f \). In particular, \( f \) does not need to be bijective.
Example 1.6 Consider the case $f(x) = x^2$. The inverse image of $B_1 = [-2, -1]$ is then

$$f^{-1}(B_1) = \{ x \in \mathbb{R} : x^2 \in B_1 \} = \{ x \in \mathbb{R} : -2 \leq x^2 \leq -1 \} = \emptyset.$$ 

On the other hand, the inverse image of $B_2 = [1, 4]$ can be computed as

$$f^{-1}(B_2) = \{ x \in \mathbb{R} : x^2 \in B_2 \} = \{ x \in \mathbb{R} : 1 \leq x^2 \leq 4 \} = [1, 2] \cup [-2, -1].$$

Theorem 1.7 – Properties of inverse images

Let $f : A \to B$ be a function and let $B_1, B_2 \subseteq B$ be arbitrary.

(a) If $B_1 \subseteq B_2$, then $f^{-1}(B_1) \subseteq f^{-1}(B_2)$.

(b) One has $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.

(c) One has $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

(d) One has $f^{-1}(B_1 - B_2) = f^{-1}(B_1) - f^{-1}(B_2)$.

- Thus, inverse images preserve inclusions, unions, intersections and also differences.

Proof. To prove the first part, we assume that $B_1 \subseteq B_2$ and we note that

$$x \in f^{-1}(B_1) \implies f(x) \in B_1 \implies f(x) \in B_2 \implies x \in f^{-1}(B_2).$$

This implies that $f^{-1}(B_1) \subseteq f^{-1}(B_2)$, as needed. For the second part, one has

$$x \in f^{-1}(B_1 \cup B_2) \iff f(x) \in B_1 \cup B_2 \iff f(x) \in B_1 \text{ or } f(x) \in B_2 \iff x \in f^{-1}(B_1) \text{ or } x \in f^{-1}(B_2) \iff x \in f^{-1}(B_1) \cup f^{-1}(B_2).$$

This proves the statement in the second part, while the other two parts are similar. ■

Theorem 1.8 – Images and inverse images

Let $f : A \to B$ be a function. Let $A_1 \subseteq A$ and $B_1 \subseteq B$ be arbitrary.

(a) One has $f^{-1}(f(A_1)) \supseteq A_1$ and equality holds whenever $f$ is injective.

(b) One has $f(f^{-1}(B_1)) \subseteq B_1$ and equality holds whenever $f$ is surjective.

Proof. We only establish part (b), as part (a) is similar. First of all, we note that

$$y \in f(f^{-1}(B_1)) \implies y = f(x) \text{ for some } x \in f^{-1}(B_1) \implies y = f(x) \text{ and also } f(x) \in B_1 \implies y \in B_1.$$
This proves the inclusion \( f(f^{-1}(B_1)) \subseteq B_1 \). If we also assume that \( f \) is surjective, then

\[
y \in B_1 \implies y = f(x) \text{ for some } x \in A \text{ (by surjectivity)}
\]

\[
\implies y = f(x) \text{ for some } x \in A \text{ and } f(x) \in B_1
\]

\[
\implies y = f(x) \text{ and } x \in f^{-1}(B_1)
\]

\[
\implies y \in f(f^{-1}(B_1)).
\]

Thus, the inclusion \( B_1 \subseteq f(f^{-1}(B_1)) \) also holds and the two sets are actually equal. \( \blacksquare \)

**Example 1.9** Consider the case \( f(x) = x^2 \). If we take \( A_1 = [0, 1] \) and \( B_1 = [-1, 1] \), then

\[
f(A_1) = [0, 1] \implies f^{-1}(f(A_1)) = \{x \in \mathbb{R} : 0 \leq x^2 \leq 1\} = [-1, 1].
\]

In particular, \( A_1 \) is a proper subset of \( f^{-1}(f(A_1)) \) and one similarly has

\[
f^{-1}(B_1) = \{x \in \mathbb{R} : -1 \leq x^2 \leq 1\} = [-1, 1] \implies f(f^{-1}(B_1)) = [0, 1] \neq B_1.
\]

\( \square \)

## 2 Infimum and supremum

**Definition 2.1 – Minimum and maximum**

If a set \( A \subseteq \mathbb{R} \) has a smallest element, then we call that element the minimum of \( A \) and we denote it by \( \min A \). If a set \( A \subseteq \mathbb{R} \) has a largest element, then we call that element the maximum of \( A \) and we denote it by \( \max A \).

