Analysis Solutions #1

1. Let $f\colon A \to B$ be a function and let $B_1, B_2 \subseteq B$ be arbitrary. Show that

$$f^{-1}(B_1 - B_2) = f^{-1}(B_1) - f^{-1}(B_2).$$

Two sets are equal if and only if they have the same elements. In this case,

$$\begin{aligned} x \in f^{-1}(B_1 - B_2) \quad &\iff \quad f(x) \in B_1 - B_2 \\ &\iff \quad f(x) \in B_1, \text{ but } f(x) \notin B_2 \\ &\iff \quad x \in f^{-1}(B_1), \text{ but } x \notin f^{-1}(B_2) \\ &\iff \quad x \in f^{-1}(B_1) - f^{-1}(B_2). \end{aligned}$$

2. Let $f\colon A \to B$ be a function and let $A_1 \subseteq A$ be arbitrary. Show that

$$f^{-1}(f(A_1)) \supseteq A_1$$

and that equality holds whenever the function f is injective.

To prove the inclusion for any function f, it suffices to note that

$$x \in A_1 \quad \implies \quad f(x) \in f(A_1) \quad \implies \quad x \in f^{-1}(f(A_1)).$$

If it happens that f is injective, then the opposite inclusion also holds. In that case,

$$\begin{aligned} x \in f^{-1}(f(A_1)) \quad &\implies \quad f(x) \in f(A_1) \\ &\implies \quad f(x) = f(z) \text{ for some } z \in A_1 \\ &\implies \quad x = z \text{ for some } z \in A_1 \\ &\implies \quad x \in A_1. \end{aligned}$$

3. Show that the set $A = \left\{ \frac{2n+1}{n+3} : n \in \mathbb{N} \right\}$ has a minimum but no maximum.

To show that $\frac{3}{4}$ is the minimum of A, we note that $\frac{3}{4} = \frac{2+1}{1+3} \in A$ and that

$$\frac{3}{4} \leq \frac{2n+1}{n+3} \quad \iff \quad 3n + 9 \leq 8n + 4 \quad \iff \quad 5 \leq 5n \quad \iff \quad 1 \leq n.$$

Since the rightmost inequality holds for each $n \in \mathbb{N}$, the leftmost inequality also does and this means that $\frac{3}{4}$ is the least element. To show that no largest element exists, let $x \in A$ be arbitrary. Then $x = \frac{2n+1}{n+3}$ for some $n \in \mathbb{N}$ and we claim that $y = \frac{2n+3}{n+4}$ is an element of A which is larger than x. In fact, one can easily verify that

$$\begin{aligned} \frac{2n+3}{n+4} > \frac{2n+1}{n+3} \quad &\iff \quad (2n+3)(n+3) > (2n+1)(n+4) \\ &\iff \quad 2n^2 + 6n + 3n + 9 > 2n^2 + 8n + n + 4 \\ &\iff \quad 9 > 4. \end{aligned}$$

These inequalities are all obviously valid, so $y > x$ and x is not the largest element.

4. Let A, B be nonempty subsets of $\mathbb{R}$ such that $\sup A < \sup B$. Show that there exists an element $b \in B$ which is an upper bound of A.

By definition, $\sup B$ is the least upper bound of B. Since $\sup A$ is even smaller, we find that $\sup A$ is not an upper bound of B. In other words, there exists an element $b \in B$ such that $b > \sup A$. On the other hand, $\sup A \geq a$ for all $a \in A$ by definition. Combining these two facts, we now get $b > \sup A \geq a$ for all $a \in A$, so b is an upper bound of A.

5. Show that $(A \cap B) \cup (A - B) = A$ for any sets A, B.

First, suppose that $x \in (A \cap B) \cup (A - B)$. Then either $x \in A \cap B$ or $x \in A - B$. In the former case, $x \in A$ and $x \in B$, so $x \in A$. In the latter case, $x \in A$ but $x \notin B$, so $x \in A$. This gives $x \in A$ in any case.

Conversely, suppose that $x \in A$. If it happens that $x \in B$, then $x \in A \cap B$. Otherwise, $x \in A$ but $x \notin B$, so $x \in A - B$. This gives $x \in (A \cap B) \cup (A - B)$ in any case.

6. Let $f \colon A \to B$ and $g \colon B \to C$ be two functions and let $g \circ f \colon A \to C$ denote their composition. Given a set $C_1 \subseteq C$, show that $(g \circ f)^{-1}(C_1) = f^{-1}(g^{-1}(C_1))$.

Two sets are equal if and only if they have the same elements. In this case,

$$x \in (g \circ f)^{-1}(C_1) \quad \iff \quad (g \circ f)(x) \in C_1 \quad \iff \quad g(f(x)) \in C_1.$$

Using the definition of the inverse image, one may thus conclude that

$$x \in (g \circ f)^{-1}(C_1) \quad \iff \quad f(x) \in g^{-1}(C_1) \quad \iff \quad x \in f^{-1}(g^{-1}(C_1)).$$

7. Determine the minimum of the set $A = \{2x^2 - 3x : x \in \mathbb{R}\}$.

The derivative of $f(x) = 2x^2 - 3x$ is $f'(x) = 4x - 3$. This is negative when $x < 3/4$ and it is positive when $x > 3/4$. In other words, f is decreasing when $x < 3/4$ and it is increasing when $x > 3/4$, so the smallest value that it attains is $f(3/4) = -9/8$.

8. Determine the maximum of the set $A = \{x \in \mathbb{R} : x^3 \leq 7x - 6\}$.

First, we find all numbers x such that $x^3 - 7x + 6 \leq 0$. Noting that $x = 1$ is a root of the left hand side, we see that $x - 1$ must be a factor, so it easily follows that

$$x^3 - 7x + 6 = (x-1)(x^2 + x - 6) = (x-1)(x-2)(x+3).$$

Since the set A consists of all numbers x such that $(x-1)(x-2)(x+3) \leq 0$, one has

$$A = (-\infty, -3] \cup [1, 2] \quad \Longrightarrow \quad \max A = 2.$$

9. Determine the min, inf, max and sup of the following sets, noting that some of these quantities may fail to exist. You do not need to justify your answers.

(a) $A = \left\{n \in \mathbb{N} : \frac{n}{n+1} < \frac{2019}{2020}\right\}$

(b) $B = \{x \in \mathbb{R} : x > 1 \text{ and } 2x \leq 5\}$

(c) $C = \{x \in \mathbb{Z} : x > 1 \text{ and } 2x \leq 5\}$

(d) $D = \{x \in \mathbb{R} : x < y \text{ for all } y > 0\}$

The first set consists of all natural numbers n such that $\frac{n}{n+1} < \frac{2019}{2020}$, while

$$\frac{n}{n+1} < \frac{2019}{2020} \quad \Longleftrightarrow \quad 2020n < 2019n + 2019 \quad \Longleftrightarrow \quad n < 2019.$$

This gives $A = \{1, 2, \ldots, 2018\}$ so $\min A = \inf A = 1$ and $\max A = \sup A = 2018$.

The second set consists of all real numbers x such that $1 < x \leq \frac{5}{2}$. This gives $B = (1, \frac{5}{2}]$ so $\max B = \sup B = \frac{5}{2}$ and $\inf B = 1$, while $\min B$ does not exist.

The third set consists of all integers x with $1 < x \leq \frac{5}{2}$. This gives $C = \{2\}$ and

$$\min C = \max C = \inf C = \sup C = 2.$$

Finally, the fourth set consists of all real numbers x that are smaller than every positive number. Thus, $D = (-\infty, 0]$ and $\max D = \sup D = 0$, while $\inf D$ and $\min D$ do not exist.

10. Show that the set $A = \left\{x + \frac{1}{x} : x > 0\right\}$ is such that $\inf A = 2$.

First, we show that 2 is a lower bound of A. Assuming that $x > 0$, one has

$$x + \frac{1}{x} \geq 2 \quad \Longleftrightarrow \quad x^2 + 1 \geq 2x \quad \Longleftrightarrow \quad (x-1)^2 \geq 0.$$

Since the rightmost inequality holds, the leftmost one holds as well, so 2 is a lower bound of A. To show that it is the greatest lower bound, we note that $2 = 1 + \frac{1}{1} \in A$. If $z > 2$, then 2 is an element of A which is smaller than z, so z is not a lower bound of A.

11. Show that the set $B = \{x \in \mathbb{R} : |2x - 3| < 5\}$ is such that $\sup B = 4$.

First of all, we simplify the given inequality to find that

$$|2x-3| < 5 \iff -5 < 2x-3 < 5 \iff -2 < 2x < 8 \iff -1 < x < 4.$$

Thus, $B = (-1, 4)$ and $4 \geq x$ for all $x \in B$, so 4 is an upper bound of B. To show that it is the least upper bound, we assume that $y < 4$ and we consider two cases.

Case 1. If $y \leq -1$, then $z = 1$ is an element of B such that $z > y$.

Case 2. If $y > -1$, then $-1 < y < 4$ and we can look at the average $z = \frac{y+4}{2}$. This is a number that satisfies $-1 < y < z < 4$, so $z \in B$ and also $z > y$.

In either case then, B contains an element $z > y$, so y is not an upper bound of B.

12. Suppose that A, B are nonempty subsets of $\mathbb{R}$ which are bounded from above. Show that $A \cup B$ is also bounded from above and $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

Let $\alpha = \sup A$ and $\beta = \sup B$ for convenience. We shall only treat the case $\alpha \leq \beta$, as the case $\beta \leq \alpha$ is similar. Since $\alpha \leq \beta$, we need to show that

$$\sup(A \cup B) = \max\{\sup A, \sup B\} = \max\{\alpha, \beta\} = \beta.$$

First, we show that β is an upper bound of $A \cup B$. If $x \in A \cup B$, then $x \in A$ or $x \in B$. In the former case, we have $x \leq \sup A = \alpha \leq \beta$. In the latter case, we have $x \leq \sup B = \beta$. This gives $x \leq \beta$ in any case, so β is an upper bound of $A \cup B$.

Next, we show that β is the least upper bound of $A \cup B$. Indeed, suppose that $y < \beta$. Since y is smaller than $\beta = \sup B$, there exists an element $b \in B$ such that $y < b$. This element of B is also an element of $A \cup B$, so there exists some $b \in A \cup B$ such that $b > y$. In particular, y is not an upper bound of $A \cup B$ and the least upper bound is β.

Analysis Solutions #2

1. Show that $A = \left\{\frac{4n+3}{2n-1} : n \in \mathbb{N}\right\}$ is bounded from below and that $\inf A = 2$.

First of all, we note that 2 is a lower bound of A because

$$2 \leq \frac{4n+3}{2n-1} \quad \Longleftrightarrow \quad 4n - 2 \leq 4n + 3 \quad \Longleftrightarrow \quad -2 \leq 3.$$

To show that 2 is the greatest lower bound, we assume that $x > 2$ and we try to find an element of A which is smaller than x. Solving the inequality $\frac{4n+3}{2n-1} < x$, we get

$$\begin{aligned} \frac{4n+3}{2n-1} < x \quad &\Longleftrightarrow \quad 4n + 3 < 2nx - x \\ &\Longleftrightarrow \quad x + 3 < (2x - 4)n \quad \Longleftrightarrow \quad \frac{x+3}{2x-4} < n. \end{aligned}$$

Pick a natural number n that satisfies the rightmost inequality. Then $\frac{4n+3}{2n-1} < x$, so there is an element of A which is smaller than x. This means that x is not a lower bound of A.

2. Let $A \subseteq \mathbb{R}$ be nonempty and bounded from above. Fix some real number $x < 0$ and consider the set $B = \{ax : a \in A\}$. Show that $\inf B = x \sup A$.

