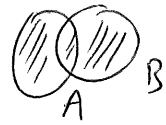
Notation. We use A, B, C etc. for sets and x, y, z etc. for elements

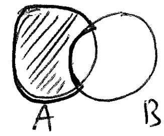
$A \subseteq B$: means every element of A is an element of B

union $A \cup B$: $x \in A \cup B$ means $x \in A$ or $x \in B$



intersection $A \cap B$: $x \in A \cap B$ means $x \in A$ and $x \in B$

difference $A - B$: $x \in A - B$ means $x \in A$ but $x \notin B$.



$\boxed{\text{De Morgan's Laws}}$ One has $A - (B \cup C) = (A-B) \cap (A-C)$

and $A - (B \cap C) = (A-B) \cup (A-C)$.

Proof. We check the first statement. ~~The~~

We prove $A - (B \cup C) \subseteq (A-B) \cap (A-C)$

Suppose $x \in A - (B \cup C)$.

Then $x \in A$ but $x \notin B \cup C$.

So $x \in A$ but $x \notin B$ and $x \notin C$.

So $x \in A - B$ and $x \in A - C$.

So $x \in (A-B) \cap (A-C)$.

We prove $(A-B) \cap (A-C) \subseteq A - (B \cup C)$.

Suppose $x \in (A-B) \cap (A-C)$.

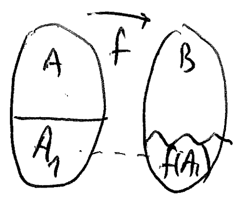
Then $x \in A - B$ and $x \in A - C$

$\Rightarrow$ $x \in A$ but $x \notin B$, $x \notin C$

$\Rightarrow$ $x \in A$ but $x \notin B \cup C$

$\Rightarrow$ $x \in A - (B \cup C)$. ▨

$\boxed{\text{Image of a set}}$ Let $f : A \to B$ be a function.



Given a set $A_1 \subseteq A$, we define $f(A_1) = \{ f(x) : x \in A_1 \}$.

① Images preserve inclusions and unions:

ⓐ $A_1 \subseteq A_2$ implies $f(A_1) \subseteq f(A_2)$

ⓑ $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

② For intersections, one has $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.

Moreover, equality holds when f is injective.

③ For differences, one has $f(A_1 - A_2) \supseteq f(A_1) - f(A_2)$.

Moreover, equality holds when f is injective.

Proof of 1a. Suppose $A_1 \subseteq A_2$. We need to show $f(A_1) \subseteq f(A_2)$.

Indeed, $y \in f(A_1) \Rightarrow y = f(x)$ for some $x \in A_1$

$\Rightarrow y = f(x)$ for some $x \in A_2$

$\Rightarrow y \in f(A_2)$. ▨

Proof of 1b. We show $f(A_1 \cup A_2) \subseteq f(A_1) \cup f(A_2)$ first.

Let $y \in f(A_1 \cup A_2)$. Then $y = f(x)$ for some $x \in A_1 \cup A_2$.

So $y = f(x)$ with $x \in A_1$ or $x \in A_2$

So $y \in f(A_1)$ or $y \in f(A_2)$

So $y \in f(A_1) \cup f(A_2)$. This proves the inclusion.

The same reasoning gives $f(A_1) \cup f(A_2) \subseteq f(A_1 \cup A_2)$. ∎

Proof of ~~1c~~ 2. We check $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.

Indeed, $y \in f(A_1 \cap A_2) \Rightarrow y = f(x)$ for some $x \in A_1 \cap A_2$

$\Rightarrow y = f(x)$ with $x \in A_1$ and $x \in A_2$

$\Rightarrow y \in f(A_1)$ and $y \in f(A_2)$

$\Rightarrow y \in f(A_1) \cap f(A_2)$. ∎

Note: consider the case $f(x) = x^2$ and $A_1 = [-1, 0]$ and $A_2 = [0, 1]$.

Then $A_1 \cap A_2 = \{0\}$ and $f(A_1 \cap A_2) = \{0\}$.

However, $f(A_1) = [0,1] = f(A_2)$ so $f(A_1) \cap f(A_2) = [0,1]$.

Proof of ~~1c~~ 2 assuming f is injective. We check $f(A_1) \cap f(A_2) \subseteq f(A_1 \cap A_2)$.

Indeed, $y \in f(A_1) \cap f(A_2) \Rightarrow y \in f(A_1)$ and $y \in f(A_2)$

$\Rightarrow y = f(x_1)$ with $x_1 \in A_1$, $y = f(x_2)$ with $x_2 \in A_2$

$\Rightarrow y = f(x_1) = f(x_2)$ with $x_1 = x_2 \in A_1 \cap A_2$

$\Rightarrow y = f(x)$ for some $x \in A_1 \cap A_2$

$\Rightarrow y \in f(A_1 \cap A_2)$. ∎

Proof of ~~1d~~ 3. To check $f(A_1 - A_2) \supseteq f(A_1) - f(A_2)$, we note that

$y \in f(A_1) - f(A_2) \Rightarrow y \in f(A_1)$ but $y \notin f(A_2)$

$\Rightarrow y = f(x)$ for some $x \in A_1$, but $y \notin f(A_2)$

$\Rightarrow y = f(x)$ for some $x \in A_1 - A_2$

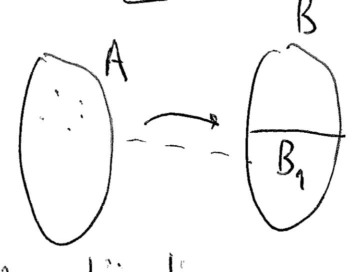
$\Rightarrow y \in f(A_1 - A_2)$. ∎

<u>Proof of ③ assuming f injective</u>. We check $f(A_1 - A_2) \subseteq f(A_1) - f(A_2)$ so that the sets are equal. Let $y \in f(A_1 - A_2)$. Then

$$y = f(x) \text{ for some } x \in A_1 - A_2 \implies y = f(x) \text{ with } x \in A_1 \text{ but } x \notin A_2$$
$$\implies y \in f(A_1) \text{ but } y \notin f(A_2) \text{ by inject.}$$

Namely, were $y = f(z)$ for some $z \in A_2$ implies $x = z \in A_2$, a contradiction. This proves $y \in f(A_1) - f(A_2)$, as needed. ▨

$\boxed{\text{Inverse images}}$ Let $f: A \to B$ and $B_1 \subseteq B$.



We define $f^{-1}(B_1) = \{x \in A : f(x) \in B_1\}$.

This makes sense for any function f, not necessarily bijective.

<u>Example</u>. Take $f(x) = x^2$ and $B_1 = [-2, -1]$. Then ~~$f^{-1}(B_1) =$~~

$$f^{-1}(B_1) = \{x \in \mathbb{R} : x^2 \in B_1\} = \{x \in \mathbb{R} : -2 \le x^2 \le -1\} = \emptyset.$$

And if $B_2 = [1, 4]$, then $f^{-1}(B_2) = \{x \in \mathbb{R} : 1 \le x^2 \le 4\} = [1,2] \cup [-2,-1]$.

$\boxed{\text{Properties of inverse images}}$ Inverse images preserve inclusions/unions/intersections/differences.

① $B_1 \subseteq B_2$ implies $f^{-1}(B_1) \subseteq f^{-1}(B_2)$, ② $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$

③ $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$, ④ $f^{-1}(B_1 - B_2) = f^{-1}(B_1) - f^{-1}(B_2)$.

<u>Proof of ②</u> To show $f^{-1}(B_1 \cup B_2) \subseteq f^{-1}(B_1) \cup f^{-1}(B_2)$, we note that

$$x \in f^{-1}(B_1 \cup B_2) \implies f(x) \in B_1 \cup B_2$$
$$\implies f(x) \in B_1 \text{ or } f(x) \in B_2$$
$$\implies x \in f^{-1}(B_1) \text{ or } x \in f^{-1}(B_2)$$
$$\implies x \in f^{-1}(B_1) \cup f^{-1}(B_2), \text{ as needed.}$$

All of these steps are reversible, so the opposite inclusion also holds. ▨

<u>Proof of ③</u>. To show $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$, we note

$$x \in f^{-1}(B_1 \cap B_2) \iff f(x) \in B_1 \cap B_2 \iff f(x) \in B_1 \text{ and } f(x) \in B_2$$
$$\iff x \in f^{-1}(B_1) \text{ and } x \in f^{-1}(B_2) \iff x \in f^{-1}(B_1) \cap f^{-1}(B_2). \text{ ▨}$$

$\boxed{\text{Images and inverse images}}$ Suppose $f: A \to B$ and $A_1 \subseteq A$ and $B_1 \subseteq B$.

① One has $f^{-1}(f(A_1)) \supseteq A_1$. Equality holds when f is <u>injective</u>.

② One has $f(f^{-1}(B_1)) \subseteq B_1$. Equality holds when f is <u>sur</u>jective.

<u>Proof of ②</u>. We prove the inclusion first:

$$\begin{aligned} y \in f(f^{-1}(B_1)) &\Rightarrow y = f(x) \text{ for some } x \in f^{-1}(B_1) \\ &\Rightarrow y = f(x) \text{ and also } f(x) \in B_1 \\ &\Rightarrow y \in B_1. \end{aligned}$$

This proves $f(f^{-1}(B_1)) \subseteq B_1$. Suppose now f is surjective.
To prove $B_1 \subseteq f(f^{-1}(B_1))$, note that

$$\begin{aligned} y \in B_1 &\Rightarrow y = f(x) \text{ for some } x \in A \text{ by surjectivity} \\ &\Rightarrow y = f(x) \text{ for some } x \in A \text{ and } f(x) = y \in B_1 \\ &\Rightarrow y = f(x) \text{ and } x \in f^{-1}(B_1) \\ &\Rightarrow y \in f(f^{-1}(B_1)), \text{ as needed.} \end{aligned}$$

▨

$\boxed{\text{Minimum and maximum}}$ The minimum of a set A is the smallest element of A, should such an element exist. The maximum of A is the largest element, should one exist.

<u>Examples</u>. $A = \{1, 2, 3\}$ has $\min A = 1$ and $\max A = 3$

$A = [1, 3]$ has $\min A = 1$ and $\max A = 3$

$A = [1, 3)$ has $\min A = 1$ and $\max A$ does not exist

$A = (1, 3)$ has no min and no max.

<u>Showing that $\min A = 1$</u>: amounts to showing ⓐ $1 \in A$ and ⓑ $1 \le x$ for all $x \in A$.