**Example 2.2** When it comes to the interval \( A = [1, 2] \), one has \( \min A = 1 \) and \( \max A = 2 \). When it comes to the interval \( B = [1, 2) \), however, \( \min B = 1 \) and \( \max B \) does not exist. \( \square \)

**Example 2.3** Consider the set \( A = \{\frac{1}{n} : n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \). To show that \( \max A = 1 \), one checks that \( 1 \) is an element of \( A \) and that \( 1 \) is at least as large as any other element. In this case, it is clear that \( 1 \in A \), while \( 1 \geq x \) for all \( x \in A \) because \( 1 \geq \frac{1}{n} \) for all \( n \in \mathbb{N} \). \( \square \)

**Example 2.4** Consider the set \( A = \{\frac{1}{n} : n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \) as before. To show that \( A \) has no minimum, one checks that \( A \) has no smallest element. Given any element of \( A \), we must thus be able to find another element of \( A \) which is smaller. Now, let \( x \in A \) be given. Then \( x = \frac{1}{n} \) for some \( n \in \mathbb{N} \) and \( y = \frac{1}{n+1} \) is an element of \( A \) such that \( y < x \). This shows that the original element \( x \) was not the smallest, so \( A \) does not have a minimum. \( \square \)

**Definition 2.5 – Upper bounds and supremum**

We say that \( A \subseteq \mathbb{R} \) is bounded from above, if there exists a number \( x \) such that \( x \geq a \) for all \( a \in A \). In that case, we also say that \( x \) is an upper bound of \( A \). The least upper bound of \( A \) is called the supremum of \( A \) and it is denoted by \( \sup A \).

- Both the maximum and the supremum of \( A \) must be at least as large as all elements of \( A \). However, \( \max A \) must itself be an element of \( A \), whereas \( \sup A \) need not be.
**Axiom of completeness**

If $A \subseteq \mathbb{R}$ is nonempty and bounded from above, then $A$ has a least upper bound.

**Example 2.6** We show that the interval $A = (-\infty, 1)$ has no maximum. Indeed, let $x \in A$ be given and note that $x < 1$. The number $y = \frac{x + 1}{2}$ is the average of $x$ and $1$ which is easily seen to satisfy $x < y < 1$. This implies that $y$ is an element of $A$ which is larger than the original element $x$. Thus, $x$ was not the largest element and $A$ has no maximum. $\Box$

**Example 2.7** Consider the interval $A = (-\infty, 1)$ once again. Upper bounds of $A$ must be at least as large as every element of $A$, so the least upper bound should be sup $A = 1$. To prove this, we check (a) that 1 is an upper bound of $A$ and (b) that 1 is the least upper bound. The first part is clear, as $1 \geq a$ for all $a \in A$. To establish the second part, we need to show that no number $x < 1$ is an upper bound of $A$. Given any $x < 1$, we must thus be able to find an element of $A$ which is bigger than $x$. If we let $y = \frac{x + 1}{2}$ once again, then we have $x < y < 1$ and so $y$ is an element of $A$ which is bigger than $x$, as needed. $\Box$

**Definition 2.8 – Lower bounds and infimum**

We say that $A \subseteq \mathbb{R}$ is bounded from below, if there exists a number $x$ such that $x \leq a$ for all $a \in A$. In that case, we also say that $x$ is a lower bound of $A$. The greatest lower bound of $A$ is called the infimum of $A$ and it is denoted by inf $A$.

- Both the minimum and the infimum of $A$ must be at least as small as all elements of $A$. However, min $A$ must itself be an element of $A$, whereas inf $A$ need not be.

**Example 2.9** It is easy to see that $A = (0, \infty)$ has no minimum. Given any element $x \in A$, one has $x > 0$ and then $y = \frac{x}{2}$ satisfies $0 < y < x$, so it is an element of $A$ which is smaller than $x$. To show that the infimum of $A$ is inf $A = 0$, one needs to check (a) that 0 is a lower bound of $A$ and (b) that 0 is the greatest lower bound. The first part is clear, as $0 \leq a$ for all $a \in A$. To establish the second part, we need to show that no number $z > 0$ is a lower bound of $A$. Given any $z > 0$, we must thus be able to find an element of $A$ which is smaller than $z$. In fact, $y = \frac{z}{2}$ is such an element because $0 < y < z$, so $y \in A$ and also $y < z$. $\Box$

**Theorem 2.10 – Relation between inf/min and sup/max**

Suppose that $A$ is a nonempty subset of $\mathbb{R}$.