To show that $x \sup A$ is a lower bound of B, we note that

$$\begin{aligned} \sup A \geq a \ \text{ for all } a \in A \quad &\Longrightarrow \quad x \sup A \leq ax \ \text{ for all } a \in A \\ &\Longrightarrow \quad x \sup A \leq b \ \text{ for all } b \in B. \end{aligned}$$

To show that it is the greatest lower bound of B, suppose $y > x \sup A$. Then $y/x < \sup A$, so y/x is not an upper bound of A and there exists some $a \in A$ such that $y/x < a$. This makes $b = ax$ an element of B that satisfies $y > ax = b$, so y is not a lower bound of B.

3. Let $A \subseteq \mathbb{R}$ be nonempty, open and bounded from above. Show that $\sup A \notin A$.

Suppose that $\sup A \in A$. Since A is open, there exists some $\varepsilon > 0$ such that

$$(\sup A - \varepsilon, \sup A + \varepsilon) \subseteq A.$$

This implies that $\sup A + \frac{\varepsilon}{2} \in A$ and that $\sup A + \frac{\varepsilon}{2} \leq \sup A$, a contradiction.

4. Let $\{x_n\}$ be a sequence of real numbers such that $x_n \to x$ as $n \to \infty$ and consider the sequence $\{y_n\}$ defined by $y_n = \frac{1}{2}(3x_n + x_{n+1})$ for each $n \geq 1$. Use the definition of convergence to show that $y_n \to 2x$ as $n \to \infty$.

Let $\varepsilon > 0$ be given. Since $x_n \to x$ as $n \to \infty$, there exists some $N \in \mathbb{N}$ such that

$$x - \varepsilon/2 < x_n < x + \varepsilon/2 \quad \text{for all } n \geq N.$$

As this statement holds for all $n \geq N$, it also implies that

$$x - \varepsilon/2 < x_{n+1} < x + \varepsilon/2 \quad \text{for all } n \geq N.$$

Multiplying the first inequality by 3 and then adding, we conclude that

$$\begin{aligned} 3(x-\varepsilon/2) < 3x_n < 3(x+\varepsilon/2) \quad &\implies \quad 4(x-\varepsilon/2) < 3x_n + x_{n+1} < 4(x+\varepsilon/2) \\ &\implies \quad 2(x-\varepsilon/2) < y_n < 2(x+\varepsilon/2) \\ &\implies \quad 2x - \varepsilon < y_n < 2x + \varepsilon \end{aligned}$$

for all $n \geq N$. This gives $y_n \in (2x - \varepsilon, 2x + \varepsilon)$ for all $n \geq N$, so $y_n \to 2x$ as $n \to \infty$.

5. Show that $A = \left\{\frac{m}{m+n} : m, n \in \mathbb{N}\right\}$ is bounded from above and that $\sup A = 1$.

To check that 1 is an upper bound of A, it suffices to note that

$$\frac{m}{m+n} \leq 1 \quad \iff \quad m \leq m + n \quad \iff \quad 0 \leq n$$

for all $m, n \in \mathbb{N}$. Since the rightmost inequality holds, the leftmost one holds as well. To check that 1 is the least upper bound of A, we assume that $x < 1$ and we try to find an element of A which is larger than x. Looking for an element of the form $\frac{m}{m+1}$, we get

$$\frac{m}{m+1} > x \quad \iff \quad m > mx + x \quad \iff \quad m(1-x) > x \quad \iff \quad m > \frac{x}{1-x}.$$

Pick a natural number m that satisfies the rightmost inequality. Then $\frac{m}{m+1} > x$, so there exists an element of A which is larger than x and x is not an upper bound of A.

6. Suppose that A, B are subsets of $\mathbb{R}$ such that $\inf A < \sup B$. Show that there exist an element $a \in A$ and an element $b \in B$ such that $\inf A \leq a < b \leq \sup B$.

Since $\inf A$ is smaller than the least upper bound of B, it is not an upper bound of B, so there exists some $b \in B$ such that $\inf A < b$. Since b is larger than $\inf A$, we can use the same argument to find some $a \in A$ such that $a < b$. This implies $\inf A \leq a < b \leq \sup B$.

7. Let $A \subseteq \mathbb{Z}$ be nonempty and bounded from below. Show that A has a minimum.

Since $\inf A + 1$ is larger than $\inf A$, there exists an element $x \in A$ such that

$$\inf A \leq x < \inf A + 1.$$

If it happens that $\inf A = x$, then $\inf A$ is an element of A, so $\min A$ exists. Otherwise, x is larger than $\inf A$, so the same argument gives another element $y \in A$ such that

$$\inf A \leq y < x \quad \Longrightarrow \quad \inf A \leq y < x < \inf A + 1.$$

This is a contradiction since the integers x, y cannot both lie in an interval of length 1.

8. Show that each of the following sets is open in $\mathbb{R}$.

$$A = \left\{x \in \mathbb{R} : x^3 > 13x - 12\right\}, \qquad B = \left\{0 < x < 1 : \frac{1}{x} \notin \mathbb{N}\right\}.$$

The first set consists of all numbers x such that $f(x) = x^3 - 13x + 12$ is positive. Noting that $x = 1$ is a root of this polynomial, we find that $x - 1$ is a factor and that

$$f(x) = x^3 - 13x + 12 = (x-1)(x^2 + x - 12) = (x-1)(x-3)(x+4).$$

When $x < -4$, all three factors are negative, so the product is negative. When $-4 < x < 1$, only two factors are negative, so the product is positive. Arguing in this manner, one finds that $A = (-4, 1) \cup (3, \infty)$, so A is the union of open intervals and thus open.

The second set is the interval $(0, 1)$ with the points $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ removed. This gives

$$B = \left(\frac{1}{2}, 1\right) \cup \left(\frac{1}{3}, \frac{1}{2}\right) \cup \left(\frac{1}{4}, \frac{1}{3}\right) \cup \cdots = \bigcup_{n \in \mathbb{N}} \left(\frac{1}{n+1}, \frac{1}{n}\right),$$

so B is also a union of open intervals and thus open itself.

9. Do there exist sets $A, B \subseteq \mathbb{R}$ such that $A, B, A - B$ are all nonempty and open?

Yes. For instance, let $A = (0, 1) \cup (1, 2)$ and $B = (0, 1)$ so that $A - B = (1, 2)$. Each of these sets is nonempty and they are all open because they are unions of open intervals.

10. Suppose that $A \subseteq \mathbb{R}$ is nonempty and bounded from above. Show that there exists a sequence of points $x_n \in A$ such that $x_n \to \sup A$ as $n \to \infty$.

Since $\sup A - \frac{1}{n}$ is smaller than the least upper bound of A for each $n \in \mathbb{N}$, there is an element $x_n \in A$ such that $\sup A - \frac{1}{n} < x_n$ for each $n \in \mathbb{N}$. This also implies that

$$\sup A - \frac{1}{n} < x_n \leq \sup A \quad \text{for each } n \in \mathbb{N}.$$

Since $\sup A - \frac{1}{n} \to \sup A$ as $n \to \infty$, it follows by the Squeeze Theorem that $x_n \to \sup A$.

11. Define a sequence $\{a_n\}$ by setting $a_1 = 1$ and $a_{n+1} = \sqrt{2a_n + 1}$ for each $n \geq 1$. Show that $a_n < a_{n+1} < 3$ for all $n \in \mathbb{N}$ and that the sequence $\{a_n\}$ converges.

Since $a_1 = 1$ and $a_2 = \sqrt{3}$, the statement $a_n < a_{n+1} < 3$ is certainly true when $n = 1$. Suppose that this statement is true for some n. It then easily follows that

$$\begin{aligned} a_n < a_{n+1} < 3 \quad \Longrightarrow \quad 2a_n < 2a_{n+1} < 6 \quad &\Longrightarrow \quad 2a_n + 1 < 2a_{n+1} + 1 < 7 \\ &\Longrightarrow \quad a_{n+1} < a_{n+2} < \sqrt{7} < 3, \end{aligned}$$

so the statement holds for all $n \in \mathbb{N}$ by induction. Thus, the given sequence is increasing and bounded from above, so it converges by the monotone convergence theorem.

12. Suppose that $A, B \subseteq \mathbb{R}$ are nonempty and bounded from above. Show that the set

$$C = \{x \in \mathbb{R} : x = a + b \text{ for some } a \in A \text{ and } b \in B\}$$

is also bounded from above and that $\sup C = \sup A + \sup B$.

To show that $\sup A + \sup B$ is an upper bound of C, we note that

$$\begin{aligned} x \in C \quad &\Longrightarrow \quad x = a + b \text{ for some } a \in A \text{ and some } b \in B \\ &\Longrightarrow \quad x = a + b \leq \sup A + \sup B. \end{aligned}$$

To show that $\sup A + \sup B$ is the least upper bound of C, suppose that $z < \sup A + \sup B$. Since $z - \sup B$ is smaller than the least upper bound of A, it is not an upper bound of A, so there exists some $a \in A$ such that $z - \sup B < a$. Since $z - a < \sup B$, one may use the same argument to find some $b \in B$ such that $z - a < b$. In particular, $z < a + b$ and $a + b$ is an element of C which is larger than z. This means that z is not an upper bound of C.

Analysis Solutions #3

1. Let $A \subseteq \mathbb{R}$ be nonempty, closed and bounded from above. Show that $\max A$ exists.

It suffices to show that $\sup A$ is an element of A, as this implies $\max A = \sup A$. If it is not an element of A, then it is an element of A^c. Since A is closed by assumption, its complement A^c is open and $(\sup A - \varepsilon, \sup A + \varepsilon) \subseteq A^c$ for some $\varepsilon > 0$. Since $\sup A - \frac{\varepsilon}{2}$ is smaller than the least upper bound of A, there also exists some $x \in A$ such that

$$\sup A - \frac{\varepsilon}{2} < x \leq \sup A \quad \Longrightarrow \quad x \in (\sup A - \varepsilon, \sup A + \varepsilon) \subseteq A^c.$$

This means that x is an element of A which is also an element of A^c, a contradiction.

2. Show that $(A \cap B)^\circ = A^\circ \cap B^\circ$ for any sets $A, B \subseteq \mathbb{R}$.

To prove the inclusion $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$, we note that

$$\begin{aligned} A \cap B \subseteq A \text{ and } A \cap B \subseteq B \quad &\Longrightarrow \quad (A \cap B)^\circ \subseteq A^\circ \text{ and } (A \cap B)^\circ \subseteq B^\circ \\ &\Longrightarrow \quad (A \cap B)^\circ \subseteq A^\circ \cap B^\circ. \end{aligned}$$

To prove the opposite inclusion $A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$, we note that

$$A^\circ \subseteq A \text{ and } B^\circ \subseteq B \quad \Longrightarrow \quad A^\circ \cap B^\circ \subseteq A \cap B.$$

This makes $A^\circ \cap B^\circ$ an open set which is contained in $A \cap B$, while the interior $(A \cap B)^\circ$ is the largest open set which is contained in $A \cap B$. We conclude that $A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$.

3. Let $A, B \subseteq \mathbb{R}$ be arbitrary. Show that $(A \cup B)^\circ$ and $A^\circ \cup B^\circ$ are not necessarily equal, but one of these sets is always contained in the other.

To show that the two sets need not be equal, we let $A = [0, 1]$ and $B = [1, 2]$. Then

$$A \cup B = [0, 2], \qquad (A \cup B)^\circ = (0, 2), \qquad A^\circ \cup B^\circ = (0, 1) \cup (1, 2) \neq (A \cup B)^\circ.$$

To show that $A^\circ \cup B^\circ$ is always contained in $(A \cup B)^\circ$, we note that

$$A^\circ \subseteq A \text{ and } B^\circ \subseteq B \quad \Longrightarrow \quad A^\circ \cup B^\circ \subseteq A \cup B.$$

This makes $A^\circ \cup B^\circ$ an open set which is contained in $A \cup B$, while the interior $(A \cup B)^\circ$ is the largest open set which is contained in $A \cup B$. We conclude that $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$.