<u>Showing that $\min A$ does not exist</u>: amounts to showing A has <u>no</u> smallest element.
We do that by starting with any element $x \in A$ and exhibit an element $y \in A$ with $y < x$.

Example. Take $A = \{1, 1/2, 1/3, 1/4, \dots\} = \{1/n : n \in \mathbb{N}\}$.

Then $\max A = 1$ and $\min A$ does not exist.

In fact, $1 \in A$ and $1 \geq \frac{1}{n}$ for all $n \in \mathbb{N}$, so $\max A = 1$.

To show $\min A$ does not exist, pick $x \in A$. Then $x = 1/n$ for some $n \in \mathbb{N}$.

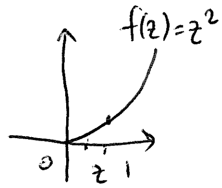
Since $\frac{1}{n+1} < \frac{1}{n}$, the element $y = \frac{1}{n+1}$ satisfies $y < x$, so no min exists.

Example. Take $A = \{x^2 : 0 \leq x < 1\} = [0, 1)$.

We claim $\min A = 0$ and no maximum exists.

In fact, $0 \in A$ and $0 \leq x$ for all $x \in A$ $\Rightarrow$ $\min A = 0$.

To show A has no max, we pick $x \in A$ and find some $y \in A$ with $y > x$. Now, $x \in A \Rightarrow x = z^2$ for some $0 \leq z < 1$.



~~Consider~~ $\cancel{y = \left(\frac{z}{2}\right)^2}$. ~~Since~~ $\cancel{0 \leq \frac{z}{2} < \frac{1}{2} < 1}$,

~~we have~~ $\cancel{y = \left(\frac{z}{2}\right)^2 \in A}$ ~~as well~~.

Consider $w = \left(\frac{z+1}{2}\right)^2$, instead .... we need a larger element.

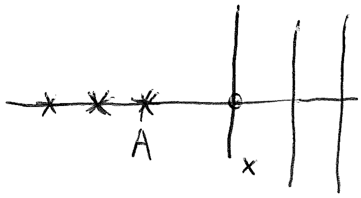
Since $0 \leq z < 1$, we have $0 \leq z < \frac{z+1}{2} < 1$ and so

$w = \left(\frac{z+1}{2}\right)^2$ is an element of A which satisfies

$$\left(\frac{z+1}{2}\right)^2 > (z)^2, \text{ namely } w > x.$$

infimum and supremum

We say that $A \subseteq \mathbb{R}$ is bounded from above when the elements of A are not arbitrarily large, namely if there exists some $x \in \mathbb{R}$ such that

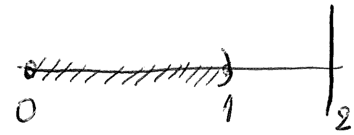


$$x \geq a \text{ for all } a \in A.$$

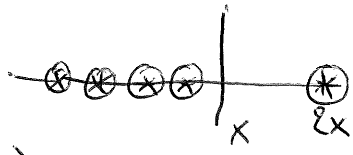
We call x an upper bound of A. The smallest upper bound is called the supremum of A = $\sup A$ = least upper bound.

Example. Let $A = [0,1)$. Then $\max A$ does not exist, but $\sup A = 1$.

Example. Let $A = [0,1]$. Then $\max A = 1$ and $\sup A = 1$.

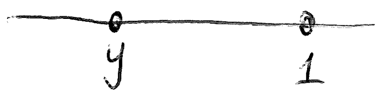


Example. Let $A = (0, \infty)$. Then $\max A$ does not exist, $\sup A$ does not exist.
If x were an upper bound, then $x > 1$ and then $\cancel{2x > x}$ $2x$ would be an element of A, which is impossible since $2x > x$.



Example with proof. We show $\sup A = 1$ when $A = [0,1)$.
To show this, we check ⓐ that 1 is an upper bound of A
ⓑ that 1 is the least upper bound of A

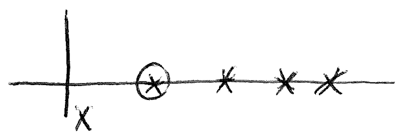
For ⓐ, we have $x \in A \Rightarrow x < 1$. Thus, 1 is an upper bound.
For ⓑ, we need to show that no $y < 1$ can be an upper bound.
That is, we need to find some element of A which is bigger than y.



Since $y < 1$, we can have
$y \leq 0$ in which case $y \leq 0 < \frac{1}{2}$ and $\frac{1}{2}$ is an element of A that's bigger
OR $0 < y < 1$ in which case $0 < y < \frac{y+1}{2} < 1$ and $\frac{y+1}{2}$ is an element of A that's bigger.

Axiom of completeness.	If A is a nonempty set that is bounded from above, then A has an upper bound and $\sup A$ exists.

We similarly define $\inf A$, the infimum, corresponding to lower bounds.
Namely, x is a lower bound of A,
if $x \leq a$ for all $a \in A$.

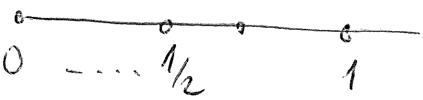


The greatest lower bound of A is called the infimum of A.

Example. Let $A = \{ \frac{1}{n} : n \in \mathbb{N} \} = \{ 1, 1/2, 1/3, 1/4, \dots \}$.

Note that $\max A = 1$ and $\min A$ does not exist.

We claim that $\inf A = 0$.

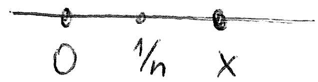


To check this, we check ⓐ 0 is a lower bound.

Namely, $0 < \frac{1}{n}$ for all $n \in \mathbb{N}$. This part is clear.

We now check ⓑ 0 is the greatest lower bound.

Namely, no $x > 0$ can be a lower bound.

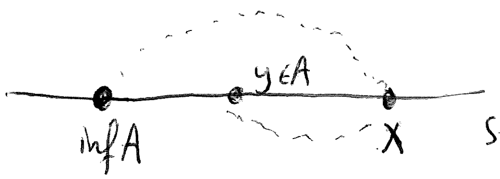


We need to find an element of A that is smaller than x. That is some $z = \frac{1}{n} \in A$ with $\frac{1}{n} < x$.

Pick an integer $n > 1/x$ to get $nx > 1$ and $x > 1/n$.

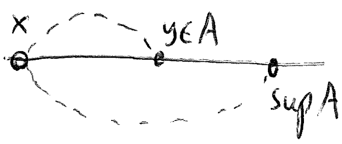
Then $1/n$ is an element of A which is smaller than x.

infimum



Start with $x > \inf A$
and find an element of A
which is smaller than x

supremum



Example. Take $B = \{ \frac{n}{n+1} : n \in \mathbb{N} \} = \{ \frac{1}{2}, \frac{2}{3}, 3/4, \dots \}$.

This is an increasing sequence with $\lim_{n \to \infty} \frac{n}{n+1} = 1$.

We check ⓐ $\min B = 1/2$ by checking $1/2 \in B$

and $\frac{1}{2} \leq \frac{n}{n+1} \iff n+1 \leq 2n \iff 1 \leq n$. ✓

ⓑ $\max B$ does not exist by checking

$$x \in B \Rightarrow x = \frac{n}{n+1} \text{ for some } n.$$

To check $y = \frac{n+1}{n+2}$ is even larger, we check

$$\frac{n}{n+1} < \frac{n+1}{n+2} \iff n^2 + 2n < (n+1)^2$$

$$\iff n^2 + 2n < n^2 + 2n + 1 \iff 0 < 1.$$ ✓

ⓒ $\sup B = 1$ (least upper bound $= 1$) by checking

- 1 is an upper bound, namely

$$1 \geq x \quad \text{for all } x \in B$$

$$\Leftrightarrow 1 \geq \frac{n}{n+1} \quad \text{for all } n \in \mathbb{N}$$

$$\Leftrightarrow n+1 \geq n \quad \text{for all } n \in \mathbb{N}. \quad ✓$$

- 1 is the least upper bound of B.

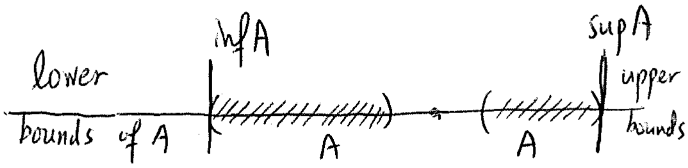
Suppose $y < 1$. We need to show y is not an upper bound, so we need to find an element bigger:

$$\frac{n}{n+1} > y \Leftrightarrow n > yn + y \Leftrightarrow n - yn > y$$

$$\Leftrightarrow n(1-y) > y \Leftrightarrow n > \frac{y}{1-y}.$$

As long as $n > \frac{y}{1-y}$, we get $\frac{n}{n+1} > y$, as needed.

$\boxed{\text{Inf / Min / Sup / Max}}$

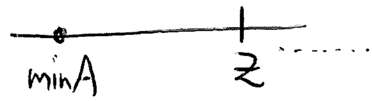


Relation between inf/min and sup/max.

① If a set A has a minimum, then it has an inf and $\inf A = \min A$.
Also, if $\inf A$ exists and $\inf A \in A$, then $\min A$ exists and $\inf A = \min A$.

② If $\max A$ exists $\Rightarrow$ $\sup A$ exists and $\sup A = \max A$.
If $\sup A$ exists and $\sup A \in A$, then $\sup A = \max A$.

Proof of ①. Suppose $\min A$ exists. Then $\min A \le x$ for all $x \in A$ and $\min A \in A$. The former implies $\min A$ is a lower bound of A. To show it is the greatest, suppose $z > \min A$.



We need to show $z \neq$ lower bound of A, so we need an element of A smaller than z. But $\min A$ is such.

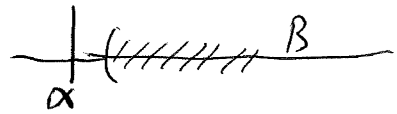
Similarly, suppose $\inf A$ exists and $\inf A \in A$.
Then $\inf A \le x$ for all $x \in A$ and $\inf A \in A \Rightarrow \inf A$ is smallest elm. ■

Existence of inf/sup

- If $A \subseteq \mathbb{R}$ is nonempty and bounded above, then A has a sup ---- Axiom
- If $A \subseteq \mathbb{R}$ is nonempty and bounded below, then A has an inf --- Theorem

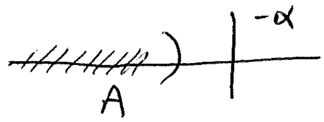
Proof of theorem. The idea is to replace elements of A by their negatives.
If $A = \{1, 3, 5, 10\}$ has $\min A = 1$, $\max A = 10$. If B consists of the negatives, then $B = \{-1, -3, -5, -10\}$ has $\max B = -1$ and $\min B = -10$.