(a) If min $A$ exists, then inf $A$ also exists and the two are equal. If inf $A$ exists and it happens to be an element of $A$, then min $A$ exists and the two are equal.

(b) If max $A$ exists, then sup $A$ also exists and the two are equal. If sup $A$ exists and it happens to be an element of $A$, then max $A$ exists and the two are equal.
Proof. We only prove the first part, as the second part is similar. If min \( A \) exists, then \( \min A \leq x \) for all \( x \in A \) and so \( \min A \) is a lower bound of \( A \). To show that it is the greatest lower bound, suppose \( y > \min A \). Then \( \min A \) is an element of \( A \) which is smaller than \( y \), so \( y \) is not a lower bound of \( A \) and the greatest lower bound is \( \min A \).

Similarly, suppose that \( \inf A \) exists and that \( \inf A \in A \). Then \( \inf A \leq x \) for all \( x \in A \) and \( \inf A \) is itself an element of \( A \), so \( \inf A \) is the smallest element of \( A \). ■

Theorem 2.11 – Existence of infimum

If \( A \subseteq \mathbb{R} \) is nonempty and bounded from below, then \( A \) has a greatest lower bound.

Proof. We consider the set \( B = \{ x \in \mathbb{R} : -x \in A \} \). This consists of the negatives of the elements of \( A \), so any lower bound of \( A \) should be an upper bound of \( B \) and vice versa.

First of all, we show that \( B \) is bounded from above. Since \( A \) is bounded from below, there exists a real number \( z \) such that \( z \leq a \) for all \( a \in A \). This implies that \( -z \geq -a \) for all \( a \in A \), so \( -z \geq b \) for all \( b \in B \). We conclude that \( -z \) is an upper bound of \( B \).

Since \( B \) is bounded from above, \( \sup B \) exists by the axiom of completeness. We now show that \( -\sup B \) is the greatest lower bound of \( A \). In fact, we have

\[
\sup B \geq b \text{ for all } b \in B \implies -\sup B \leq -b \text{ for all } b \in B \\
-\sup B \leq a \text{ for all } a \in A
\]

and this means that \( -\sup B \) is a lower bound of \( A \). To show that it is the greatest one, suppose \( z > -\sup B \) and note that \( -z < \sup B \). Then \( -z \) is not an upper bound of \( B \), so there exists some \( b \in B \) such that \( -z < b \). This gives \( z > -b \), so \( -b \) is an element of \( A \) which is smaller than \( z \). In particular, \( z \) is not a lower bound of \( A \), as needed. ■

Theorem 2.12 – Inf/Sup of a subset

(a) Suppose that \( A \subseteq \mathbb{R} \) is nonempty and bounded from below. If \( B \subseteq A \), then \( B \) is bounded from below as well and one has \( \inf B \geq \inf A \).

(b) Suppose that \( A \subseteq \mathbb{R} \) is nonempty and bounded from above. If \( B \subseteq A \), then \( B \) is bounded from above as well and one has \( \sup B \leq \sup A \).

• Plainly stated, larger sets must have a larger supremum, but a smaller infimum.

Proof. We only prove the first part, as the second part is similar. Since \( A \) has a lower bound by assumption, its infimum \( \inf A \) exists and one has

\[
\inf A \leq x \text{ for all } x \in A \implies \inf A \leq x \text{ for all } x \in B.
\]

Thus, \( \inf A \) is a lower bound of \( B \), so \( B \) is bounded from below and \( \inf B \) exists. As \( \inf A \) is a lower bound of \( B \) and \( \inf B \) is the greatest lower bound of \( B \), one has \( \inf A \leq \inf B \). ■
Theorem 2.13 – Archimedean property

The set $\mathbb{N}$ of natural numbers is not bounded from above. Given any real number $x$, that is, there exists a natural number $n$ such that $n > x$.

Proof. To prove the first statement, suppose $\mathbb{N}$ is bounded from above and let $\alpha = \sup \mathbb{N}$ be its least upper bound. Since $\alpha - 1$ is smaller, it is not an upper bound of $\mathbb{N}$, so there exists some $x \in \mathbb{N}$ such that $\alpha - 1 < x$. This gives $x + 1 > \alpha$ which means that $x + 1$ is a natural number that is actually larger than $\alpha = \sup \mathbb{N}$, a contradiction.