4. Show that the closure of the complement is the complement of the interior. In other words, show that $\overline{A^c} = (A^\circ)^c$ for any set $A \subseteq \mathbb{R}$.

According to the second part of Theorem 4.12 in the notes, one has

$$\begin{aligned} x \in \overline{A^c} \quad &\iff \quad \text{every neighbourhood of } x \text{ intersects } A^c \\ &\iff \quad \text{no neighbourhood of } x \text{ is contained in } A. \end{aligned}$$

According to the first part of Theorem 4.12 in the notes, this also implies that

$$x \in \overline{A^c} \quad \iff \quad x \notin A^\circ \quad \iff \quad x \in (A^\circ)^c.$$

5. Suppose $A \subseteq \mathbb{R}$ is open in $\mathbb{R}$ and $B \subseteq \mathbb{R}$ is closed. Show that $A - B$ is open in $\mathbb{R}$.

The set $A - B$ consists of all points $x \in A$ with $x \notin B$ and so $A - B = A \cap B^c$. Since A is open and B is closed, both A and B^c are open, so their intersection is open as well.

6. Show that each of the following sets is closed in $\mathbb{R}$.

$$A = \{x \in \mathbb{R} : x^4 \leq 5x^2 - 4\}, \qquad B = \{x \in \mathbb{R} : x^3 \leq 3x - 2\}.$$

The first set consists of all real numbers x such that $x^4 - 5x^2 + 4 \leq 0$. Since

$$x^4 - 5x^2 + 4 = (x^2 - 1)(x^2 - 4) = (x - 1)(x + 1)(x - 2)(x + 2),$$

one easily finds that $A = [-2, -1] \cup [1, 2]$. Thus, A is the union of two closed intervals and so A is itself closed. To show that B is closed as well, we note that

$$x^3 - 3x + 2 = (x - 1)(x^2 + x - 2) = (x - 1)^2(x + 2).$$

It easily follows that $B = (-\infty, -2] \cup \{1\}$ and this also implies that B is closed.

7. Find a sequence of nested intervals I_n such that their intersection $\bigcap_{n=1}^\infty I_n$ is empty.

Two simple examples are provided by the intervals $I_n = \left(0, \frac{1}{n}\right]$ and $I_n = \left(0, \frac{1}{n}\right)$. Note that these are nested because they satisfy $I_{n+1} \subseteq I_n$ for each $n \in \mathbb{N}$. If their intersection contains some number x, then $0 < x \leq \frac{1}{n}$ for all $n \in \mathbb{N}$ and thus $n \leq \frac{1}{x}$ for each $n \in \mathbb{N}$. This is not possible, however, because the set $\mathbb{N}$ does not have an upper bound.

8. Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$ for any sets $A, B \subseteq \mathbb{R}$.

To prove the inclusion $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$, we note that

$$\begin{aligned} A \subseteq A \cup B \text{ and } B \subseteq A \cup B \quad &\implies \quad \overline{A} \subseteq \overline{A \cup B} \text{ and } \overline{B} \subseteq \overline{A \cup B} \\ &\implies \quad \overline{A} \cup \overline{B} \subseteq \overline{A \cup B}. \end{aligned}$$

To prove the opposite inclusion $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$, we note that

$$A \subseteq \overline{A} \text{ and } B \subseteq \overline{B} \quad \implies \quad A \cup B \subseteq \overline{A} \cup \overline{B}.$$

This makes $\overline{A} \cup \overline{B}$ a closed set which contains $A \cup B$, while the closure $\overline{A \cup B}$ is the smallest closed set which contains $A \cup B$. We conclude that $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

9. Let $A, B \subseteq \mathbb{R}$ be arbitrary. Show that $\overline{A \cap B}$ and $\overline{A} \cap \overline{B}$ are not necessarily equal, but one of these sets is always contained in the other.

To show that the two sets need not be equal, we let $A = (0, 1)$ and $B = (1, 2)$. Then

$$A \cap B = \varnothing, \qquad \overline{A \cap B} = \varnothing, \qquad \overline{A} \cap \overline{B} = [0, 1] \cap [1, 2] = \{1\} \neq \overline{A \cap B}.$$

To show that $\overline{A \cap B}$ is always contained in $\overline{A} \cap \overline{B}$, we note that

$$A \subseteq \overline{A} \text{ and } B \subseteq \overline{B} \quad \implies \quad A \cap B \subseteq \overline{A} \cap \overline{B}.$$

This makes $\overline{A} \cap \overline{B}$ a closed set which contains $A \cap B$, while the closure $\overline{A \cap B}$ is the smallest closed set which contains $A \cap B$. We conclude that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

10. Show that a set $A \subseteq \mathbb{R}$ is closed in $\mathbb{R}$ if and only if A contains its limit points.

If the set A is closed, then $A = \overline{A}$ and this implies that

$$A = \overline{A} = A \cup A' \supseteq A'.$$

Conversely, if a set A contains its limit points, then its closure satisfies

$$\overline{A} = A \cup A' \subseteq A.$$

Since we also have $A \subseteq \overline{A}$ by definition, this gives $A = \overline{A}$ and so A is closed.

11. Suppose that $A \subseteq \mathbb{R}$ is nonempty and $x \in \mathbb{R}$ is a limit point of A. Show that every neighbourhood of x must contain infinitely many points of A.

Since x is a limit point of A, every neighbourhood of x intersects A at a point other than x. Suppose there is a neighbourhood U which intersects A at finitely many points other than x and let $B = \{x_1, x_2, \ldots, x_n\}$ consist of these points. Being a finite set, B is then closed in $\mathbb{R}$ and B^c is open. This implies that $U - B = U \cap B^c$ is an open set that contains x, so it is a neighbourhood of x. On the other hand, $U \cap B^c$ does not intersect A at a point other than x. This contradicts the assumption that x is a limit point of A.

12. Suppose that $A \subseteq \mathbb{R}$ is open in $\mathbb{R}$. Show that the set of limit points A' is equal to the closure $\overline{A}$. Is this statement true for an arbitrary subset of $\mathbb{R}$?

The closure always contains the limit points, as $\overline{A} = A \cup A' \supseteq A'$ for any set $A \subseteq \mathbb{R}$. To show that the opposite inclusion does not hold in general, let $A = \{0\}$ and note that

$$A' = \varnothing, \qquad \overline{A} = A \cup A' = A.$$

It remains to show that $\overline{A} = A \cup A' \subseteq A'$ whenever A is open. To prove this inclusion, we need to check that $A \subseteq A'$. If $x \in A$, then there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq A$. Suppose now that U is a neighbourhood of x. Then U is open and there exists $\varepsilon' > 0$ such that $(x - \varepsilon', x + \varepsilon') \subseteq U$. Letting $\delta = \min\{\varepsilon, \varepsilon'\}$, we find that

$$(x - \delta, x + \delta) = (x - \varepsilon, x + \varepsilon) \cap (x - \varepsilon', x + \varepsilon') \subseteq A \cap U.$$

In particular, $x - \frac{\delta}{2} \in A \cap U$ and U intersects A at a point other than x. Since this is true for every neighbourhood U of x, we conclude that $x \in A'$, as needed.

Analysis Solutions #4

1. Show that the function $f\colon \mathbb{R} \to \mathbb{R}$ is not continuous at all points when

$$f(x) = \left\{ \begin{array}{ll} x^2 & \text{if } x \leq 1 \\ 2x & \text{if } x > 1 \end{array} \right\}.$$

It suffices to find an open set U whose inverse image $f^{-1}(U)$ is not open. Consider an open interval such as $U = (\frac{1}{4}, 2)$. Using the piecewise definition of f, one finds that

$$\begin{aligned} f^{-1}(U) &= \{x \in \mathbb{R} : 1/4 < f(x) < 2\} \\ &= \{x \leq 1 : 1/4 < x^2 < 2\} \cup \{x > 1 : 1/4 < 2x < 2\} \\ &= (-\sqrt{2}, -1/2) \cup (1/2, 1]. \end{aligned}$$

This set is not open because it contains the endpoint $x = 1$. Thus, f is not continuous.

2. Suppose that $B \subseteq \mathbb{R}$ is open in $\mathbb{R}$ and let $A \subseteq B \subseteq \mathbb{R}$. Show that A is open in B if and only if A is open in $\mathbb{R}$.

First, suppose that A is open in B. Then $A = U \cap B$ for some set U which is open in $\mathbb{R}$. Since both U and B are open in $\mathbb{R}$, their intersection A is open in $\mathbb{R}$ as well.

Conversely, suppose that A is open in $\mathbb{R}$. Then $A \cap B$ is open in B. Since $A \subseteq B$ by assumption, this means that A is open in B.

3. Let $A, B \subseteq \mathbb{R}$. Show that a function $f\colon A \to B$ is continuous at all points if and only if the inverse image $f^{-1}(K)$ is closed in A whenever K is closed in B.

First, suppose that $f^{-1}(K)$ is closed in A whenever K is closed in B. To show that f is continuous, we let U be open in B. Then $B - U$ is closed in B and

$$f^{-1}(B - U) = f^{-1}(B) - f^{-1}(U) = A - f^{-1}(U)$$

is closed in A. This implies that $f^{-1}(U)$ is open in A and so f is continuous.

Conversely, suppose that f is continuous at all points and K is closed in B. Then $B - K$ is open in B and its inverse image is open in A. On the other hand,

$$f^{-1}(B - K) = f^{-1}(B) - f^{-1}(K) = A - f^{-1}(K).$$

Since this set is open in A, we conclude that $f^{-1}(K)$ is closed in A.

4. Show that $f\colon [0,1] \to \mathbb{R}$ is uniformly continuous when $f(x) = x^3$ for all x.

Let $\varepsilon > 0$ be given. Assuming that $x, y \in [0,1]$, one can easily check that

$$|f(x) - f(y)| = |x^3 - y^3| = |x - y| \cdot |x^2 + xy + y^2| \leq 3|x - y|.$$

Taking $\delta = \varepsilon/3$, we conclude that f is uniformly continuous because

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| \leq 3|x - y| < 3\delta = \varepsilon.$$

5. Show that $A = \{x \in \mathbb{R} : f(x) \neq 0\}$ is open in $\mathbb{R}$ whenever $f\colon \mathbb{R} \to \mathbb{R}$ is continuous.

First of all, we note that A can be expressed as the union of the sets

$$A^+ = \{x \in \mathbb{R} : f(x) > 0\}, \qquad A^- = \{x \in \mathbb{R} : f(x) < 0\}.$$

To see that the former set is open in $\mathbb{R}$, we let $U = (0, \infty)$ and we note that

$$A^+ = \{x \in \mathbb{R} : f(x) \in U\} = f^{-1}(U).$$

Since U is open in $\mathbb{R}$, its inverse image $f^{-1}(U) = A^+$ is also open in $\mathbb{R}$ by continuity. A similar argument shows that A^- is open as well, so the same is true for $A = A^+ \cup A^-$.

6. Suppose that $f\colon [0,1] \to [0,1]$ is continuous. Show that $f(x) = x$ for some $x \in [0,1]$.

The result is clear, if $f(0) = 0$ or $f(1) = 1$. Suppose $f(0) > 0$ and $f(1) < 1$. Being the difference of continuous functions, $g(x) = f(x) - x$ is then continuous with

$$g(0) = f(0) > 0, \qquad g(1) = f(1) - 1 < 0.$$

It follows by Bolzano's theorem that $g(x) = 0$ for some $x \in (0,1)$ and thus $f(x) = x$.

7. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and $|f(x)| \leq 3$ for all $x \in \mathbb{R}$. Show that there exists some real number x such that $f(x) = x$.