Formally, let $B = \{x \in \mathbb{R} : -x \in A\}$. Since B bounded below, there is a lower bound α : $\alpha \le x$ for all $x \in B$



$$\Rightarrow -\alpha \ge -x \text{ for all } x \in B$$
$$\Rightarrow -\alpha \ge -x \text{ for all } -x \in A$$
$$\Rightarrow -\alpha \ge y \text{ for all } y \in A.$$

This makes $-\alpha$ an upper bound of A.



Thus B bounded below $\Rightarrow$ A bounded above $\Rightarrow$ $\sup A$ exists.

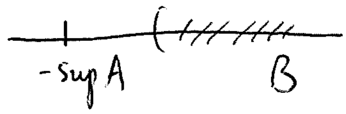
We claim that $-\sup A$ is the greatest lower bound for B.

ⓐ Why lower bound? $\sup A \geq x$ for all $x \in A$

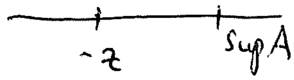
$$\Rightarrow -\sup A \leq -x \quad \text{for all } x \in A$$

$$\Rightarrow -\sup A \leq y \quad \text{for all } y \in B.$$

ⓑ Why greatest? Suppose $z > -\sup A$.

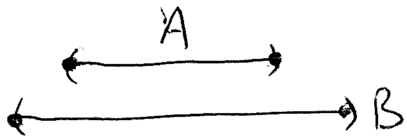


Then $-z < \sup A$ so there exists an element of A, say x, such that $x > -z$.



Then $-x < z$ so $-x$ is an element of B with $-x < z$. ▨

<u>Theorem: Inf/Sup of subsets</u>



⊙ If $B \subseteq \mathbb{R}$ is nonempty and bounded from below and if $A \subseteq B$, then $\inf A \geq \inf B$

⊙ If $B \subseteq \mathbb{R}$ is nonempty and bounded from above, and if $A \subseteq B$, then $\sup A \leq \sup B$.

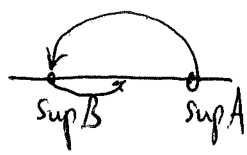
<u>Proof</u>. We prove the second statement: B has an upper bound $\Rightarrow$ $\sup B$ exists. Then $\sup B \geq x$ for all $x \in B$

$$\Rightarrow \sup B \geq x \text{ for all } x \in A \Rightarrow \sup B \text{ is up. bound of } A$$

$$\Rightarrow \sup A \text{ exists as well.}$$

<u>Let's show $\sup A \leq \sup B$</u>. If not, then $\sup A > \sup B$.

Thus $\sup B$ is <u>smaller</u> than least u.b. of A



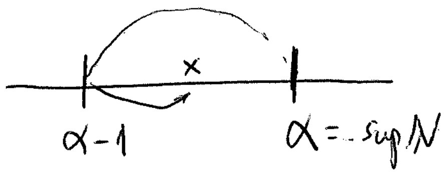
$\therefore$ $\sup B$ is <u>not</u> an u.b. of A

$\therefore$ there is $a \in A$ such that $a > \sup B$

Then $a \in B$ as well and $a > \sup B$, a contradiction. ▨

Theorem (Archimedean property) The set $\mathbb{N}$ is not bounded from above.
Thus, given any $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$.

Proof. If $\mathbb{N}$ has an upper bound,
then it has a sup: $\alpha = \sup \mathbb{N}$.



Since $\alpha - 1$ is smaller, there exists $x \in \mathbb{N}$ with $x > \alpha - 1$.
This gives $\alpha < x + 1$, a contradiction.... since $x + 1 \in \mathbb{N}$ implies $x + 1 \leq \alpha$. ▨

Example. Consider $A = \left\{ \frac{2n+1}{n+3} : n \in \mathbb{N} \right\}$. We find $\inf A$, $\sup A$.
In this case, $A = \left\{ \frac{3}{4}, \frac{5}{5}, \frac{7}{6}, \frac{9}{7}, \dots \right\}$. We show $\inf A = 3/4$, $\sup A = 2$.

ⓐ To show $\min A = 3/4$: $3/4 \in A$ and $\frac{3}{4} \leq \frac{2n+1}{n+3}$ for all $n \in \mathbb{N}$.

This means $3n + 9 \leq 8n + 4$
$5 \leq 5n$ ✓

ⓑ To show $\sup A = 2$: we need 2 to be an upper bound

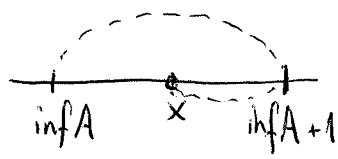
$$2 \geq \frac{2n+1}{n+3} \iff 2n + 6 \geq 2n + 1 \quad ✓$$

And the least upper bound: given $x < 2$, we should find an element

$$\frac{2n+1}{n+3} > x \iff 2n + 1 > nx + 3x$$
$$\iff (2 - x)n > 3x - 1 \iff n > \frac{3x-1}{2-x}.$$

Theorem (Subsets of $\mathbb{N}$). Every nonempty subset of $\mathbb{N}$ has a minimum.

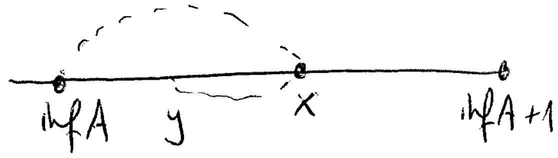
Proof. Let $A \subseteq \mathbb{N}$ be nonempty. Then $x \geq 1$ for all $x \in A$.
Thus 1 is a lower bound $\Rightarrow$ $\inf A$ exists.



Since $\inf A + 1$ is larger than $\inf A$,
$\inf A + 1$ is not a lower bound
so there exists $x \in A$ such that $\inf A \leq x < \inf A + 1$.
Note that $x \in A \Rightarrow x \in \mathbb{N}$.

Case 1. If $x = \inf A$, then the fact that $x \in A$ implies $x = \min A$.

Case 2. Suppose $\inf A < x < \inf A + 1$



Since x is larger than $\inf A$, there exists an element $y \in A$ such that $\inf A \le y < x$.

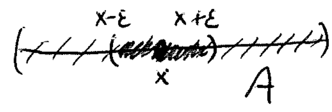
This gives $\inf A \le y < x < \inf A + 1$ and so x, y are distinct integers in an interval of length 1, a contradiction. ▨

Theorem (Mathematical induction) Suppose $P(n)$ is a statement involving natural numbers $n \in \mathbb{N}$. Suppose $P(1)$ holds and $P(n)$ implies $P(n+1)$. Then $P(n)$ holds for all $n \in \mathbb{N}$.

Proof. Let $A =$ the set of all n for which $P(n)$ fails. We need $A = \emptyset$. If not, then A has a min by the previous theorem. By assumption, $1 \notin A$ so $m = \min A > 1$. This is the smallest element of A, so $m-1 \notin A$ so $P(m-1)$ holds so $P(m)$ holds (by assumption) so $m \notin A$, a contradiction. ▨

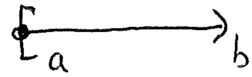
Open sets

Definition. We say that $A \subseteq \mathbb{R}$ is open in $\mathbb{R}$ if, given any $x \in A$ there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq A$ as well.



⊙ Intuitively, (a,b) is open but $[a,b)$ is not open.

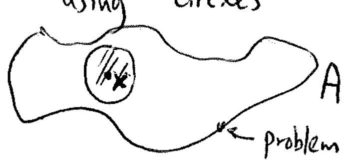
Namely, take $A = [a,b)$. This is not open because the point $x = a$ does not satisfy the definition:



$(a-\varepsilon, a+\varepsilon)$ is not a subset of $[a,b)$

because $a - \varepsilon/2$ lies in $(a-\varepsilon, a+\varepsilon)$ but not in $[a,b)$.

Digression Open sets in $\mathbb{R}^2$ are defined similarly using circles



Theorem 1. Arbitrary unions of open sets are open.

Namely, if the sets $\mathcal{U}_\alpha$ are open in $\mathbb{R}$, then $\bigcup_\alpha \mathcal{U}_\alpha$ is also.

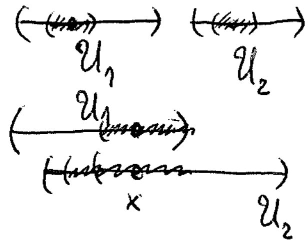
Proof. Suppose the sets $\mathcal{U}_\alpha$ are open in $\mathbb{R}$. We need $\bigcup \mathcal{U}_\alpha$ to be open.

Let $x \in \bigcup \mathcal{U}_\alpha$. Then $x \in \mathcal{U}_\alpha$ for some α.

Since $\mathcal{U}_\alpha$ is open, and $x \in \mathcal{U}_\alpha$, there exists $\varepsilon > 0$ such that

$$(x-\varepsilon, x+\varepsilon) \subseteq \mathcal{U}_\alpha \subseteq \bigcup \mathcal{U}_\alpha .$$

This proves that the union is open in $\mathbb{R}$. $\blacksquare$



Theorem 2. Finite intersections of open sets are open.

Proof. Suppose $\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_n$ are open in $\mathbb{R}$.

We need $\mathcal{U}_1 \cap \mathcal{U}_2 \cap \dots \cap \mathcal{U}_n$ to be open. Let $x \in \mathcal{U}_1 \cap \dots \cap \mathcal{U}_n$.

Then $x \in \mathcal{U}_i$ for each $1 \le i \le n$ and each $\mathcal{U}_i$ is open so there exists $\varepsilon_i > 0$ such that $(x-\varepsilon_i, x+\varepsilon_i) \subseteq \mathcal{U}_i$ for each i.

Take ε to be the smallest radius, $\underline{\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}}$.

Note that $\varepsilon > 0$ since these are finitely many. Then we claim

$$(x-\varepsilon, x+\varepsilon) \subseteq \mathcal{U}_1 \cap \mathcal{U}_2 \cap \dots \cap \mathcal{U}_n .$$

Indeed, if $y \in (x-\varepsilon, x+\varepsilon)$, then

$x - \varepsilon < y < x + \varepsilon$ so $x - \varepsilon_i \le x - \varepsilon < y < x + \varepsilon \le x + \varepsilon_i$ for each i

so $x - \varepsilon_i < y < x + \varepsilon_i$

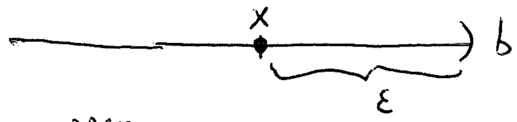
so $y \in (x-\varepsilon_i, x+\varepsilon_i) \subset \mathcal{U}_i$

for each i. Thus $y \in \mathcal{U}_i$ for all i and $y \in \mathcal{U}_1 \cap \dots \cap \mathcal{U}_n$. $\blacksquare$

Example. Arbitrary intersections of open sets need not be open.