To prove the second statement, suppose $n \leq x$ for all $n \in \mathbb{N}$. Then $x$ is an upper bound of $\mathbb{N}$ and this contradicts the first statement. Thus, there exists $n \in \mathbb{N}$ such that $n > x$. ■

Example 2.14 Consider the set $A = \{ \frac{2n+1}{n+3} : n \in \mathbb{N} \}$. To show that $\sup A = 2$, we check that 2 is an upper bound and that it is the least upper bound. The first part is clear, as

$$2 \geq \frac{2n+1}{n+3} \quad \iff \quad 2n + 6 \geq 2n + 1 \quad \iff \quad 6 \geq 1.$$  

To check the second part, suppose that $x < 2$. We need to find an element of $A$ which is larger than $x$ and this amounts to ensuring that $\frac{2n+1}{n+3} > x$. On the other hand, one has

$$\frac{2n+1}{n+3} > x \quad \iff \quad 2n + 1 > nx + 3x$$

$$\iff \quad (2-x)n > 3x - 1 \quad \iff \quad n > \frac{3x - 1}{2-x}.$$  

Pick a natural number $n$ that satisfies the rightmost inequality. Then $\frac{2n+1}{n+3} > x$, so there is an element of $A$ which is larger than $x$. This shows that $x$ is not an upper bound of $A$. ■

Theorem 2.15 – Nonempty subsets of $\mathbb{N}$

Every nonempty subset of $\mathbb{N}$ must have a minimum.

Proof. Suppose that $A \subseteq \mathbb{N}$ is nonempty. Since $x \geq 1$ for all $x \in A$, the set $A$ is then bounded from below and $\inf A$ exists. If we can show that $\inf A \in A$, then $\min A$ also exists and the two are equal. Thus, it suffices to show that $\inf A \in A$.

Since $\inf A + 1 > \inf A$, there exists an element $x \in A$ such that $\inf A \leq x < \inf A + 1$. If it happens that $\inf A = x$, then $\inf A \in A$ and the proof is complete. Otherwise, we must have $\inf A < x$ and we may proceed as before to find some element $y \in A$ such that

$$\inf A \leq y < x \quad \implies \quad \inf A \leq y < x < \inf A + 1.$$  

This is impossible because two integers $x, y$ cannot lie in an interval of length 1. ■
Consider a statement \( P(n) \) involving the natural numbers \( n \in \mathbb{N} \). Suppose that \( P(1) \) holds and that \( P(n) \) implies \( P(n + 1) \) for each \( n \in \mathbb{N} \). Then \( P(n) \) holds for all \( n \in \mathbb{N} \).

Proof. We study the set \( A = \{ n \in \mathbb{N} : P(n) \text{ does not hold} \} \). If we show that \( A \) is empty, then \( P(n) \) holds for all \( n \in \mathbb{N} \) and the result follows. Suppose then that \( A \) is nonempty. According to the previous theorem, it must have a least element \( m = \min A \).

Since \( P(1) \) holds by assumption, \( 1 \notin A \) and so \( m > 1 \). In particular, \( m - 1 \) is a natural number which is smaller than the least element of \( A \), so \( m - 1 \notin A \) and \( P(m - 1) \) holds. It follows by assumption that \( P(m) \) also holds and this gives \( m \notin A \), a contradiction. ■

3 Open sets and convergence

**Definition 3.1 – Open set**

We say that a set \( A \subseteq \mathbb{R} \) is open in \( \mathbb{R} \) if, given any point \( x \in A \), there exists some \( \varepsilon > 0 \) such that \( (x - \varepsilon, x + \varepsilon) \subseteq A \).

**Example 3.2** Consider the interval \( A = [a, b) \) which contains its endpoint \( x = a \). If \( A \) was actually open in \( \mathbb{R} \), then we would have \( (a - \varepsilon, a + \varepsilon) \subseteq A \) for some \( \varepsilon > 0 \). This is not the case, however, because points such as \( a + \frac{\varepsilon}{2} \) lie in \( (a - \varepsilon, a + \varepsilon) \) but not in \( A \). ■

**Theorem 3.3 – Unions and intersections of open sets**

Every union of open sets is open and every finite intersection of open sets is open.