We note that $g(x) = f(x) - x$ is a continuous function on $[-4, 4]$ and that

$$g(4) = f(4) - 4 \leq 3 - 4 < 0, \qquad g(-4) = f(-4) + 4 \geq -3 + 4 > 0.$$

It follows by Bolzano's theorem that $g(x) = 0$ for some $x \in (-4, 4)$ and thus $f(x) = x$.

8. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and f has a root in every open interval (a, b). Show that f is the zero function, namely that $f(x) = 0$ for all $x \in \mathbb{R}$.

Let $x \in \mathbb{R}$ be arbitrary. Since f has a root in the interval $(x, x + \frac{1}{n})$ for each $n \in \mathbb{N}$, there exists a sequence of real numbers $\{x_n\}$ such that

$$x < x_n < x + \frac{1}{n}, \qquad f(x_n) = 0.$$

In view of the Squeeze Theorem, we must have $x_n \to x$ as $n \to \infty$. It follows by continuity that $f(x_n) \to f(x)$ as well. Since $f(x_n) = 0$ for all n, we conclude that $f(x) = 0$.

9. Show that every subset of A is open in A when $A \subseteq \mathbb{R}$ has finitely many elements.

It suffices to show that every subset of A is closed in A, as this also implies that every subset of A is open in A. Suppose that $B \subseteq A$ and A has finitely many elements. Being a finite set, B is closed in $\mathbb{R}$ and $B \cap A$ is closed in A. This means that B is closed in A.

10. Suppose $A \subseteq \mathbb{R}$ and $f\colon A \to \mathbb{R}$ is uniformly continuous with $|f(x)| \geq 2$ for all x. Show that $g\colon A \to \mathbb{R}$ is also uniformly continuous when $g(x) = 1/f(x)$ for all x.

Let $\varepsilon > 0$ be given. Since f is uniformly continuous, there exists $\delta > 0$ such that

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon$$

for all $x, y \in A$. Using the fact that $|f(x)| \geq 2$ for all $x \in A$, one finds that

$$|g(x) - g(y)| = \left| \frac{1}{f(x)} - \frac{1}{f(y)} \right| = \frac{|f(y) - f(x)|}{|f(x)| \cdot |f(y)|} \leq \frac{1}{4} \cdot |f(x) - f(y)|$$

for all $x, y \in A$. Once we now combine the last two equations, we may conclude that

$$|x - y| < \delta \quad \Longrightarrow \quad |g(x) - g(y)| \leq \frac{1}{4} \cdot |f(x) - f(y)| < \frac{\varepsilon}{4} < \varepsilon$$

for all $x, y \in A$. This verifies the definition of uniform continuity for the function g.

11. Show that $f\colon (0, 1) \to \mathbb{R}$ is not uniformly continuous when $f(x) = 1/x$ for all x.

Suppose that f is uniformly continuous. Then there exists $\delta > 0$ such that

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < 1$$

for all $0 < x, y < 1$. Consider the points $x = \frac{1}{n+1}$ and $y = \frac{1}{n}$ for some $n \in \mathbb{N}$. Since

$$|x - y| \leq |x| + |y| = \frac{1}{n+1} + \frac{1}{n} < \frac{2}{n},$$

we have $|x - y| < \delta$ for all $n > \frac{2}{\delta}$ and this leads to the contradiction

$$1 > |f(x) - f(y)| = |n + 1 - n| = 1.$$

12. Let $A, B \subseteq \mathbb{R}$ and let $i\colon B \to \mathbb{R}$ be the inclusion map which is defined by $i(x) = x$ for all $x \in B$. Show that a function $f\colon A \to B$ is continuous at all points if and only if the composition $i \circ f\colon A \to \mathbb{R}$ is continuous at all points.

Inclusions are always continuous. If f is continuous, then $i \circ f$ is the composition of continuous functions and thus continuous. Conversely, suppose $i \circ f$ is continuous. To show that f is continuous as well, we let U be open in B and we show that its inverse image is open in A. Since U is open in B, one has $U = V \cap B$ for some set V which is open in $\mathbb{R}$. Note that the inverse image $i^{-1}(V)$ is given by

$$i^{-1}(V) = \{x \in B : i(x) \in V\} = \{x \in B : x \in V\} = V \cap B = U.$$

In particular, the inverse image $(i \circ f)^{-1}(V)$ can be expressed in the form

$$\begin{aligned}(i \circ f)^{-1}(V) &= \{x \in A : i(f(x)) \in V\} \\ &= \{x \in A : f(x) \in i^{-1}(V)\} = \{x \in A : f(x) \in U\} = f^{-1}(U).\end{aligned}$$

Since $i \circ f\colon A \to \mathbb{R}$ is continuous, this set is open in A, and thus $f^{-1}(U)$ is open in A.

Analysis Solutions #5

1. Suppose that $\{x_n\}$ is a sequence of real numbers such that $x_n \leq \alpha$ for all $n \in \mathbb{N}$. If it happens that $\{x_n\}$ converges to some number x, show that $x \leq \alpha$ as well.

Assume that $x > \alpha$ and let $\varepsilon = x - \alpha > 0$. Then there exists some $N \in \mathbb{N}$ such that

$$x - \varepsilon < x_n < x + \varepsilon$$

for all $n \geq N$. This is a contradiction because $x - \varepsilon = \alpha \geq x_n$ for all $n \in \mathbb{N}$.

2. What can you say about a Cauchy sequence which consists entirely of integers?

Suppose that $\{x_n\}$ is a Cauchy sequence which consists entirely of integers. Using the definition of a Cauchy sequence with $\varepsilon = 1$, we can then find some $N \in \mathbb{N}$ such that

$$|x_m - x_n| < 1 \quad \text{for all } m, n \geq N.$$

Letting $m = N$ in the last equation, we conclude that

$$|x_N - x_n| < 1 \quad \Longrightarrow \quad x_N - 1 < x_n < x_N + 1 \quad \text{for all } n \geq N.$$

Since all the terms x_n are integers, however, this actually gives $x_n = x_N$ for all $n \geq N$.

3. Let $A, B \subseteq \mathbb{R}$ and suppose that $f \colon A \to B$ is uniformly continuous. Given a Cauchy sequence $\{x_n\}$ of elements of A, show that $\{f(x_n)\}$ is a Cauchy sequence as well.

Let $\varepsilon > 0$ be given. Since f is uniformly continuous, there exists $\delta > 0$ such that

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon \quad \text{for all } x, y \in A.$$

Since the sequence $\{x_n\}$ is Cauchy, there also exists a natural number N such that

$$|x_m - x_n| < \delta \quad \text{for all } m, n \geq N.$$

Once we now combine the last two equations, we may conclude that

$$|x_m - x_n| < \delta \quad \Longrightarrow \quad |f(x_m) - f(x_n)| < \varepsilon \quad \text{for all } m, n \geq N.$$

This means that the sequence $\{f(x_n)\}$ is also Cauchy, as needed.

4. Let $A \subseteq \mathbb{R}$. Show that A is a dense subset of $\mathbb{R}$, if and only if every nonempty open subset of $\mathbb{R}$ intersects A at some point.

First, suppose that A is a dense subset of $\mathbb{R}$, in which case $\overline{A} = \mathbb{R}$. Let U be a nonempty open subset of $\mathbb{R}$ and let $x \in U$. Since x is in the closure of A, every neighbourhood of x must intersect A by Theorem 4.12 and thus U intersects A.

Conversely, suppose that every nonempty open subset of $\mathbb{R}$ intersects A. If x is any real number, then every neighbourhood of x intersects A and so $x \in \overline{A}$ by Theorem 4.12. This shows that every real number is in the closure of A and thus $\overline{A} = \mathbb{R}$.

5. Show that the sequence $\{x_n\}$ is Cauchy, and thus convergent, when

$$x_n = \frac{\sin 1}{2} + \frac{\sin 2}{4} + \ldots + \frac{\sin n}{2^n} \quad \text{for each } n \geq 1.$$

We need to show that $|x_m - x_n|$ becomes arbitrarily small for large enough m, n. Let us assume that $m > n$ without loss of generality. Since $|\sin x| \leq 1$ for all $x \in \mathbb{R}$, one has

$$|x_m - x_n| = \left| \frac{\sin(n+1)}{2^{n+1}} + \frac{\sin(n+2)}{2^{n+2}} + \ldots + \frac{\sin m}{2^m} \right| \leq \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \ldots + \frac{1}{2^m}.$$

The right hand side is a geometric series with ratio $r = 1/2$, so it easily follows that

$$|x_m - x_n| \leq \sum_{i=n+1}^{m} \frac{1}{2^i} < \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^n}.$$

Thus, $|x_m - x_n|$ becomes arbitrarily small for large enough m, n and the result follows.

6. Which of the following subsets of $\mathbb{R}$ are complete? Explain.

$$A = [0, 1), \qquad B = \mathbb{Z}, \qquad C = \mathbb{Q}, \qquad D = \{x \in \mathbb{R} : x^2 \geq \sin x\}.$$

A subset of $\mathbb{R}$ is complete if and only if it is closed in $\mathbb{R}$. In this case, A is not closed because it fails to contain a limit point and C is not closed because $\overline{C} = \mathbb{R} \neq C$. On the other hand, it is easy to check that B is closed in $\mathbb{R}$ because its complement is

$$\mathbb{R} - B = \bigcup_{x \in \mathbb{Z}} (x, x+1)$$

which is a union of open intervals and thus open. To show that D is closed in $\mathbb{R}$ as well, we note that $f(x) = x^2 - \sin x$ is a continuous function and that $U = [0, \infty)$ is closed in $\mathbb{R}$. It follows by continuity that $D = f^{-1}(U)$ is closed in $\mathbb{R}$ as well.

7. Show that the Bolzano-Weierstrass theorem implies the nested interval property.

Consider a nested sequence of closed intervals $I_n = [a_n, b_n]$. Since the sequence $\{a_n\}$ is contained in $[a_1, b_1]$, it is a bounded sequence, so it has a convergent subsequence $\{a_{n_k}\}$. We claim that its limit a is in I_n for all n. Since the terms $a_{n_1}, a_{n_2}, \ldots$ are all elements of the closed interval $I_{n_1} = [a_{n_1}, b_{n_1}]$, their limit a must also be an element of I_{n_1}. More generally, the terms $a_{n_k}, a_{n_{k+1}}, \ldots$ are all elements of I_{n_k} and this implies that $a \in I_{n_k}$ for all k. Now, let $n \in \mathbb{N}$ be given and pick some $n_k > n$. Then $I_{n_k} \subseteq I_n$ and $a \in I_{n_k}$, so $a \in I_n$ as well.

8. Let $0 < \alpha < 1$. Show that a sequence $\{x_n\}$ of real numbers is Cauchy, if it satisfies

$$|x_{n+1} - x_n| \leq \alpha \cdot |x_n - x_{n-1}| \quad \text{for each } n \geq 2.$$

We need to show that $|x_m - x_n|$ becomes arbitrarily small for large enough m, n. Let us assume that $m > n$ without loss of generality. The given assumption implies that

$$|x_{k+1} - x_k| \leq \alpha \cdot |x_k - x_{k-1}| \leq \alpha^2 \cdot |x_{k-1} - x_{k-2}| \leq \cdots \leq \alpha^{k-1} \cdot |x_2 - x_1|$$

for all integers $k \geq 2$. Using this fact along with the triangle inequality, one finds that

$$\begin{aligned} |x_m - x_n| &\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \ldots + |x_{n+1} - x_n| \\ &\leq \alpha^{m-2} \cdot |x_2 - x_1| + \alpha^{m-3} \cdot |x_2 - x_1| + \ldots + \alpha^{n-1} \cdot |x_2 - x_1|. \end{aligned}$$

The right hand side is a geometric series with ratio $r = \alpha$, so it easily follows that

$$|x_m - x_n| \leq \sum_{i=n-1}^{m-2} \alpha^i \cdot |x_2 - x_1| \leq \sum_{i=n-1}^{\infty} \alpha^i \cdot |x_2 - x_1| = \frac{\alpha^{n-1}}{1-\alpha} \cdot |x_2 - x_1|.$$

Thus, $|x_m - x_n|$ becomes arbitrarily small for large enough m, n and the result follows.