For instance, let $A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ for each $n \geq 1$. Then $\bigcap_{n\geq 1}^{\infty} A_n = \{0\}$ is not open.

Theorem. The open intervals $(-\infty, b)$, (a, ∞) and (a,b) are all open in $\mathbb{R}$.



Proof. Let $A = (-\infty, b)$. To show A is open, let $x \in A$ be given. Then $x < b$. We take $\varepsilon = b - x$ and claim that $(x-\varepsilon, x+\varepsilon) \subseteq A$. Indeed,

$$\begin{aligned} y \in (x-\varepsilon, x+\varepsilon) &\Rightarrow x - \varepsilon < y < x + \varepsilon \\ &\Rightarrow x - \varepsilon < y < b \\ &\Rightarrow y < b \Rightarrow y \in A. \end{aligned}$$

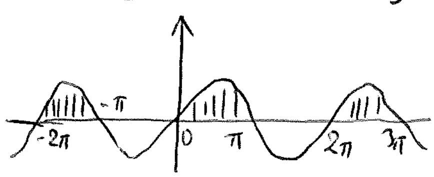
Thus $(-\infty, b)$ is open. So is (a, ∞) and $(a,b) = (-\infty, b) \cap (a, \infty)$. ▨

Theorem A is open in $\mathbb{R} \Leftrightarrow A$ is a union of open intervals (a,b).

Proof. If A = union of open intervals $\Rightarrow A$ = union of open sets
$\Rightarrow A$ is itself open (by previous theorems).

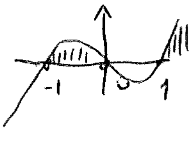
Conversely, suppose A is open in $\mathbb{R}$.
Then for each $x \in A$ there exists $\varepsilon_x > 0$ such that $(x-\varepsilon_x, x+\varepsilon_x) \subseteq A$.
This gives
$$A = \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} (x - \varepsilon_x, x + \varepsilon_x) \subseteq A$$
and thus equality holds so $A = \bigcup_{x \in A} (x - \varepsilon_x, x + \varepsilon_x)$. ▨

Example. Consider $A = \{ x \in \mathbb{R} : \sin x > 0 \}$ or $B = \{ x \in \mathbb{R} : x^3 > x \}$.



In the first case, $\sin x > 0$ within the intervals $(0, \pi)$, $(2\pi, 3\pi)$, $(4\pi, 6\pi)$ and also
$\cancel{(-\pi, -2\pi)}$, $\cancel{(-3\pi, -4\pi)}$ ~~etc~~ $(-2\pi, -\pi)$, $(-4\pi, -3\pi)$.
This gives $A = \bigcup_{k \in \mathbb{Z}} (2k\pi, 2k\pi + \pi) \Rightarrow A$ is open. Similarly,

$x^3 > x \Leftrightarrow x^3 - x > 0 \Leftrightarrow x(x^2-1) > 0 \Leftrightarrow x(x-1)(x+1) > 0$



and so $B = (-1, 0) \cup \underbrace{(1, \infty)} = (-1, 0) \cup \bigcup_{n \geq 2} (1, n)$.

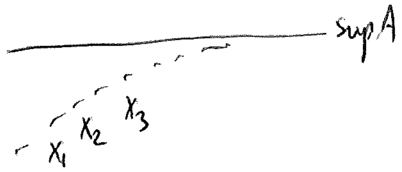
Definition (Convergence of sequences) We say $\{x_n\}$ converges to x as $n \to \infty$ if, given $\varepsilon > 0$, there exists an integer $N \in \mathbb{N}$ s.t.
$$x_n \in (x-\varepsilon, x+\varepsilon) \quad \text{for all} \quad n \geq N.$$

In this case, we write $\lim\limits_{n\to\infty} x_n = x$ and we call x the limit of x_n.

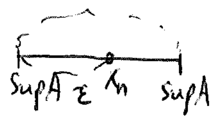
Theorem 1. (Monotone convergence theorem)

(a) If $\{x_n\}$ is increasing and bounded from above, then $\{x_n\}$ converges to $\sup\{x_1, x_2, x_3, \dots\}$.

(b) If $\{x_n\}$ is decreasing and bounded from below, then $\{x_n\}$ converges to $\inf\{x_1, x_2, \dots\}$.



Proof. We only prove (a). We need to show $\{x_n\}$ converges to $\sup\{x_1, x_2, \dots\}$.
Let $\varepsilon > 0$ be given. Since $\sup A - \varepsilon$ is not an upper bound there exists $\sup A - \varepsilon < x_N$ for some N.



Since $\{x_n\}$ is increasing, we have $x_N \leq x_n$ for all $n \geq N$. This gives $x - \varepsilon < x_N \leq x_n \leq x < x + \varepsilon$ and so $x_n \in (x-\varepsilon, x+\varepsilon)$ for all $n \geq N$. ∎

Theorem 2. (Squeeze Law) Suppose $x_n \leq y_n \leq z_n$ for all $n \geq 1$. Suppose $x_n, z_n \to \alpha$ as $n \to \infty$. Then $y_n \to \alpha$ as $n \to \infty$.

Proof. We need $y_n \to \alpha$ as $n \to \infty$. Let $\varepsilon > 0$ be given.
We need an integer N s.t. $\quad y_n \in (\alpha - \varepsilon, \alpha + \varepsilon) \ \forall n \geq N$.
Since $x_n \to \alpha$ we know $\exists N_1$: $\quad x_n \in (\alpha-\varepsilon, \alpha+\varepsilon) \ \forall n \geq N_1$.
Since $z_n \to \alpha$ we know $\exists N_2$: $\quad z_n \in (\alpha-\varepsilon, \alpha+\varepsilon) \ \forall n \geq N_2$.
We take $N = \max\{N_1, N_2\}$ and then $n \geq N$ implies
$$\begin{Bmatrix} \alpha - \varepsilon < x_n < \alpha + \varepsilon \\ \alpha - \varepsilon < z_n < \alpha + \varepsilon \end{Bmatrix} \Rightarrow \alpha - \varepsilon < x_n \leq y_n \leq z_n < \alpha + \varepsilon$$
$$\Rightarrow y_n \in (\alpha - \varepsilon, \alpha + \varepsilon). \quad ∎$$

Example. We show that $\lim_{n\to\infty} \frac{1}{n} = 0$ and that $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

- To prove the former, let $\varepsilon > 0$ be given. We need $N \in \mathbb{N}$ such that $\frac{1}{n} \in (0-\varepsilon, 0+\varepsilon)$ for all $n \geq N$, namely $-\varepsilon < \frac{1}{n} < \varepsilon$ for all $n \geq N$.

  Indeed, pick $N > 1/\varepsilon$ and note that $\frac{1}{n} \leq \frac{1}{N} < \varepsilon$ for all $n \geq N$. We get $-\varepsilon < 0 < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$ for all $n \geq N$.

- To prove the latter, we note that $-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$ and that $\pm\frac{1}{n}$ approaches zero. Thus $\frac{\sin n}{n} \to 0$ as well.

Theorem 3. The following are equivalent.

(a) One has $x_n \to x$ as $n \to \infty$.

(b) Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $x_n \in (x-\varepsilon, x+\varepsilon)$ for all $n \geq N$.

(c) Given any open set $\mathcal{U}$ that contains x, $\exists N \in \mathbb{N}$ s.t. $x_n \in \mathcal{U}$ for all $n \geq N$.

Proof. (a) $\Leftrightarrow$ (b) by definition.

(b) $\Rightarrow$ (c): Suppose $\mathcal{U}$ open and $\mathcal{U}$ contains x.
Since $\mathcal{U}$ open and $x \in \mathcal{U}$ $\exists \varepsilon > 0 : (x-\varepsilon, x+\varepsilon) \subseteq \mathcal{U}$.
By (b) $\exists N \in \mathbb{N} :$ $x_n \in (x-\varepsilon, x+\varepsilon)$ $\forall n \geq N$.
Thus $\exists N \in \mathbb{N}$: $x_n \in (x-\varepsilon, x+\varepsilon) \subseteq \mathcal{U}$.
Thus (c) follows.

(c) $\Rightarrow$ (b): Let $\varepsilon > 0$ be given. We let $\mathcal{U} = (x-\varepsilon, x+\varepsilon)$.
Then $\mathcal{U}$ open containing x $\Rightarrow$ by (c) $\exists N \in \mathbb{N}$:
$x_n \in \mathcal{U} = (x-\varepsilon, x+\varepsilon)$ $\forall n \geq N$. ▨

Closed sets A set $A \subseteq \mathbb{R}$ is closed in $\mathbb{R}$, if its complement $A^c = \mathbb{R} - A$ is open in $\mathbb{R}$. BEING CLOSED IS NOT THE OPPOSITE OF BEING OPEN.

Example 1. We know $(-\infty, b)$ and (a, ∞) and (a, b) are open in $\mathbb{R}$. These intervals do not include their endpoints. Examples of closed sets are then $[b, \infty)$ and $(-\infty, a]$ and $(-\infty, a] \cup [b, \infty)$. Those include their endpoints.

Example 2. Consider $A = [a, b)$ which includes a but not b.

- This is not open because of $x = a$: if we take $\varepsilon > 0$, then $(a-\varepsilon, a+\varepsilon)$ is not contained in A.
- It is also not closed because $A^c = (-\infty, a) \cup [b, \infty)$ is not open. This is due to the presence of $x = b$.

Theorem (Unions and intersections of closed sets)

Any union of open sets is open.	Any intersection of closed sets is closed.
Finite intersections of open sets are open.	Finite unions of closed sets are closed.

Proof. Suppose the sets A_i are closed and let $A = \bigcap_i A_i$.

To say A is closed is to say $A^c = \mathbb{R} - \bigcap_i A_i$ is open.

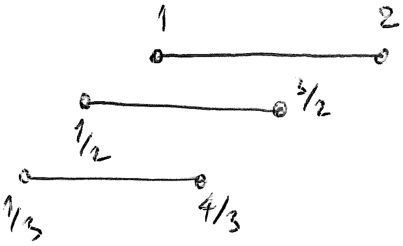
De Morgan's Laws ---- $A^c = \mathbb{R} - \bigcap_i A_i = \bigcup_i (\mathbb{R} - A_i)$.