- Infinite intersections of open sets need not be open. For instance, \( U_n = (-\frac{1}{n}, \frac{1}{n}) \) is open in \( \mathbb{R} \) for each \( n \in \mathbb{N} \), but one has \( \bigcap_{n=1}^{\infty} U_n = \{0\} \) and this is not open in \( \mathbb{R} \).

Proof. Let us worry about unions first. We assume that the sets \( U_i \) are open in \( \mathbb{R} \) and we look at their union \( A = \bigcup_i U_i \). To show that \( A \) is open in \( \mathbb{R} \), let \( x \in A \) be given. Since \( x \) belongs to the union of the sets \( U_i \), we have \( x \in U_i \) for some \( i \). We can thus find some \( \varepsilon > 0 \) such that \( (x - \varepsilon, x + \varepsilon) \subseteq U_i \) and this implies that \( (x - \varepsilon, x + \varepsilon) \subseteq \bigcup_i U_i = A \).

Next, we prove the statement for intersections. Assume that the sets \( U_i \) are open in \( \mathbb{R} \) and let \( B = \bigcap_{i=1}^{n} U_i \). To show that \( B \) is open in \( \mathbb{R} \), let \( x \in B \) be given. Then \( x \in U_i \) for each \( i \) and there exist \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n > 0 \) such that \( (x - \varepsilon_i, x + \varepsilon_i) \subseteq U_i \) for each \( i \). If we now take \( \varepsilon > 0 \) to be the smallest of the numbers \( \varepsilon_i \), then \( \varepsilon \leq \varepsilon_i \) for each \( i \) and so

\[
(x - \varepsilon, x + \varepsilon) \subseteq (x - \varepsilon_i, x + \varepsilon_i) \subseteq U_i
\]

for each \( i \). It easily follows that \( (x - \varepsilon, x + \varepsilon) \subseteq \bigcap_{i=1}^{n} U_i = B \), as needed. ■
**Theorem 3.4 – Examples of open sets**

(a) The intervals \((a, \infty)\), \((-\infty, b)\) and \((a, b)\) are open in \(\mathbb{R}\) for all \(a, b\).

(b) A set \(A \subseteq \mathbb{R}\) is open in \(\mathbb{R}\) if and only if it is a union of open intervals.

**Proof.** First, consider the interval \(A = (a, \infty)\). Given a point \(x \in A\), we have \(x > a\) and we need to find some \(\varepsilon > 0\) such that \((x - \varepsilon, x + \varepsilon) \subseteq A\). Letting \(\varepsilon = x - a\), we get

\[
y \in (x - \varepsilon, x + \varepsilon) \implies y > x - \varepsilon = a \implies y \in A.
\]

This shows that \(A = (a, \infty)\) is open in \(\mathbb{R}\). A similar argument shows that \(B = (-\infty, b)\) is also open in \(\mathbb{R}\), so their intersection \(A \cap B = (a, b)\) is open in \(\mathbb{R}\) as well.

Let us now turn to part (b). If a set is a union of open intervals, then it is a union of open sets, so it is open. Conversely, suppose \(A \subseteq \mathbb{R}\) is open. Given any \(x \in A\), we can find some \(\varepsilon_x > 0\) such that \((x - \varepsilon_x, x + \varepsilon_x) \subseteq A\). Since \(A\) is the union of its elements, we get

\[
A = \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} (x - \varepsilon_x, x + \varepsilon_x) \subseteq A.
\]

Thus, the above sets are all equal and \(A\) itself is a union of open intervals. \(\square\)

**Example 3.5** Consider the set \(A = \{x \in \mathbb{R} : x^3 > x\}\). To show that \(A\) is open in \(\mathbb{R}\), we first find the values of \(x\) such that \(x^3 > x\). Note that \(x^3 - x\) can be factored as

\[
x^3 - x = x(x^2 - 1) = x(x - 1)(x + 1).
\]

When \(x < -1\), all three factors are negative, so the product is negative. When \(-1 < x < 0\), only two factors are negative, so the product is positive. Arguing in this manner, one finds that \(A = (-1, 0) \cup (1, \infty)\). Thus, \(A\) is open in \(\mathbb{R}\) by the previous theorem. \(\square\)

**Definition 3.6 – Convergence of sequences**

A sequence \(\{x_n\}\) of real numbers converges to \(x\) as \(n \to \infty\) if, given any \(\varepsilon > 0\), there exists a natural number \(N\) such that \(x_n \in (x - \varepsilon, x + \varepsilon)\) for all \(n \geq N\). In that case, we call \(x\) the limit of the sequence and we write \(x_n \to x\) as \(n \to \infty\).