9. Suppose that $\{x_n\}$ is an increasing sequence of real numbers which has a convergent subsequence. Show that the whole sequence $\{x_n\}$ converges as well.

Let us denote the convergent subsequence by $\{x_{n_k}\}$. Since this is convergent, it is also bounded, so there exists a real number M such that $x_{n_k} \leq M$ for all $k \in \mathbb{N}$. If we can show that $x_n \leq M$ for all $n \in \mathbb{N}$, then the whole sequence will be increasing and bounded, so the whole sequence will converge by the monotone convergence theorem. Now, let $n \in \mathbb{N}$ be given and pick some $n_k > n$. Since the sequence $\{x_n\}$ is increasing, we conclude that

$$x_{n_k} \geq x_n \quad \Longrightarrow \quad x_n \leq x_{n_k} \leq M.$$

10. Show that there exists an irrational number between any two real numbers.

Consider two real numbers $x < y$ and pick a rational number z such that $x < z < y$. If we define $w_n = z + \frac{1}{n}\sqrt{2}$, then w_n is irrational for each $n \in \mathbb{N}$ and we also have

$$w_n < y \quad \Longleftrightarrow \quad z + \frac{1}{n}\sqrt{2} < y \quad \Longleftrightarrow \quad \frac{\sqrt{2}}{n} < y - z \quad \Longleftrightarrow \quad n > \frac{\sqrt{2}}{y-z}.$$

Once we now fix some $n \in \mathbb{N}$ that satisfies this inequality, we find that $x < z < w_n < y$.

11. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is both continuous and surjective. Given a set $A \subseteq \mathbb{R}$ which is a dense subset of $\mathbb{R}$, show that its image $f(A)$ is also a dense subset of $\mathbb{R}$.

By Problem 4, we need to show that every nonempty open subset of $\mathbb{R}$ intersects $f(A)$. Suppose that U is such a subset and let $y \in U$. Then $y = f(x)$ for some $x \in \mathbb{R}$ and the inverse image $f^{-1}(U)$ is open in $\mathbb{R}$. Since $f^{-1}(U)$ contains x, this open set is nonempty, so it intersects the dense subset A. In other words, there exists $a \in A$ such that $f(a) \in U$. This means that $f(a) \in U \cap f(A)$ and thus U intersects $f(A)$, as needed.

12. Suppose that $f, g\colon \mathbb{R} \to \mathbb{R}$ are continuous and let $A \subseteq \mathbb{R}$ be a dense subset of $\mathbb{R}$ such that $f(x) = g(x)$ for all $x \in A$. Show that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

The difference $h(x) = f(x) - g(x)$ is continuous and it satisfies $h(x) = 0$ for all $x \in A$. Suppose that $h(x_0) > 0$ for some $x_0 \in \mathbb{R}$. Since $U = (0, 2h(x_0))$ is open, its inverse image is also open and it contains x_0, but it does not intersect A at any point because

$$x \in A \quad \Longrightarrow \quad h(x) = 0 \quad \Longrightarrow \quad h(x) \notin U \quad \Longrightarrow \quad x \notin h^{-1}(U).$$

In particular, $h^{-1}(U)$ is a neighbourhood of x_0 that does not intersect A, so x_0 is not in the closure of A. This contradicts the assumption that $\overline{A} = \mathbb{R}$, while a similar contradiction arises when $h(x_0) < 0$ for some $x_0 \in \mathbb{R}$. We conclude that $h(x) = 0$ for all $x \in \mathbb{R}$.

Analysis Solutions #6

1. Show that a set $A \subseteq \mathbb{R}$ is connected if and only if there is no function $f\colon A \to \{0, 1\}$ which is both continuous and surjective.

First, we note that $B_1 = \{0\}$ and $B_2 = \{1\}$ are both open in $B = \{0, 1\}$ because

$$B_1 = (-1, 1) \cap B, \qquad B_2 = (0, 2) \cap B.$$

If such a function exists, then $f^{-1}(B_1)$ and $f^{-1}(B_2)$ are nonempty by surjectivity and they are also open in A by continuity. Since the union of these sets is

$$f^{-1}(B_1) \cup f^{-1}(B_2) = f^{-1}(B_1 \cup B_2) = f^{-1}(B) = A,$$

we conclude that A is not connected. Conversely, suppose that A is not connected and write $A = A_1 \cup A_2$ for some nonempty disjoint sets A_1, A_2 which are both open in A. Then

$$f(x) = \left\{ \begin{array}{ll} 0 & \text{if } x \in A_1 \\ 1 & \text{if } x \in A_2 \end{array} \right\}$$

defines a surjective function $f\colon A \to B$. To show that f is also continuous, we need to show that every open subset of B has an inverse image which is open in A. Now, the only subsets of B are $\varnothing, B_1, B_2$ and B. Since their inverse images are $\varnothing, A_1, A_2$ and A, respectively, the inverse images are all open in A. This means that f is continuous.

2. Suppose that $A \subseteq \mathbb{R}$ is connected and $f\colon A \to \mathbb{R}$ is continuous with $f(x) \neq 1$ for all $x \in A$. Show that either $f(x) > 1$ for all $x \in A$ or else $f(x) < 1$ for all $x \in A$.

Since $f(x) \neq 1$ for all $x \in A$, the set A can be expressed as the union of

$$A_1 = \{x \in A : f(x) > 1\}, \qquad A_2 = \{x \in A : f(x) < 1\}.$$

These sets are obviously disjoint. To show that they are also open in A, we note that

$$A_1 = \{x \in A : f(x) \in (1, \infty)\} = \{x \in A : f(x) \in U\} = f^{-1}(U),$$

where $U = (1, \infty)$. Since U is open in $\mathbb{R}$, its inverse image is open in A, so A_1 is open in A. The same argument shows that A_2 is open in A as well. Since their union $A = A_1 \cup A_2$ is connected, either A_1 or A_2 must be empty. If A_1 is empty, then A_2 is all of A and $f(x) < 1$ for all $x \in A$. If A_2 is empty, then A_1 is all of A and $f(x) > 1$ for all $x \in A$.

3. Show that the union of two countable sets is countable.

Suppose that A, B are countable. Then there exist surjections $f\colon \mathbb{N} \to A$ and $g\colon \mathbb{N} \to B$. To show that the union $A \cup B$ is countable, we need to find a surjection $h\colon \mathbb{N} \to A \cup B$. Let us then associate the even integers with the elements of A and the odd integers with the elements of B. More precisely, let us define the function h using the formula

$$h(x) = \left\{ \begin{array}{cl} f(x/2) & \text{if } x \text{ is even} \\ g((x+1)/2) & \text{if } x \text{ is odd} \end{array} \right\}.$$

Then every element $a \in A$ is in the image of h because $a = f(m)$ for some $m \in \mathbb{N}$ and this gives $a = h(2m)$. Similarly, every element $b \in B$ is in the image of h because $b = g(n)$ for some $n \in \mathbb{N}$ and this gives $b = h(2n-1)$. We conclude that h is surjective.

4. Show that the set A consisting of all subsets of $\mathbb{N}$ is uncountable.

Suppose that A is countable and $A_1, A_2, A_3, \ldots$ are the only subsets of $\mathbb{N}$. To construct another subset of $\mathbb{N}$ which does not appear in this list, consider the set

$$S = \{n \in \mathbb{N} : n \notin A_n\}.$$

Note that $n \in S$ whenever $n \notin A_n$ and thus $n \notin S$ whenever $n \in A_n$. This means that S is a subset of $\mathbb{N}$ which differs from each A_n, so it is a subset of $\mathbb{N}$ which does not appear in our list. We conclude that the set of all subsets of $\mathbb{N}$ cannot be countable.

5. Is the set $A = \{x \in \mathbb{R} : x^4 - 12x^2 + 16x \leq 0\}$ complete? Is it connected?

First of all, we simplify the given inequality. Factoring the polynomial, one finds that

$$x^4 - 12x^2 + 16x = x(x^3 - 12x + 16) = x(x-2)(x^2 + 2x - 8) = x(x-2)^2(x+4).$$

This expression is zero when $x = -4, 0, 2$ and it is negative when $x(x+4) < 0$, so

$$A = [-4, 0] \cup \{2\}.$$

Since A is the union of two closed sets, A is closed in $\mathbb{R}$ and thus complete. On the other hand, A is not connected because it is not one of the sets listed in Theorem 7.6.

6. Suppose that the sets $A, B \subseteq \mathbb{R}$ are nonempty, disjoint and open in $\mathbb{R}$. If there is a connected set U such that $U \subseteq A \cup B$, show that either $U \subseteq A$ or else $U \subseteq B$.

Consider the sets $U_1 = U \cap A$ and $U_2 = U \cap B$. These sets are obviously disjoint and open in U. Moreover, the former contains the elements of U which lie in A and the latter contains the elements of U which lie in B. Since all elements of U lie in either A or B, we conclude that $U_1 \cup U_2 = U$. On the other hand, U is connected, so this can only happen, if either U_1 or U_2 is empty. If U_1 is empty, then U_2 is all of U and so $U = U_2 \subseteq B$. If U_2 is empty, then U_1 is all of U and so $U = U_1 \subseteq A$.

7. Consider two functions $f\colon A \to B$ and $g\colon B \to C$. If f, g are both surjective, then show that $g \circ f$ is surjective. If $g \circ f$ is surjective, then show that g is surjective.

For the first part, suppose f, g are surjective and $c \in C$. Then there exists some $b \in B$ such that $g(b) = c$ and there exists some $a \in A$ such that $f(a) = b$. This gives

$$(g \circ f)(a) = g(f(a)) = g(b) = c$$

and so $g \circ f$ is surjective. For the second part, suppose $g \circ f$ is surjective and $c \in C$. Then there exists some $a \in A$ such that $g(f(a)) = c$. Once we now let $b = f(a) \in B$, we may conclude that $g(b) = g(f(a)) = c$. This shows that g is surjective.

8. Find a bijective function $f\colon (0,1] \to (0,1] \cup (2,3]$. Is such a function continuous?

Since $(0,1]$ is connected and its image is not, the function f is not continuous. To find a specific example, one may consider a function such as

$$f(x) = \left\{ \begin{array}{cc} 2x & \text{if } 0 < x \leq 1/2 \\ 2x+1 & \text{if } 1/2 < x \leq 1 \end{array} \right\}.$$

This is a piecewise linear function which maps $(0, 1/2]$ to $(0,1]$ and $(1/2, 1]$ to $(2,3]$.

9. Find a bijective function $f\colon A \to A - \{x_0\}$ when A is an infinite set and $x_0 \in A$.

Since A is infinite, it contains a sequence $x_0, x_1, x_2, \ldots$ of distinct points. Consider a function f that shifts all these terms to the right. More precisely, consider the function

$$f(x) = \left\{ \begin{array}{cc} x_{n+1} & \text{if } x = x_n \text{ for some } n \geq 0 \\ x & \text{otherwise} \end{array} \right\}.$$

This maps the points $x_0, x_1, x_2, \ldots$ to the points $x_1, x_2, x_3, \ldots$ and leaves all other points fixed. It is thus a bijective function between the set A and the set $A - \{x_0\}$.