We know A_i closed $\Rightarrow$ $\mathbb{R} - A_i$ is open $\Rightarrow$ $\bigcup_i (\mathbb{R} - A_i)$ open

$\Rightarrow A^c$ open $\Rightarrow$ A closed.

This settles any intersection. Unions are similar. ∎

Example 3. Arbitrary unions of closed sets need not be closed. For instance, consider $[a_n, b_n] = \left[\frac{1}{n}, 1 + \frac{1}{n}\right]$. This starts out as $[1, 2]$ and $1/n$ decreases to 0 while $1 + 1/n$ decreases to 1. We want to look at the ~~intersection~~ union of those which is



$$\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1+\frac{1}{n}\right] = (0, 2]$$

and thus not closed.

A simpler example: $\left[\frac{1}{n}, 2\right]$ and $\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 2\right] = (0, 2]$.

<u>Example 4.</u> A set $A = \{x\}$ with one element is always closed.
This is because $A^c = (-\infty, x) \cup (x, \infty)$ is union of open $\Rightarrow$ open.
More generally, A = finite set = finite union of closed $\Rightarrow$ closed.

<u>Example 5.</u> Consider the following sets, where $a < b < c$.

$[a,b] \cup \{c\}$ --- not open due to a,b,c --- but closed

$(a,b) \cup \{c\}$ --- not open due to c --- not closed due to a,b

$\{a,b,c\}$ --- not open due to a,b,c --- closed

$\mathbb{Z}$ --- not open ~~no integers~~ --- closed

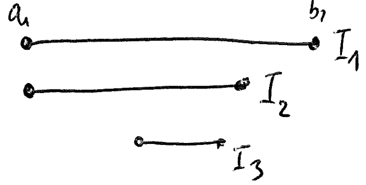
since $\mathbb{Z}^c$ is the union of $(1,2), (2,3), (3,4)$ etc $(0,1), (-1,0)$ etc.
$\mathbb{Z}^c = \bigcup_{k \in \mathbb{Z}} (k, k+1)$

<u>Theorem. (Closed sets and convergence)</u>

Suppose $A \subseteq \mathbb{R}$ is closed in $\mathbb{R}$ and $\{x_n\}$ is a sequence of points in A. If $\{x_n\}$ converges to some limit x, then $x \in A$ as well.

① For instance ~~x_n~~ $x_n = 1/n$ is a sequence in $(0,2)$ but
$\lim_{n\to\infty} x_n = 0$ and $0 \notin (0,2)$.
On the other hand, $x_n \in [0,1]$ does approach a limit in $[0,1]$.

<u>Proof.</u> We assume A closed and $x_n \to x$. We need to show $x \in A$.
If not, $x \in A^c$ and A^c is open. Thus $\exists \varepsilon > 0 : (x-\varepsilon, x+\varepsilon) \subseteq A^c$.
Since $x_n \to x$, there is $N \in \mathbb{N}$ such that $x_n \in (x-\varepsilon, x+\varepsilon)$ $\forall n \geq N$.
This gives $x_n \in A^c$ for all $n \geq N$, a contradiction ∎



Theorem (Nested interval property)

Suppose $I_n = [a_n, b_n]$ is a sequence of closed intervals which satisfy $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$. Then the intersection $\bigcap_{n=1}^{\infty} I_n$ is nonempty.

Proof. Note that $a_n = \min I_n = \inf I_n$ and $b_n = \max I_n = \sup I_n$.

ⓐ We claim a_n is increasing and ~~increas~~ bounded from above.

Namely, $I_{n+1} \subseteq I_n \Rightarrow \inf I_{n+1} \geq \inf I_n \Rightarrow a_{n+1} \geq a_n$ and so $\{a_n\}$ is increasing, while $a_n \leq b_1$ for all $n \in \mathbb{N}$.

Thus a_n converges to some number $a = \sup\{a_1, a_2, \dots\}$.

ⓑ We claim b_n is decreasing and bounded from below.

In fact $I_{n+1} \subseteq I_n \Rightarrow \sup I_{n+1} \leq \sup I_n \Rightarrow b_{n+1} \leq b_n$ and $\{b_n\}$ is decreasing with $b_n \geq a_n \geq a_1$ for all $n \in \mathbb{N}$

ⓒ We check that $a_n \leq \sup\{a_1, \dots, a_n, \dots\} \leq \inf\{b_1, b_2, \dots, b_n, \dots\} \leq b_n$ for all $n \in \mathbb{N}$. This means $a_n \leq \underline{\sup A \leq \inf B} \leq b_n$ for all n

so $[\sup A, \inf B] \subseteq [a_n, b_n] = I_n$ for all n.

The part $a_n \leq \sup A$ is clear. The part $\inf B \leq b_n$ is clear.

To prove $\sup A \leq \inf B$, we first show $a_m \leq b_n$ for all m, n.

This is because
$$
\begin{aligned}
a_m &\leq a_{m+n} && \text{(since } a\text{'s increase)}\\
&\leq b_{m+n} && \text{(by definition)}\\
&\leq b_n && \text{(since } b\text{'s decrease)}.
\end{aligned}
$$

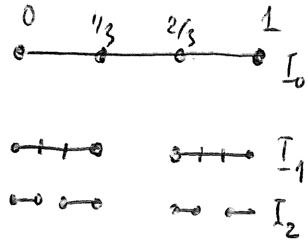
Once we know $\underline{a_m \leq b_n \text{ for all } m, n}$:

$\Rightarrow$ b_n is an upper bound of A
but $\sup A$ is the least upper bound of A

$\Rightarrow$ $\underline{\sup A \leq b_n}$ for all n

$\Rightarrow$ ~~b_n is~~ $\sup A$ is a lower bound of B
but $\inf B$ is the greatest lower bound of B

$\Rightarrow$ $\underline{\sup A \leq \inf B}$, as claimed. ▨



<u>Example</u> (Cantor set) We start with $I_0 = [0,1]$ and divide it into three equal parts and remove the middle part to get $I_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. For each interval of I_1, we repeat the same process to obtain $I_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}], \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$ by removing the middle thirds. This gives a sequence of sets I_n such that (a) $I_{n+1} \subseteq I_n$ (b) I_n is consisting of 2^n intervals that have length $(\frac{1}{3})^n$ each so the total length is $(\frac{2}{3})^n$ which approaches zero and (c) the intersection $\bigcap_{n=0}^{\infty} I_n$ is closed and nonempty. Also, (d) each subinterval I_n contains a point in the intersection so $\bigcap_{n=0}^{\infty} I_n$ contains infinitely many points.

$\boxed{\text{Interior \& Closure}}$ Let $A \subseteq \mathbb{R}$ be arbitrary. We define

- the closure $\bar{A}$ as the <u>smallest</u> <u>closed</u> <u>set</u> that contains A. It is the intersection of all closed sets that contain A.
- the interior A° as the <u>largest</u> <u>open</u> <u>set</u> that is contained in A. It is the union of all open sets that are contained in A.

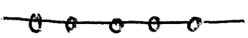
<u>Examples.</u>

$A = [a,b)$ ..... $\bar{A} = [a,b]$ ... $A^\circ = (a,b)$

$A = (a,b) \cup \{c\}$ .... $\bar{A} = [a,b] \cup \{c\}$ ... $A^\circ = (a,b)$

$(a<b<c)$ $A = (a,b) \cup (c,\infty)$ ... $\bar{A} = [a,b] \cup [c,\infty)$ .... $A^\circ = A$

 $A = \{a,b,c\}$ ..... $\bar{A} = A$ .... $A^\circ = \emptyset$

 $A = \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ .. $\bar{A} = A$ ...... $A^\circ = \emptyset$

<u>Theorem</u> (Properties of closure/interior)

(a) $A \subseteq \bar{A}$ for all sets A ..... $A^\circ \subseteq A$ for all sets A

(b) $A \subseteq B$ implies $\bar{A} \subseteq \bar{B}$ ..... $A^\circ \subseteq B^\circ$ for all A, B

(*)(c) $A = \bar{A} \iff A$ is closed ..... $A = A^\circ \iff A$ is open

(d) $\bar{\bar{A}} = \bar{A}$ for all sets A ... $(A^\circ)^\circ = A^\circ$ for all sets A.

Proof. We look at closures only.

(a) $\bar{A}$ = smallest closed set containing A so $\bar{A} \supseteq A$.

(b) $A \subseteq B \subseteq \bar{B}$ so $\bar{B}$ is <u>a</u> closed set containing A
and $\bar{A}$ is the smallest such set $\Rightarrow$ $\bar{A} \subseteq \bar{B}$.

(c) $A = \bar{A}$ implies A is closed... by definition.
A closed implies A is <u>smallest</u> closed set containing A, so $A = \bar{A}$.

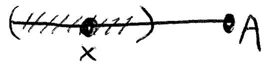
(d) $\bar{A}$ is closed by definition so (c) gives $\bar{\bar{A}} = \bar{A}$. ∎

Closure $\bar{A}$ ... smallest closed set containing A = intersection of closed sets containing A

Interior A° ... largest open set that contains A = union of open sets contained in A

Thus ... $A^\circ \subset A \subset \bar{A}$ with A° open and $\bar{A}$ closed.

<u>Neighbourhood of x</u>: An open set that contains x.

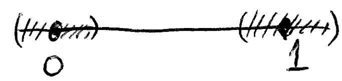


<u>Theorem.</u> (Closure/Interior in terms of open sets)

(a) One has $x \in A^\circ \iff$ there exists <u>some</u> neighb. of x contained in A.

(b) One has $x \in \bar{A} \iff$ <u>all</u> neighbourhoods of x intersect A.

<u>Example</u>. $A = [0,1)$ gives $A^\circ = (0,1)$ and $[0,1]$.



In this case $0 \notin A^\circ$ because no neighb. of 0 is contained in A.
Also, the closure $\bar{A}$ contains 1 because ~~some~~ <u>all</u> neighb. of 1 intersect A.

<u>Proof of (a)</u>: Suppose $x \in A^\circ \Rightarrow x$ lies in the largest open set contained in A.

Then $x \in A^\circ$ <u>and</u> A° open $\Rightarrow (x-\varepsilon, x+\varepsilon) \subseteq A^\circ$ for some $\varepsilon > 0$
$\Rightarrow (x-\varepsilon, x+\varepsilon)$ is a nbd. of x contained in $A^\circ \subseteq A$.