**Theorem 3.7 – Monotone convergence theorem**

(a) If a sequence \(\{x_n\}\) is increasing and bounded from above, then \(\{x_n\}\) converges.

(b) If a sequence \(\{x_n\}\) is decreasing and bounded from below, then \(\{x_n\}\) converges.

**Proof.** We only prove the first part, as the second part is similar. Our goal is to show that the sequence converges to \(\sup A\), where \(A = \{x_1, x_2, \ldots\}\). Let \(\varepsilon > 0\) be given. As \(\sup A - \varepsilon\) is smaller than the least upper bound of \(A\), there exists \(x_N \in A\) such that \(x_N > \sup A - \varepsilon\). Since the sequence is increasing, this actually gives \(x_n \geq x_N > \sup A - \varepsilon\) for all \(n \geq N\). On the other hand, \(\sup A\) is an upper bound of \(A\), so \(\sup A \geq x_n\) for all \(n\). We thus have

\[
\sup A - \varepsilon < x_N \leq x_n \leq \sup A < \sup A + \varepsilon
\]

for all \(n \geq N\). In other words, \(x_n \in (\sup A - \varepsilon, \sup A + \varepsilon)\) for all \(n \geq N\), as needed. \(\square\)
Theorem 3.8 – Squeeze theorem

If \( x_n \leq y_n \leq z_n \) for all \( n \in \mathbb{N} \) and \( x_n, z_n \to \alpha \) as \( n \to \infty \), then \( y_n \to \alpha \) as \( n \to \infty \).

Proof. Let \( \varepsilon > 0 \) be given. Since \( x_n \to \alpha \) as \( n \to \infty \), there exists a natural number \( N_1 \) such that \( x_n \in (\alpha - \varepsilon, \alpha + \varepsilon) \) for all \( n \geq N_1 \). Since \( z_n \to \alpha \) as \( n \to \infty \), there also exists a natural number \( N_2 \) such that \( z_n \in (\alpha - \varepsilon, \alpha + \varepsilon) \) for all \( n \geq N_2 \). We must thus have

\[
\alpha - \varepsilon < x_n, z_n < \alpha + \varepsilon
\]

for all \( n \geq \max\{N_1, N_2\} \). Since \( x_n \leq y_n \leq z_n \) by assumption, this implies that

\[
\alpha - \varepsilon < x_n \leq y_n \leq z_n < \alpha + \varepsilon
\]

for all \( n \geq \max\{N_1, N_2\} \). In other words, it implies that \( y_n \to \alpha \) as \( n \to \infty \). \( \blacksquare \)

Theorem 3.9 – Convergence in terms of open intervals/sets

The following statements are equivalent whenever \( \{x_n\} \) is a sequence and \( x \in \mathbb{R} \).

(a) One has \( x_n \to x \) as \( n \to \infty \).

(b) Given any \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that \( x_n \in (x - \varepsilon, x + \varepsilon) \) for all \( n \geq N \).

(c) Given any open \( U \) with \( x \in U \), there exists \( N \in \mathbb{N} \) such that \( x_n \in U \) for all \( n \geq N \).

Proof. The first two parts are equivalent by definition.

To show that (b) implies (c), suppose \( U \) is open and \( x \in U \). Then \( (x - \varepsilon, x + \varepsilon) \subseteq U \) for some \( \varepsilon > 0 \) and one may use part (b) to find some \( N \in \mathbb{N} \) such that \( x_n \in (x - \varepsilon, x + \varepsilon) \) for all \( n \geq N \). This implies that \( x_n \in (x - \varepsilon, x + \varepsilon) \subseteq U \) for all \( n \geq N \), so part (c) follows.

To prove that (c) implies (b), let \( \varepsilon > 0 \) be given and take \( U = (x - \varepsilon, x + \varepsilon) \). Then \( U \) is an open set that contains \( x \), so one may use part (c) to find some \( N \in \mathbb{N} \) such that \( x_n \in U \) for all \( n \geq N \). This gives \( x_n \in (x - \varepsilon, x + \varepsilon) \) for all \( n \geq N \), so part (b) follows. \( \blacksquare \)