10. Show that every subset of a countable set is countable.

Suppose A is countable and $B \subseteq A$. Then there exists an injective map $g\colon A \to \mathbb{N}$ and this gives a bijective map $g\colon B \to g(B)$. Since $g(B)$ is a subset of $\mathbb{N}$, it is countable by Theorem 8.3. One can thus find an injective map $h\colon g(B) \to \mathbb{N}$. This implies that the composition $h \circ g\colon B \to \mathbb{N}$ is injective and that the set B is countable.

11. Suppose A is a countable set. Show that there is no surjective map $f\colon A \to (0,1)$.

Suppose that such a map exists. Since A is countable by assumption, there also exists a surjective map $g\colon \mathbb{N} \to A$. The composition $f \circ g\colon \mathbb{N} \to (0,1)$ is then surjective as well. Using Theorem 8.4, we conclude that $(0,1)$ is countable. This contradicts Theorem 8.2.

12. A set $A \subseteq \mathbb{R}$ is called path connected if, given any two points $x, y \in A$, there exists a continuous function $f\colon [0,1] \to A$ such that $f(0) = x$ and $f(1) = y$. Show that every path connected subset of $\mathbb{R}$ is connected.

Suppose that $A \subseteq \mathbb{R}$ is path connected, but not connected. Then A can be expressed as a union $A = A_1 \cup A_2$ of two nonempty disjoint sets A_1, A_2 which are both open in A. Pick some points $x \in A_1$ and $y \in A_2$. Since A is path connected, there exists a continuous function $f\colon [0,1] \to A$ such that $f(0) = x$ and $f(1) = y$. Now, consider the inverse images

$$U_1 = f^{-1}(A_1), \qquad U_2 = f^{-1}(A_2).$$

Since $f(0) = x \in A_1$ and $f(1) = y \in A_2$, one has $0 \in U_1$ and $1 \in U_2$. Thus, the above sets are nonempty. To show that they are also disjoint, we note that

$$U_1 \cap U_2 = f^{-1}(A_1) \cap f^{-1}(A_2) = f^{-1}(A_1 \cap A_2) = \varnothing.$$

To show that their union is $[0,1]$, we use a similar computation to check that

$$U_1 \cup U_2 = f^{-1}(A_1) \cup f^{-1}(A_2) = f^{-1}(A_1 \cup A_2) = f^{-1}(A) = [0,1].$$

We conclude that $[0,1]$ is the union of two nonempty, disjoint sets U_1, U_2 which are both open in $[0,1]$. This contradicts Theorem 7.6 which asserts that $[0,1]$ is connected.

Analysis Solutions #7

1. Show that the set A consisting of all functions $f\colon \{0,1\} \to \mathbb{N}$ is countable.

Any such function is determined by its values $f(0)$ and $f(1)$. In other words, there is a bijective map $\varphi\colon A \to \mathbb{N}\times\mathbb{N}$ which is defined by $\varphi(f) = (f(0), f(1))$. Since $\mathbb{N}\times\mathbb{N}$ is countable by Example 8.6, there also exists a bijective map $g\colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$. It easily follows that the composition $g\circ\varphi\colon A \to \mathbb{N}$ is bijective and that A is countable.

2. Show that the set B consisting of all functions $f\colon \mathbb{N} \to \{0,1\}$ is uncountable.

Suppose that B is countable and $f_1, f_2, f_3, \ldots$ are the only such functions. In order to find another function which does not appear in this list, we define $f\colon \mathbb{N} \to \{0,1\}$ by

$$f(n) = \left\{ \begin{array}{ll} 0 & \text{if } f_n(n) = 1 \\ 1 & \text{if } f_n(n) = 0 \end{array} \right\}.$$

Since $f(n) \neq f_n(n)$ for each $n \in \mathbb{N}$, the function f differs from each of the given functions, so it does not appear in the original list and this is a contradiction.

3. Show that the union of two compact subsets of $\mathbb{R}$ is compact.

Suppose that A, B are compact subsets of $\mathbb{R}$. To show that their union is also compact, consider some sets U_i which form an open cover of $A\cup B$. The sets U_i cover $A\cup B$, so they certainly cover both A and B. Since A, B are compact, they are covered by finitely many of these sets. Suppose A is covered by m of them and B is covered by n of them. Then $A\cup B$ is covered by at most $m+n$ of the sets, so $A\cup B$ is compact as well.

4. Are the following subsets of $\mathbb{R}$ compact? Why or why not?

$$A = \{x\in\mathbb{R} : x^4 - 2x^2 - 8 \leq 0\}, \qquad B = \{x \in \mathbb{R} : x + \sin x \geq 0\}.$$

When it comes to the first set, one may factor the given polynomial to find that

$$x^4 - 2x^2 - 8 = (x^2 - 4)(x^2 + 2).$$

Since $x^2 + 2$ is always positive, this gives $A = [-2, 2]$ and so A is compact by Theorem 9.9. On the other hand, it is easy to see that B is not compact because

$$x \geq 1 \quad\Longrightarrow\quad x + \sin x \geq 1 + \sin x \geq 0 \quad\Longrightarrow\quad x \in B.$$

This means that the set B is unbounded, so B is not compact by Theorem 9.4.

5. Let $\{x_n\}$ be a sequence of real numbers such that x_n converges to x as $n \to \infty$ and consider the set $A = \{x, x_1, x_2, x_3, \ldots\}$. Show that A is a compact subset of $\mathbb{R}$.

Suppose that the sets U_i form an open cover of A. Then A is contained in the union of these sets, so $x \in U_{i_0}$ for some index i_0. Since U_{i_0} is open, it follows by Theorem 3.9(c) that U_{i_0} must contain $x_N, x_{N+1}, \ldots$ for some natural number N. Thus, the only terms which are not contained in U_{i_0} are the terms x_k with $1 \leq k < N$. These are finitely many terms and each of them lies in A, so each of them lies in U_{i_k} for some index i_k. In particular, all the elements of A are contained in finitely many of the given sets and A is compact.

6. Show that none of the following sets are compact.

$$A = (0, \infty), \qquad B = (1, 3), \qquad C = \{x \in \mathbb{R} : x^2 \geq x\}, \qquad D = \{1/n : n \in \mathbb{N}\}.$$

The sets A, C contain every real number $x \geq 1$, so these sets are unbounded and they are not compact by Theorem 9.4. To show that B is not compact, consider the function

$$f \colon B \to \mathbb{R}, \qquad f(x) = 1/(3 - x).$$

Then f is continuous on B and $f(3 - \frac{1}{n}) = n$ for each $n \in \mathbb{N}$, so f is unbounded. In view of the Extreme Value Theorem, this means that B is not compact. For the set D, we take

$$g \colon D \to \mathbb{R}, \qquad g(x) = 1/x.$$

Then g is continuous on D and $g(1/n) = n$ for each $n \in \mathbb{N}$, so D is not compact, either.

7. Show that there exists no continuous surjective function $f \colon [0, 1] \to A$ when

$$A = [0, 1] \cup [2, 3], \qquad A = [0, 1), \qquad A = \mathbb{Q} \cap [0, 1], \qquad A = (0, \infty).$$

Suppose that such a function exists. Since the interval $[0, 1]$ is connected, the image A must then be connected, so it must have the intermediate point property. This is not true for either the first or the third set. For instance, the third set contains $0, 1$ but not $\frac{1}{2}\sqrt{2}$.

A similar argument applies for the other two sets as well. Since $[0, 1]$ is compact, the image A must also be compact. However, $(0, \infty)$ is not compact because it is unbounded and $[0, 1)$ is not compact since $f(x) = 1/(1 - x)$ is continuous, but not bounded, on $[0, 1)$.

8. Find a bijective function $f \colon [0, 1] \to [0, 1)$. Is such a function continuous?

The first part is a special case of Problem 9 from Problem Set #6. As long as the set A is infinite, there is always a bijection $f \colon A \to A - \{x_0\}$ for any element $x_0 \in A$. In fact, one may simply pick a sequence of points in A and shift them to the right. For instance, let

$$f(x) = \left\{ \begin{array}{cc} 1/(n+1) & \text{if } x = 1/n \text{ for some } n \in \mathbb{N} \\ x & \text{otherwise} \end{array} \right\}.$$

This function maps $1, \frac{1}{2}, \frac{1}{3}, \dots$ to $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ and leaves all other points fixed, so it defines a bijection $f\colon [0,1] \to [0,1)$. We now show that such a bijection cannot be continuous. If it were continuous, then the fact that $[0,1]$ is compact would imply that the image $[0,1)$ is compact. As we saw in the previous problem, however, the interval $[0,1)$ is not compact.

9. Show that every open subset of $\mathbb{R}$ can be written as the union of open intervals (r,s) whose endpoints r,s are rational numbers with $r<s$.

Suppose that $A \subseteq \mathbb{R}$ is open. Given any $x \in A$, we can then find some $\varepsilon_x > 0$ such that

$$(x-\varepsilon_x, x+\varepsilon_x) \subseteq A.$$

Pick a rational number $r_x \in (x-\varepsilon_x, x)$ and a rational number $s_x \in (x, x+\varepsilon_x)$ so that

$$x-\varepsilon_x < r_x < x < s_x < x+\varepsilon_x \quad \Longrightarrow \quad x \in (r_x, s_x) \subseteq (x-\varepsilon_x, x+\varepsilon_x) \subseteq A.$$

Since the set A can be expressed as the union of its elements, we conclude that

$$A = \bigcup_{x\in A} \{x\} \subseteq \bigcup_{x \in A} (r_x, s_x) \subseteq A.$$

In particular, the above sets are all equal and A is the union of the intervals (r_x, s_x).

10. Show that a set $A \subseteq \mathbb{Z}$ is compact if and only if it is finite.

If $A \subseteq \mathbb{Z}$ is finite, then A is compact by Example 9.2. Conversely, suppose $A \subseteq \mathbb{Z}$ is compact. Then A must be bounded, so $A \subseteq [-N, N]$ for some positive integer N. Since A consists entirely of integers, this implies that $A \subseteq \{0, \pm 1, \dots, \pm N\}$ and so A is finite.

11. Suppose that $A \subseteq \mathbb{R}$ is nonempty and compact. Show that $\max A$ exists.

Consider the inclusion map $i\colon A \to \mathbb{R}$ which is defined by $i(x) = x$ for all $x \in A$. This is continuous by Theorem 5.10 and A is compact, so i attains a maximum value by the Extreme Value Theorem. Needless to say, this maximum value is the largest element of A.

12. Suppose that $A \subseteq \mathbb{R}$ is compact and $f\colon A \to A$ is continuous with

$$|f(x) - f(y)| < |x-y| \quad \text{for all } x \neq y.$$

Show that there exists a point $x_0 \in A$ such that $f(x_0) = x_0$.

Consider the function $g\colon A \to \mathbb{R}$ defined by $g(x) = |f(x) - x|$. Being the composition of continuous functions, g is then continuous. Since A is compact, it follows by the Extreme Value Theorem that g attains a minimum value $g(x_0)$ at some point $x_0 \in A$. If $g(x_0) = 0$, then $f(x_0) = x_0$ and the proof is complete. Otherwise, $g(x_0) \neq 0$ and so $f(x_0) \neq x_0$. Using the given inequality, one may thus conclude that

$$|f(f(x_0)) - f(x_0)| < |f(x_0) - x_0| \quad \Longrightarrow \quad g(f(x_0)) < g(x_0).$$

Since $f(x_0) \in A$, this contradicts the fact that $g(x_0)$ is the minimum value attained by g.

Analysis Solutions #8

1. Suppose $A \subseteq \mathbb{R}$ is bounded. Show that its closure $\overline{A}$ is compact.

A subset of $\mathbb{R}$ is compact if and only if it is bounded and closed in $\mathbb{R}$. Since $\overline{A}$ is closed by definition, it remains to show that $\overline{A}$ is bounded. Since A is bounded, $A \subseteq [-N, N]$ for some $N > 0$. Moreover, $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$. This gives $\overline{A} \subseteq [-N, N]$, so $\overline{A}$ is bounded.