Conversely, suppose there exists $\mathcal{U}$ open with $x \in \mathcal{U}$ and $\mathcal{U} \subseteq A$.
Then A° = largest open set contained in A implies $\mathcal{U} \subseteq A^\circ$
and since $x \in \mathcal{U}$, we find that $x \in \mathcal{U} \subseteq A^\circ$ as well. ∎

<u>Proof of (b)*</u>: We need $x \in \bar{A} \iff$ every neighb. of x intersects A.

We prove this by showing: $x \notin \bar{A} \iff$ some neighb. of x does <u>not</u> intersect A.

⊙ Suppose $x \notin \bar{A}$ $\Rightarrow$ $x \notin$ intersection of all closed sets containing A

$\Rightarrow$ $x \notin K$ for some closed set $K \supseteq A$.

Since K is closed and $x \notin K$, the complement $\mathcal{U} = K^c$ is open and contains x. Thus $\mathcal{U} = K^c$ is a neighb. of x. In fact, $\mathcal{U}$ does not intersect A because $y \in A$ implies $y \in K$ and thus $y \notin \mathcal{U}$.

⊙ Conversely, suppose $\mathcal{U}$ is open and $x \in \mathcal{U}$ and $\mathcal{U} \cap A = \emptyset$.

Then $K = \mathcal{U}^c$ is closed and $x \notin K$ and $K \supseteq A$ because

$$y \in A \Rightarrow y \notin \mathcal{U} \Rightarrow y \in \mathcal{U}^c \Rightarrow y \in K.$$

This gives $x \notin K$ for some closed set $K \supseteq A$ so $x \notin \bar{A}$. ∎

Limit point. Let $A \subseteq \mathbb{R}$. We say z is a limit point of A, if every neighb. of z intersects A at a point other than z.

Theorem. If z is a limit point, then there is $\{x_n\}$ a sequence of points $x_n \in A$ such that $x_n \to z$ as $n \to \infty$.

Proof. Consider $(z - 1/n, z + 1/n)$, an open set containing z.

It intersects A at a point $x_n \in A$ and then

$$z - \frac{1}{n} < x_n < z + 1/n \quad \text{for all } n \in \mathbb{N}.$$

Since $z \pm 1/n \to z$ as $n \to \infty$, we get $x_n \to z$ as $n \to \infty$. ∎

Theorem (Closure in terms of lim. points) One has $\bar{A} = A \cup A'$,

where A' consists of limit points of A.

Proof. Suppose $x \in A \cup A'$. Then $x \in A$ or $x \in A'$.

If $x \in A$, we get $x \in A \subseteq \bar{A}$.

If $x \in A'$, every neighb. of x intersects A (at a point $\neq x$)

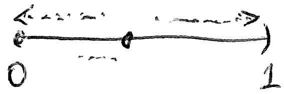
$\Rightarrow$ every neighb. of x intersects A $\Rightarrow$ $x \in \bar{A}$.

Conversely, if $x \in \bar{A}$, then every neigh. of x intersects A.

If they all intersect A at a point $\neq x$, then $x \in A'$.

Otherwise, one intersects A at x, so $x \in A$. ∎

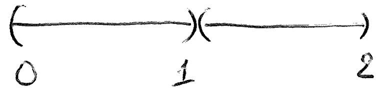
<u>Examples.</u> ① $A = [0,1)$ ... $\bar{A} = [0,1]$ ... $A^\circ = (0,1)$ ... $A' = [0,1]$.



② $A = (0,1) \cup (1,2)$ .... $A^\circ = A = (0,1) \cup (1,2)$

$A' = [0,2]$

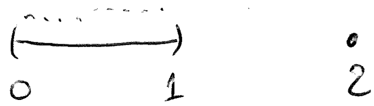
$\bar{A} = A \cup A' = [0,2]$.



③ $A = (0,1) \cup \{2\}$ .... $A^\circ = (0,1)$

$A' = [0,1]$

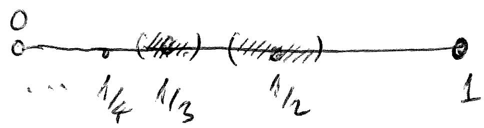
$\bar{A} = [0,1] \cup \{2\}$.



④ $A = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \dots\}$ .... $A^\circ = \emptyset$

$A' = \{0\}$

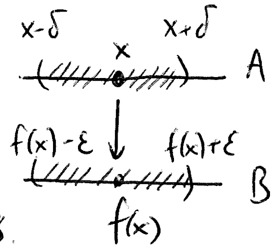
$\bar{A} = A \cup \{0\} = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \dots\} \cup \{0\}$.



⑤ $A = \{1,2,3,4\}$ $A^\circ = \emptyset$, $\bar{A} = A$ (closed) and $A' = \emptyset$.

$\boxed{\text{Continuity at a point}}$ Let $A, B \subseteq \mathbb{R}$ and $f: A \to B$ be a function.

We say f is continuous at $x \in \mathbb{R}$ if, given any $\varepsilon > 0$ there exists $\delta > 0$ such that



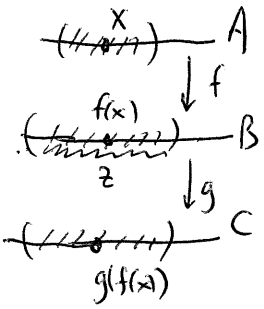
$$z \in (x-\delta, x+\delta) \cap A \implies f(z) \in (f(x)-\varepsilon, f(x)+\varepsilon).$$

Theorem 1. (Composition of continuous functions) Let $A, B, C \subseteq \mathbb{R}$.

Suppose $f: A \to B$ is cont. at $x \in A$

and $g: B \to C$ is cont. at $f(x) \in B$.

Then $g \circ f : A \to C$ is continuous at $x \in A$.



Proof. Let $\varepsilon > 0$ be given. We need $g \circ f$ cont. at x.

① Since g is continuous at $f(x)$, there exists $\delta_1 > 0$:

$$\underline{z \in (f(x)-\delta_1, f(x)+\delta_1) \cap B} \implies g(z) \in (g(f(x))-\varepsilon, g(f(x))+\varepsilon).$$

② Since f is continuous at x, there exists $\delta_2 > 0$:

$$y \in (x-\delta_2, x+\delta_2) \cap A \implies \underline{f(y) \in (f(x)-\delta_1, f(x)+\delta_1)}.$$

Combining these two, we get

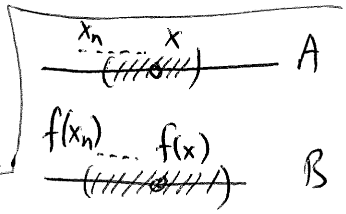
$$y \in (x-\delta_2, x+\delta_2) \cap A \implies f(y) \in (f(x)-\delta_1, f(x)+\delta_1)$$
$$\implies g(f(y)) \in (g(f(x))-\varepsilon, g(f(x))+\varepsilon).$$

This verifies the $\varepsilon\delta$-definition of continuity for $g \circ f$ at x. ▨

Theorem 2. (Continuity and sequences) Let $A, B \subseteq \mathbb{R}$ and $f: A \to B$.

Suppose $\{x_n\}$ is a sequence in A with $x_n \to x \in A$.

If f is continuous at x, then $f(x_n) \to f(x)$ as well.



▨ This can be stated as $\lim_{n\to\infty} f(x_n) = f(x) = f(\lim_{n\to\infty} x_n)$.

Thus, the square root of a limit = limit of square root etc.

Proof. We need to show $f(x_n) \to f(x)$. Let $\varepsilon > 0$ be given.

① Since f is continuous at x, there exists $\delta > 0$:

$z \in (x-\delta, x+\delta) \cap A \Rightarrow f(z) \in (f(x)-\varepsilon, f(x)+\varepsilon)$

② Since $x_n \to x$ as $n \to \infty$, there exists N:

$x_n \in (x-\delta, x+\delta)$ for all $n \geq N$.

Combining these two, we get

$$x_n \in (x-\delta, x+\delta) \cap A \Rightarrow f(x_n) \in (f(x)-\varepsilon, f(x)+\varepsilon)$$

for all $n \geq N$.

This verifies the definition of convergence $f(x_n) \to f(x)$. ▨

<u>Definition</u> (Relatively open/closed) Let $A \subseteq B \subseteq \mathbb{R}$.

We say <u>A is open in B</u>, if, given any $x \in A$ there exists some $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \cap B \subseteq A$. ....

We get an interval that is contained in A.

We say <u>A is closed in B</u>, if the complement $B-A$ is open in B.

<u>Theorem</u> (Equivalent formulation) To say that $A \subseteq B$ is open in B is to say that $A = \mathcal{U} \cap B$ for some set $\mathcal{U}$ open in $\mathbb{R}$.

To say that A is closed in B is to say $A = \mathcal{U} \cap B$ with $\mathcal{U}$ closed in $\mathbb{R}$.

<u>Examples</u>

- Since $(0,1)$ is open in $\mathbb{R}$, $(0,1) \cap \mathbb{Q}$ is open in $\mathbb{Q}$
- Since $(0,1)$ is open in $\mathbb{R}$, $(0,1) \cap [-1, \tfrac{1}{2}] = (0, \tfrac{1}{2}]$ is open in $[-1, \tfrac{1}{2}]$
- Since $[0,\infty)$ is closed in $\mathbb{R}$, $[0,\infty) \cap \mathbb{Z} = \{0,1,2,3,\ldots\}$ is closed in $\mathbb{Z}$.
- Since $[0,1]$ is closed in $\mathbb{R}$, $[0,1] \cap (-1, \tfrac{1}{2}) = [0, \tfrac{1}{2})$ is closed in $(-1, \tfrac{1}{2})$.

<u>Proof.</u> Suppose $A = \mathcal{U} \cap B$, where $\mathcal{U}$ is ~~the~~ open in $\mathbb{R}$.

We need to show A open in B: given $x \in A$, we get $x \in \mathcal{U}$ and so $(x-\varepsilon, x+\varepsilon) \subseteq \mathcal{U}$ and so $(x-\varepsilon, x+\varepsilon) \cap B \subseteq \mathcal{U} \cap B = A$.

This proves A open in B. Conversely, suppose A open in B.

Given any $x \in A$, there exists $\varepsilon_x > 0$: $(x-\varepsilon_x, x+\varepsilon_x) \cap B \subseteq A$.

Let $\mathcal{U} = \bigcup_{x \in A} (x-\varepsilon_x, x+\varepsilon_x)$. Then $\mathcal{U}$ is open in $\mathbb{R}$ and

$\cancel{A = \bigcup_{x \in A} \{x\}}$

$$\mathcal{U} \cap B = \bigcup_{x \in A} (x-\varepsilon_x, x+\varepsilon_x) \cap B \subseteq A,$$

while $$A = \bigcup_{x\in A} \{x\} \subseteq \bigcup_{x \in A} (x-\varepsilon_x, x+\varepsilon_x) = \mathcal{U}$$

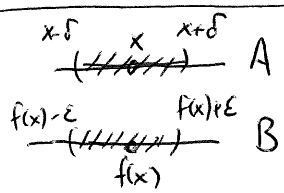
implies $A \cap B \subseteq \mathcal{U} \cap B$

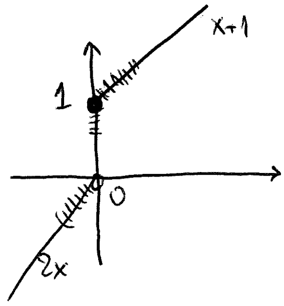
so $A \subseteq \mathcal{U} \cap B$. ∎

Recall: when $A \subseteq B \subseteq \mathbb{R}$ we say A open in B if for each $x \in A$ we have $(x-\varepsilon, x+\varepsilon) \cap B \subseteq A$.

We showed this means $\mathcal{U}$ open in $\mathbb{R} \Rightarrow \mathcal{U} \cap B$ is open in B.

Theorem (Continuity at all points) Let $f: A \to B$ be a function, $A, B \subseteq \mathbb{R}$.
To say that f is continuous (at all points) is to say that $f^{-1}(\mathcal{U})$ is open in A whenever $\mathcal{U}$ is open in B.





__Example.__ Consider $f(x) = \begin{cases} 2x & \text{if } x < 0 \\ x+1 & \text{if } x \geq 0 \end{cases}$.

We show that f is __not__ continuous $f: \mathbb{R} \to \mathbb{R}$.

For instance, $\mathcal{U} = (-1/2, 1/2)$ is an open interval around $x = 0$

and $f^{-1}(\mathcal{U}) = \{x \in \mathbb{R} : f(x) \in \mathcal{U}\} = (-1/4, 0)$ is still open in $\mathbb{R}$.

However, $\mathcal{U} = (1/2, 3/2)$ is an open interval

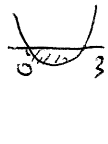
whose inverse image $f^{-1}(\mathcal{U}) = [0, 1/2)$ is not open in $\mathbb{R}$ because of the endpoint $x = 0$.

__Example.__ Suppose $f: \mathbb{R} \to \mathbb{R}$ is continuous.

Then $$A = \{x \in \mathbb{R} : f(x) > 0\} = \{x \in \mathbb{R} : f(x) \in (0, \infty)\} = f^{-1}((0,\infty)).$$

Since $(0,\infty)$ is open, $f^{-1}(0,\infty)$ is open as well. We used to look at

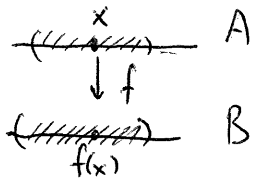
$$A = \{x \in \mathbb{R} : x^2 > 3x\} = \{x \in \mathbb{R} : x^2 - 3x > 0\} = (-\infty, 0) \cup (3, \infty).$$



Since that is a union of open intervals, it is open itself.
However, A is also open by the previous theorem as $A = f^{-1}((0,\infty))$.
Similarly, $A = \{ x \in \mathbb{R} : x^3 + \sin x > 2 \}$ is open by the theorem.

<u>Proof of theorem</u>. We need f continuous $\iff$ $f^{-1}(\mathcal{U})$ open in A whenever $\mathcal{U}$ open in B

<u>First</u> suppose $f : A \to B$ is continuous at all points.



We take a set $\mathcal{U}$ that is open in B.
To show $f^{-1}(\mathcal{U})$ is open in A, let $x \in f^{-1}(\mathcal{U})$.
Then $f(x) \in \mathcal{U}$ and $\mathcal{U}$ is open in B, so

$$\underline{(f(x)-\varepsilon, f(x)+\varepsilon) \cap B} \subseteq \cancel{B}\ \mathcal{U} \text{ ... for some } \varepsilon > 0.$$

Since f is continuous at x, there exists $\delta > 0$:

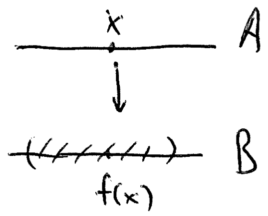
$$z \in (x-\delta, x+\delta) \cap A \implies \underline{f(z) \in (f(x)-\varepsilon, f(x)+\varepsilon) \cap B}.$$

Combining these statements, we get

$$z \in (x-\delta, x+\delta) \cap A \implies f(z) \in \mathcal{U} \implies z \in f^{-1}(\mathcal{U}).$$

This proves $(x-\delta, x+\delta) \cap A \subseteq f^{-1}(\mathcal{U})$.

<u>Conversely</u> suppose $f^{-1}(\mathcal{U})$ open in A whenever $\mathcal{U}$ is open in B.
We need to show $f : A \to B$ is continuous at any point $x \in A$.
Let $\varepsilon > 0$ be given. Since $\mathcal{U} = (f(x)-\varepsilon, f(x)+\varepsilon)$ is open, $\mathcal{U} \cap B = (f(x)-\varepsilon, f(x)+\varepsilon) \cap B$ is open in B and its inverse image is open in A: $f^{-1}(\mathcal{U} \cap B)$ is open in A.



Since $f(x) \in \mathcal{U} \cap B$, we have $x \in f^{-1}(\mathcal{U} \cap B)$ and so

$$A \cap (x-\delta, x+\delta) \subseteq f^{-1}(\mathcal{U} \cap B).$$

This gives:

$$\begin{aligned} z \in A \cap (x-\delta, x+\delta) &\implies z \in f^{-1}(\mathcal{U} \cap B) \\ &\implies f(z) \in \mathcal{U} \cap B \\ &\implies f(z) \in (f(x)-\varepsilon, f(x)+\varepsilon) \cap B. \end{aligned}$$

In other words, f is continuous at the point $x \in A$. $\blacksquare$

Example (Restrictions) Let $f: \mathbb{R} \to \mathbb{R}$ be continuous and $A \subseteq \mathbb{R}$. Consider the restriction $g: A \to \mathbb{R}$ defined by $g(x) = f(x)$ for all $x \in A$. Then g is still continuous. Namely, if $\mathcal{U}$ is open in $\mathbb{R}$, its inverse image under g is

$$\begin{aligned} g^{-1}(\mathcal{U}) &= \{x \in A : g(x) \in \mathcal{U}\} \\ &= \{x \in A : f(x) \in \mathcal{U}\} \\ &= \{x \in A : x \in f^{-1}(\mathcal{U})\} = f^{-1}(\mathcal{U}) \cap A. \end{aligned}$$

Since f is continuous, $f^{-1}(\mathcal{U})$ is open in $\mathbb{R}$

so $f^{-1}(\mathcal{U}) \cap A$ is open in A

so $g^{-1}(\mathcal{U})$ is open in A.

This proves that $g: A \to \mathbb{R}$ is continuous as well. $\blacksquare$

Example (Inclusions) Let $A \subseteq \mathbb{R}$ and consider the map $i: A \to \mathbb{R}$ defined by $i(x) = x$. This function is continuous for any $A \subseteq \mathbb{R}$.. To check this, let $\mathcal{U}$ be open in $\mathbb{R}$ and note that

$$\begin{aligned} i^{-1}(\mathcal{U}) &= \{x \in A : i(x) \in \mathcal{U}\} \\ &= \{x \in A : x \in \mathcal{U}\} = \mathcal{U} \cap A. \end{aligned}$$

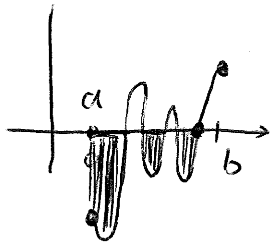
Since $\mathcal{U}$ is open in $\mathbb{R}$, $\mathcal{U} \cap A$ is open in A

so $i^{-1}(\mathcal{U})$ is open in A

so $i: A \to \mathbb{R}$ is continuous. $\blacksquare$

Theorem (Bolzano's theorem) Suppose $f: [a,b] \to \mathbb{R}$ is continuous and $f(a), f(b)$ have opposite sign. Then $f(x) = 0$ at some point $x \in (a,b)$.

Proof. Consider the case $f(a) < 0 < f(b)$, as the other case is similar. We define the sets



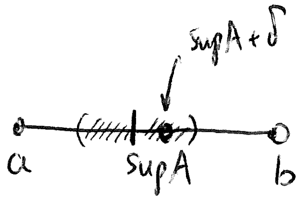
$$A = \{ a \le x \le b : f(x) < 0 \} \quad \text{and} \quad B = \{ a \le x \le b : f(x) > 0 \}.$$

Step 1. We expect $\sup A$ to be a root of f.

Note that $a \in A$ and b is an upper bound of A.

Thus $\sup A$ exists and $a \le \sup A \le b$.

Step 2. We exclude the case $f(\sup A) < 0$.



Suppose $f(\sup A) < 0$ which gives $\sup A < b$.

Since $A = \{ a \le x \le b : f(x) \in (-\infty, 0) \}$

$= f^{-1}((-\infty, 0)) \cap [a,b]$, the set A is open in $[a,b]$.

This means that $(\sup A - \varepsilon, \sup A + \varepsilon) \cap [a,b] \subseteq A$.

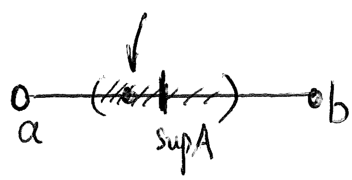
Take $\delta = \min\{\frac{\varepsilon}{2}, b - \sup A\}$ so that ~~$\sup A + \delta \le b$~~

$\delta \le b - \sup A$ and $\sup A + \delta \le b$.

Then $\sup A + \delta \in [\sup A, \sup A + \varepsilon) \cap [a,b] \subseteq A$,

a contradiction since $\sup A \ge x$ for all $x \in A$.

Step 3. We exclude the case $f(\sup A) > 0$.



Suppose $f(\sup A) > 0$ which gives $\sup A > a$.

As above, $B = \{ x \in [a,b] : f(x) > 0 \}$ is open in $[a,b]$.