2. Suppose $A \subseteq \mathbb{R}$ is compact and $B \subseteq A$ is closed in A. Show that B is compact.

A subset of $\mathbb{R}$ is compact if and only if it is bounded and closed in $\mathbb{R}$. In this case, A is compact, so it is bounded and $B \subseteq A$ is bounded as well. To show that B is closed in $\mathbb{R}$, we use the fact that B is closed in A. Write $B = A \cap C$ for some set C which is closed in $\mathbb{R}$. Since B is the intersection of two closed sets, B is then closed as well.

3. Are the following subsets of $\mathbb{R}$ compact? Why or why not?

$$A = \{x \in \mathbb{R} : \sin x + \cos x \leq 1\}, \qquad B = \{x \in \mathbb{R} : x^2 + \sin x \leq 1\}.$$

The first set is not compact because it is not bounded. For instance, $x = 0$ is an element of A and the functions $\sin x$, $\cos x$ are periodic, so any integer multiple of 2π is an element of A as well. To show that B is compact, we let $f(x) = x^2 + \sin x$ and we note that

$$B = \{x \in \mathbb{R} : f(x) \leq 1\} = \{x \in \mathbb{R} : f(x) \in U\} = f^{-1}(U),$$

where $U = (-\infty, 1]$. Since U is closed in $\mathbb{R}$ and f is continuous, the inverse image must be closed in $\mathbb{R}$, so B is closed in $\mathbb{R}$. To show that B is also bounded, we note that

$$x \in B \quad \Longrightarrow \quad x^2 + \sin x \leq 1 \quad \Longrightarrow \quad x^2 \leq 1 - \sin x \leq 2.$$

4. Suppose $A \subseteq \mathbb{R}$ is nonempty and $f\colon A \to \mathbb{R}$ is continuous. If the set A is bounded, must $f(A)$ be bounded? If the set A is closed, must $f(A)$ be closed?

The first part is not true because $A = (0, 1)$ is bounded and $f(x) = 1/x$ continuous, but the image $f(A) = (1, \infty)$ is not bounded. For the second part, one may consider sets A which are both closed and bounded. In that case, A is compact, so $f(A)$ is compact and thus closed. If one considers sets A which are closed but not bounded, however, the result is not true. For instance, $A = [1, \infty)$ is closed in $\mathbb{R}$, but its image $f(A) = (0, 1]$ is not.

5. Suppose $A \subseteq \mathbb{R}$ is compact. Show that every infinite subset of A has a limit point.

Suppose that $B \subseteq A$ is infinite and B has no limit points. Then every element $x \in A$ has a neighbourhood U_x which does not intersect B at a point other than x. Consider the collection of all the neighbourhoods U_x. Since those contain all elements of A, they form an open cover of A, so finitely many of them cover A. In other words, we have

$$B \subseteq A \subseteq U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_n}$$

for some n. Now, each neighbourhood U_{x_k} does not intersect B at a point other than x_k. Thus, the right hand side contains only a finite number of elements of B. Since the right hand side contains all elements of B, this means that B is finite, a contradiction.

6. What can you say about a set $A \subseteq \mathbb{R}$, if every subset of A is compact?

We show that only finite sets have this property. If $A \subseteq \mathbb{R}$ is finite, then A is compact by Example 9.2. Moreover, every subset of A is finite, so every subset of A is compact.

Suppose now that $A \subseteq \mathbb{R}$ is infinite and compact. Then A contains a sequence $\{x_n\}$ of distinct points. Since A is compact, it is bounded, so the sequence is bounded as well. In view of the Bolzano-Weierstrass theorem, it must thus have a convergent subsequence. Let us denote the limit of this subsequence by x. Since A is compact, it is also closed, so x is an element of A by Theorem 4.6. This implies that $A - \{x\}$ is not closed because it contains a convergent subsequence but not its limit. In particular, $A - \{x\}$ is not compact, either.

7. Show that the function $f\colon [0, a] \to \mathbb{R}$ is integrable for any $a > 0$ when $f(x) = x^2$.

Consider a partition $P = \{x_0, x_1, \ldots, x_n\}$ of equally spaced points and note that

$$\begin{aligned} m_k &= \inf \{f(x) : x_k \leq x \leq x_{k+1}\} = x_k^2, \\ M_k &= \sup \{f(x) : x_k \leq x \leq x_{k+1}\} = x_{k+1}^2 \end{aligned}$$

for each $0 \leq k \leq n - 1$. Using the definition of Darboux sums, one now finds that

$$U(f, P) - L(f, P) = \sum_{k=0}^{n-1} (M_k - m_k) \cdot (x_{k+1} - x_k) = \sum_{k=0}^{n-1} (x_{k+1} + x_k)(x_{k+1} - x_k)^2.$$

In this case, $x_{k+1} + x_k \leq 2a$ for each k and we also have $x_{k+1} - x_k = a/n$, so

$$U(f, P) - L(f, P) \leq \sum_{k=0}^{n-1} 2a \cdot \frac{a^2}{n^2} = \frac{2a^3}{n}.$$

Since the right hand side approaches zero as $n \to \infty$, we conclude that f is integrable.

8. Show that the function $f\colon [0,1] \to \mathbb{R}$ is integrable for any $a, b \in \mathbb{R}$ when

$$f(x) = \left\{ \begin{array}{ll} a & \text{if } x \neq 0 \\ b & \text{if } x = 0 \end{array} \right\}.$$

Consider the partition $P_n = \{0, 1/n, 1\}$ for any integer $n \geq 2$. In this case, one has

$$\begin{aligned} m_0 &= \inf \{f(x) : 0 \leq x \leq 1/n\} = \min\{a, b\}, \\ M_0 &= \sup \{f(x) : 0 \leq x \leq 1/n\} = \max\{a, b\} \end{aligned}$$

and also $f(x) = a$ throughout $[1/n, 1]$, so $m_1 = M_1 = a$. It easily follows that

$$U(f, P_n) - L(f, P_n) = \frac{M_0 - m_0}{n} = \frac{\max\{a, b\} - \min\{a, b\}}{n}.$$

Since the right hand side approaches zero as $n \to \infty$, we conclude that f is integrable.

9. Suppose that $f\colon [0,1] \to \mathbb{R}$ is integrable and let $a > 0$. If the function $g\colon [0,a] \to \mathbb{R}$ is defined by $g(x) = f(x/a)$, show that g is integrable and $\int_0^a g(x)\,dx = a \int_0^1 f(x)\,dx$.

Given a partition $P = \{x_0, x_1, \ldots, x_n\}$ of the interval $[0, a]$, one may let $y_k = x_k/a$ for each k to obtain a partition Q of the interval $[0, 1]$. In addition, it is easy to check that

$$\begin{aligned} m_k(g) &= \inf \{g(x) : x_k \leq x \leq x_{k+1}\} \\ &= \inf \{f(x/a) : y_k \leq x/a \leq y_{k+1}\} = m_k(f) \end{aligned}$$

and similarly $M_k(g) = M_k(f)$. Thus, the corresponding Darboux sums satisfy the relation

$$\begin{aligned} U(g, P) - L(g, P) &= \sum_{k=0}^{n-1} (M_k(g) - m_k(g)) \cdot (x_{k+1} - x_k) \\ &= \sum_{k=0}^{n-1} (M_k(f) - m_k(f)) \cdot a(y_{k+1} - y_k) \\ &= a \cdot [U(f, Q) - L(f, Q)]. \end{aligned}$$

Since f is integrable by assumption, this expression becomes arbitrarily small and so g is integrable as well. Using the definition of the integral, one may thus compute

$$\begin{aligned} \int_0^a g(x)\,dx &= \sup \{L(g, P) : P \text{ is a partition of } [0, a]\} \\ &= \sup \{aL(f, Q) : Q \text{ is a partition of } [0, 1]\} = a \int_0^1 f(x)\,dx. \end{aligned}$$

10. Suppose $f\colon [0,1] \to [0,\infty)$ is integrable with $f(x) = 0$ for all $x \in \mathbb{Q}$. Show that

$$\int_0^1 f(x)\,dx = 0.$$

Given any partition $P = \{x_0, x_1, \ldots, x_n\}$ of the interval $[0,1]$, one has

$$m_k = \inf\{f(x) : x_k \le x \le x_{k+1}\} = 0$$

for each $0 \le k \le n-1$. This is because $f(x)$ is non-negative and each interval $[x_k, x_{k+1}]$ contains a rational number. In particular, $L(f,P) = 0$ for any partition P, and thus

$$\int_0^1 f(x)\,dx = \sup\{L(f,P) : P \text{ is a partition of } [0,1]\} = 0.$$

11. Let $a < b$. Find a function $f\colon [a,b] \to \mathbb{R}$ such that f^2 is integrable, but f is not.

We give an example which is very similar to Example 10.6. Consider the function

$$f(x) = \left\{ \begin{array}{cl} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \notin \mathbb{Q} \end{array} \right\}.$$

Given any partition $P = \{x_0, x_1, \ldots, x_n\}$ of the interval $[a,b]$, it is easy to see that

$$\begin{aligned} m_k &= \inf\{f(x) : x_k \le x \le x_{k+1}\} = -1, \\ M_k &= \sup\{f(x) : x_k \le x \le x_{k+1}\} = 1 \end{aligned}$$

for each $0 \le k \le n-1$. This is because each interval $[x_k, x_{k+1}]$ contains both a rational and an irrational number. In particular, all the lower Darboux sums are equal to

$$L(f,P) = \sum_{k=0}^{n-1} m_k(x_{k+1} - x_k) = -\sum_{k=0}^{n-1}(x_{k+1} - x_k) = -(b-a)$$

and all the upper Darboux sums are equal to $U(f,P) = b - a$. This implies that

$$\mathcal{L}(f) = a - b < 0, \qquad \mathcal{U}(f) = b - a > 0 \quad \Longrightarrow \quad \mathcal{L}(f) \neq \mathcal{U}(f),$$

so f is not integrable. Since f^2 is constant, however, f^2 is integrable by Example 10.5.

12. Let $a < b$ and suppose $f\colon [a,b] \to \mathbb{R}$ is increasing. Show that f is integrable on $[a,b]$.

If the partition $P = \{x_0, x_1, \ldots, x_n\}$ consists of equally spaced points, then

$$\begin{aligned} m_k &= \inf\{f(x) : x_k \le x \le x_{k+1}\} = f(x_k), \\ M_k &= \sup\{f(x) : x_k \le x \le x_{k+1}\} = f(x_{k+1}) \end{aligned}$$

for each $0 \le k \le n-1$. In particular, the Darboux sums of f satisfy the relation

$$U(f,P) - L(f,P) = \sum_{k=0}^{n-1} (M_k - m_k) \cdot (x_{k+1} - x_k) = \frac{b-a}{n} \sum_{k=0}^{n-1} \big(f(x_{k+1}) - f(x_k) \big).$$

This relation involves a telescoping sum that can be simplified to give

$$U(f,P) - L(f,P) = \frac{b-a}{n} \cdot \Big[f(x_n) - f(x_0) \Big] = \frac{b-a}{n} \cdot \Big[f(b) - f(a) \Big].$$

Since the right hand side approaches zero as $n \to \infty$, we conclude that f is integrable.

Analysis Solutions #9

1. Suppose that $f\colon [a,b] \to \mathbb{R}$ is integrable and let I be a real number such that

$$L(f,P) \leq I \leq U(f,P)$$

for all partitions P of $[a,b]$. Show that I must be equal to $I = \int_a^b f(x)\,dx$.

Since $L(f,P) \leq I$ for any partition P, the number I is an upper bound for the lower Darboux sums. The integral is the least such upper bound by definition, so

$$\int_a^b f(x)\,dx = \sup\{L(f,P) : P \text{ is a partition of } [a,b]\} \leq I.$$

The opposite inequality follows in a similar manner. Since $I \leq U(f,P)$ for any partition P, the number I is a lower bound for the upper Darboux sums and this implies that

$$\int_a^b f(x)\,dx = \inf\{U(f,P) : P \text{ is a partition of } [a,b]\} \geq I.$$

2. Suppose f is integrable on the interval $[a,b]$ and let $a < c < b$. Show that f is also integrable on the subintervals $[a,c]$ and $[c,b]$.

Let $\varepsilon > 0$ be given. Then there exists a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon.$$

The refinement $Q = P \cup \{c\}$ satisfies $U(f,Q) \leq U(f,P)$ and also $L(f,Q) \geq L(f,P)$, so

$$U(f,Q) - L(f,Q) \leq U(f,P) - L(f,P) < \varepsilon.$$

Now, the partition Q has the form $Q = \{x_0, x_1, \ldots, x_n\}$, where $x_i = c$ for some i. One may thus decompose Q into a partition Q_1 of $[a,c]$ and a partition Q_2 of $[c,b]$. The sum

$$L(f,Q) = \sum_{k=0}^{n-1} \inf\{f(x) : x_k \leq x \leq x_{k+1}\} \cdot (x_{k+1} - x_k)$$

involves both the terms $x_k < x_i = c$ and the terms $x_k \geq x_i = c$, so one clearly has

$$L(f,Q) = L(f,Q_1) + L(f,Q_2).$$

Since a similar relation holds for the upper Darboux sums, one may then conclude that

$$[U(f,Q_1) - L(f,Q_1)] + [U(f,Q_2) - L(f,Q_2)] = U(f,Q) - L(f,Q) < \varepsilon.$$

Here, the left hand side is a sum of two non-negative terms whose sum is smaller than ε. Thus, each of the terms is smaller than ε and f is integrable on both $[a,c]$ and $[c,b]$.

3. Suppose $f\colon [a,b] \to \mathbb{R}$ is a bounded function which is zero at all points except for one point. Show that f is integrable on $[a,b]$ and $\int_a^b f(x)\,dx = 0$.

Suppose that f is zero at all points except for the point c and let $P_n = \{x_0, x_1, \ldots, x_n\}$ be a partition that consists of equally spaced points. Then one clearly has

$$m_k = \inf\{f(x) : x_k \leq x \leq x_{k+1}\} = 0,$$
$$M_k = \sup\{f(x) : x_k \leq x \leq x_{k+1}\} = 0$$

on any interval $[x_k, x_{k+1}]$ which does not contain the point c, while

$$m_i = \inf\{f(x) : x_i \leq x \leq x_{i+1}\} = \min\{f(c), 0\},$$
$$M_i = \sup\{f(x) : x_i \leq x \leq x_{i+1}\} = \max\{f(c), 0\}$$

on the remaining interval $[x_i, x_{i+1}]$ which contains c. This gives rise to the relation

$$\begin{aligned} U(f, P_n) - L(f, P_n) &= (M_i - m_i) \cdot (x_{i+1} - x_i) \\ &= \Big[\max\{f(c), 0\} - \min\{f(c), 0\}\Big] \cdot \frac{b-a}{n}. \end{aligned}$$

Since the right hand side approaches zero as $n \to \infty$, we conclude that f is integrable.

To compute the integral of f, we note that the upper Darboux sums are equal to

$$U(f, P_n) = M_i(x_{i+1} - x_i) = \max\{f(c), 0\} \cdot \frac{b-a}{n}.$$

These are all non-negative and arbitrarily small, so the definition of the integral gives

$$\int_a^b f(x)\,dx = \inf\{U(f, P) : P \text{ is a partition of } [a,b]\} = 0.$$

4. Suppose $f\colon [a,b] \to \mathbb{R}$ is bounded and $g\colon [a,b] \to \mathbb{R}$ is integrable. If $f(x) = g(x)$ at all points except for finitely many points, show that f is integrable on $[a,b]$.

The difference $h = f - g$ is zero at all points except for finitely many points. One may thus express h as a sum of functions which are zero at all points except for one. In view of the previous problem, each of these functions is integrable. It follows by Theorem 11.3 that their sum h is also integrable. In particular, $f = g + h$ is integrable as well.

5. Find two bounded functions $f, g\colon [a,b] \to \mathbb{R}$ such that $f(x) = g(x)$ at all points except for countably many points and g is integrable on $[a,b]$, while f is not.

Consider the function f defined by $f(x) = 1$ for all $x \in \mathbb{Q}$ and $f(x) = 0$ for all $x \notin \mathbb{Q}$. This is not integrable by Example 10.6, but it is bounded and it satisfies $f(x) = 0$ except for countably many points. On the other hand, $g(x) = 0$ is certainly integrable on $[a,b]$.

6. Show that the function $f\colon [0,2\pi] \to \mathbb{R}$ is integrable when

$$f(x) = \left\{ \begin{array}{ll} \sin x & \text{if } 0 \leq x \leq \pi \\ \cos x & \text{if } \pi < x \leq 2\pi \end{array} \right\}.$$

When it comes to the interval $[0,\pi]$, the function f is continuous and thus integrable by Theorem 11.1. When it comes to the interval $[\pi, 2\pi]$, one has $f(x) = \cos x$ at all points except for one, so f is integrable on $[\pi, 2\pi]$ by Problem 4. Since integrals are additive by Theorem 11.2, we conclude that f is integrable on $[0, 2\pi]$ as well.

7. Suppose $f\colon [a,b] \to \mathbb{R}$ is continuous. Show that there exists some $c \in [a,b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

Since f is continuous on the compact interval $[a,b]$, it attains both a minimum value m and a maximum value M. This gives $m \leq f(x) \leq M$ for all $x \in [a,b]$ and thus

$$\int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx \quad \Longrightarrow \quad m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

by Theorem 11.5 and Example 10.5. Note that f attains the values m, M and that

$$m \leq \frac{1}{b-a} \int_a^b f(x)\,dx \leq M.$$

The result now follows by the intermediate value theorem because f is continuous.

8. Suppose $f\colon [a,b] \to \mathbb{R}$ is continuous and $g\colon [a,b] \to \mathbb{R}$ is a non-negative, integrable function. Show that there exists some $c \in [a,b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(c) \int_a^b g(x)\,dx.$$

Since f is continuous on the compact interval $[a,b]$, it attains both a minimum value m and a maximum value M. This gives $m \leq f(x) \leq M$ for all $x \in [a,b]$ and thus

$$mg(x) \leq f(x)g(x) \leq Mg(x) \quad \Longrightarrow \quad m\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq M\int_a^b g(x)\,dx$$

by Theorem 11.5. If the integral $I = \int_a^b g(x)\,dx$ is zero, we have $\int_a^b f(x)g(x)\,dx = 0$, so the given equation holds for all $c \in [a,b]$. If I is nonzero, on the other hand, we have

$$mI \leq \int_a^b f(x)g(x)\,dx \leq MI \quad \Longrightarrow \quad m \leq \frac{1}{I} \int_a^b f(x)g(x)\,dx \leq M.$$

Since f attains the values m and M, it attains every other value that lies between them by the intermediate value theorem. In particular, there exists some $c \in [a, b]$ such that

$$\frac{1}{I}\int_a^b f(x)g(x)\,dx = f(c) \quad\Longrightarrow\quad \int_a^b f(x)g(x)\,dx = f(c)I = f(c)\int_a^b g(x)\,dx.$$

9. Given a continuous function $f\colon [a, b] \to \mathbb{R}$, show that $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$.

This inequality follows easily by Theorem 11.5. Since $\pm f(x) \leq |f(x)|$, that is, one has

$$\left|\int_a^b f(x)\,dx\right| = \pm\int_a^b f(x)\,dx = \int_a^b [\pm f(x)]\,dx \leq \int_a^b |f(x)|\,dx.$$

10. Suppose $f\colon [a, b] \to \mathbb{R}$ is continuous and let $F(z) = \int_a^z f(x)\,dx$ for each $z \in [a, b]$. Show that the function F is continuous as well.

Let $\varepsilon > 0$ be given. To prove continuity, one needs to find some $\delta > 0$ such that

$$|y - z| < \delta \quad\Longrightarrow\quad |F(y) - F(z)| < \varepsilon$$

for all $y, z \in [a, b]$. Since f is continuous on the compact interval $[a, b]$, it is bounded, so there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a, b]$. When $y \geq z$, one can write

$$F(y) - F(z) = \int_a^y f(x)\,dx - \int_a^z f(x)\,dx = \int_z^y f(x)\,dx$$

because integrals are additive by Theorem 11.2. Using the previous problem, we now get

$$|F(y) - F(z)| = \left|\int_z^y f(x)\,dx\right| \leq \int_z^y |f(x)|\,dx \leq \int_z^y M\,dx = M(y - z).$$

The exact same argument applies when $y \leq z$. In that case, one finds that

$$|F(z) - F(y)| = \left|\int_y^z f(x)\,dx\right| \leq \int_y^z |f(x)|\,dx \leq \int_y^z M\,dx = M(z - y).$$

This gives $|F(y) - F(z)| \leq M|y - z|$ in any case. Letting $\delta = \varepsilon/M$, we conclude that

$$|y - z| < \delta \quad\Longrightarrow\quad |F(y) - F(z)| \leq M|y - z| < M\delta = \varepsilon.$$

11. Suppose $f\colon [a,b] \to \mathbb{R}$ is bounded and integrable. Show that f^2 is integrable.

Since f is bounded, one has $|f(x)| \leq M$ for some $M > 0$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$ and consider the expressions

$$m_k(f) = \inf\{f(x) : x_k \leq x \leq x_{k+1}\}, \qquad M_k(f) = \sup\{f(x) : x_k \leq x \leq x_{k+1}\}.$$

Given any two points $y, z \in [x_k, x_{k+1}]$, one has $f(y), f(z) \in [m_k(f), M_k(f)]$ and thus

$$f(y)^2 - f(z)^2 = [f(y) + f(z)] \cdot [f(y) - f(z)] \leq 2M \cdot [M_k(f) - m_k(f)].$$

This gives an upper bound for the values $f(y)^2$ and $M_k(f^2)$ is the least upper bound, so

$$M_k(f^2) \leq 2M \cdot [M_k(f) - m_k(f)] + f(z)^2$$

for any $z \in [x_k, x_{k+1}]$. This gives a lower bound for the values $f(z)^2$, so we similarly get

$$M_k(f^2) - 2M \cdot [M_k(f) - m_k(f)] \leq m_k(f^2).$$

It remains to relate the lower and upper Darboux sums for f and f^2. Noting that

$$U(f^2, P) - L(f^2, P) = \sum_{k=0}^{n-1} [M_k(f^2) - m_k(f^2)] \cdot (x_{k+1} - x_k)$$

by definition, one may combine the last two equations to find that

$$\begin{aligned} U(f^2, P) - L(f^2, P) &\leq 2M \cdot \sum_{k=0}^{n-1} [M_k(f) - m_k(f)] \cdot (x_{k+1} - x_k) \\ &= 2M \cdot [U(f, P) - L(f, P)]. \end{aligned}$$

Since the right hand side is arbitrarily small, we conclude that f^2 is integrable as well.

12. Suppose $f, g\colon [a,b] \to \mathbb{R}$ are bounded and integrable. Show that fg is integrable.

The result follows easily by the previous problem. Since f, g are integrable, one finds that $f \pm g$ are also integrable, so the same is true for $(f \pm g)^2$. On the other hand,

$$(f+g)^2 - (f-g)^2 = (f^2 + 2fg + g^2) - (f^2 - 2fg + g^2) = 4fg.$$

This gives $fg = \frac{1}{4}(f+g)^2 - \frac{1}{4}(f-g)^2$, so fg is integrable by Theorems 11.3 and 11.4.