Since $\sup A \in B$, we have $(\sup A - \varepsilon, \sup A + \varepsilon) \cap [a,b] \subseteq B$

for some $\varepsilon > 0$. Take $\delta = \frac{1}{2}\min\{\varepsilon, \sup A - a\}$ so that

$\delta \le \sup A - a$ and $a \le \sup A - \delta$. Then

$\sup A - \delta \in (\sup A - \varepsilon, \sup A] \cap [a,b] \subseteq B$

and this is smaller than $\sup A$, so there is $\tilde{a} \in A$ such that $\sup A - \delta < \tilde{a} < \sup A$, a contradiction since $f(\tilde{a}) > 0$ but $\tilde{a} \in B$ by above. ∎

$f: A \to B$

Continuity at a point x: We say ~~f~~ is continuous at x if given $\varepsilon > 0$ there exists $\delta > 0$: $y \in (x-\delta, x+\delta) \cap A \Rightarrow f(y) \in (f(x)-\varepsilon, f(x)+\varepsilon)$.

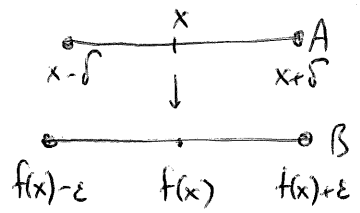
Uniform continuity on a set: We say $f: A \to B$ is uniformly continuous on A if given $\varepsilon > 0$ there exists $\delta > 0$: $y \in (x-\delta, x+\delta) \Rightarrow f(y) \in (f(x)-\varepsilon, f(x)+\varepsilon)$ for all points $x, y \in A$.

- Note that δ may depend on x in the first case.
- We can also write $\boxed{|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.}$

This is because $y \in (x-\delta, x+\delta)$ means $x - \delta < y < x + \delta$ so $-\delta < y - x < \delta$.

Example. Consider $f(x) = x^2$ as a function $f: [0,1] \to [0,1]$.

To show f is uniformly continuous on $[0,1]$, let $\varepsilon > 0$ be given and estimate $|f(x) - f(y)|$:



$$|f(x) - f(y)| = |x^2 - y^2| = |x-y| \cdot |x+y|.$$

We are assuming $|x-y| < \delta$ and we also have $|x+y| = x + y \leq 2$.

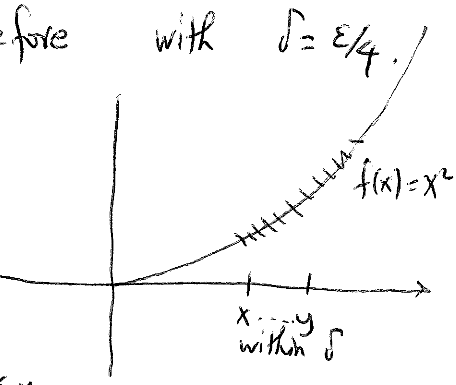
This gives $|f(x) - f(y)| = |x-y| \cdot |x+y| < 2\delta = \varepsilon$ as long as $\delta = \varepsilon/2$.

Example. Consider $f(x) = x^2$ as a function $f: [0,2] \to [0,4]$.

In this case, $|f(x) - f(y)| = |x^2 - y^2| = |x-y| \cdot |x+y|$ with $x + y \leq 2 + 2 = 4$

so $|f(x) - f(y)| < 4\delta = \varepsilon$ as before with $\delta = \varepsilon/4$.

We need a smaller choice of δ in this case.



This is because $|x-y| < \delta$ .... relates to x values

$|x^2 - y^2| < \varepsilon$ ---- relates to y values

and the choice of δ will generally depend on x, y.

Example. Consider $f(x) = x^2$ as a function $f: \mathbb{R} \to \mathbb{R}$.
This is not uniformly continuous. To see this, we let $\varepsilon = 1$ and show that the definition fails: there is no $\delta > 0$ with

$$|x-y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < 1.$$

Let's look at x and $y = x + \delta/2$, for instance.
Then $|x-y| = \delta/2$ so the first condition holds.
But $|f(x) - f(y)| = |x^2 - y^2| = |x-y| \cdot |x+y| = \frac{\delta}{2}\left(2x + \delta/2\right)$.
Thus $|f(x) - f(y)| \geq \delta x$ for any $\delta > 0$ and any $x > 0$.
If the condition holds, we get $\delta x \leq |f(x) - f(y)| < 1$ for all $x > 0$, a contradiction. More precisely, we can pick $x > 1/\delta$ so that the inequality fails.

Recall that $f: A \to \mathbb{R}$ is uniformly continuous if, given any $\varepsilon > 0$, there exists $\delta > 0$ such that $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$ for all $x, y \in A$.

Theorem (Differentiable functions with bounded derivative)
Suppose ~~f: A→R is~~ $f: [a,b] \to \mathbb{R}$ is differentiable and suppose there exists $c > 0$ such that $|f'(x)| \leq c$ for all x. Then f is uniformly continuous on $[a,b]$.

Proof. Let $\varepsilon > 0$ be given. Let $x, y \in A$ be arbitrary, $A = [a,b]$.
Then $|f(x) - f(y)| = |f'(z)| \cdot |x-y|$ for some point z between x and y (by the Mean Value Theorem).

We can let $\delta = \varepsilon / c$ to conclude that

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| = |f'(z)| \cdot |x-y| \leq c\delta = \varepsilon .$$

This proves uniform continuity. ▨

Examples:
- $f(x) = \sin x$ on any interval $[a,b]$ is unif. cont. because $|f'(x)| = |\cos x| \leq 1$ for all x.
- $f(x) = \cos x$ on any interval $[a,b]$ since $|f'(x)| = |\sin x| \leq 1$.
- $f(x) = x^2$ on any interval $[a,b]$ since $|f'(x)| = 2x$ is bounded on $[a,b]$.

Completeness

Convergent sequence. We say $\{x_n\}$ converges to x if, given any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for all $n \geq N$.

Cauchy sequence. We say $\{x_n\}$ is Cauchy if, given any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ for all $m, n \geq N$.

Bounded sequence. We say $\{x_n\}$ is bounded, if there exists $c > 0$ such that $|x_n| < c$ for all $n \geq 1$.

$$\boxed{\text{Convergent}} \Longrightarrow \boxed{\text{Cauchy}} \Longrightarrow \boxed{\text{Bounded}}$$

Note that $(-1)^n$ is an alternating sequence that is bounded, oscillating between -1 and 1, so it is neither Cauchy nor convergent.

<u>Theorem</u> (Convergent implies Cauchy) Suppose $\{x_n\}$ converges to x. Then $\{x_n\}$ is Cauchy.

<u>Proof</u>. Let $\varepsilon > 0$ be given. Then there exists $N \in \mathbb{N}$:

$$|x_n - x| < \varepsilon/2 \quad \text{for all } n \geq N.$$

This gives

$$|x_m - x| < \varepsilon/2 \quad \text{for all } m \geq N$$

and so

$$|x_m - x_n| = |x_m - x + x - x_n| \leq |x_m - x| + |x - x_n| < \varepsilon \quad \text{for all } m, n \geq N.$$

<u>Theorem</u> (Cauchy implies bounded) If $\{x_n\}$ is Cauchy, then $\{x_n\}$ is bounded.

<u>Proof</u>. We use the definition with $\varepsilon = 1$. Then there exists $N \in \mathbb{N}$:

$$|x_m - x_n| < 1 \quad \text{for all } m, n \geq N.$$

In particular,

$$|x_m - x_N| < 1$$

and so

$$|x_m| = |x_m - x_N + x_N| \leq |x_m - x_N| + |x_N| < 1 + |x_N| \quad \text{for all } m \geq N.$$

This settles the terms x_m with $m \geq N$. The remaining terms satisfy $|x_m| \leq \max\{|x_1|, |x_2|, \ldots, |x_N|\}$ and that gives

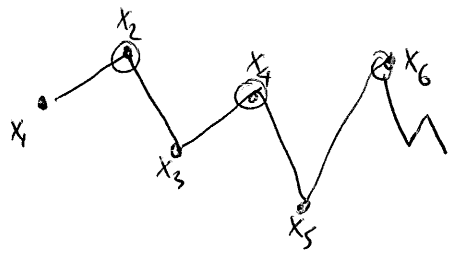
$$|x_m| \leq \max\{|x_1|, |x_2|, \ldots, |x_N|, 1 + |x_N|\} \quad \text{for all } m \geq 1.$$

<u>Definition</u> (Completeness) We say that $A \subseteq \mathbb{R}$ is complete, if every Cauchy sequence $\{x_n\}$ of elements of A actually converges to an element of A.

<u>Example</u>. $(0,1)$ is not complete because $x_n = \frac{1}{n+1}$ gives a sequence in $(0,1)$ whose terms ~~are~~ become arbitrarily small, but the limit $\lim_{n\to\infty} x_n = 0$ is <u>not</u> an element of $(0,1)$.

We'll show $[0,1]$ is complete and $\mathbb{R}$ is complete.

Theorem 1. (Bolzano – Weierstrass theorem)
If $\{x_n\}$ is a bounded sequence of real numbers, then $\{x_n\}$ has a convergent subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \dots$



Proof. The idea is to look for a monotonic subsequence. If we can find one that is monotonic (and bounded), then that will converge by the monotone convergence theorem.

Let us call x_N a peak point if all subsequent terms are smaller, namely if $x_n \le x_N$ for all $n \geq N$. If I can find infinitely many of those $x_{n_1}, x_{n_2}, x_{n_3}, x_{n_4}, \dots$ all of them being peak points, then $x_{n_1} \geq x_{n_2} \geq x_{n_3} \geq \dots$ and we get a decreasing seq.
Otherwise, there is a finite number of peak points, among ~~say~~ the first N.
Then $y_1 = x_{N+1}$ is not peak $\Rightarrow$ there exists a subsequent term that is bigger, say y_2, that is not a peak point and we can find a subsequent term y_3 that is bigger etc. This gives an increasing subsequence. ∎

Theorem 2. (Cauchy sequence with a convergent subsequence)
If $\{x_n\}$ is a Cauchy sequence of real numbers with a convergent subsequence, then the whole sequence $\{x_n\}$ converges.

Proof. Assume $x_{n_1}, x_{n_2}, x_{n_3}, \dots$ is a convergent subsequence, say $x_{n_k} \to x$.
We need to show $x_n \to x$ as well. Let $\varepsilon > 0$. Then there exists

(a) a natural number N_1: $|x_{n_k} - x| < \varepsilon/2$ for all $n_k \geq N_1$.

(b) a natural number N_2: $|x_m - x_n| < \varepsilon/2$ for all $m, n \geq N_2$.

Take ~~$m = N_2$ to get~~ $N = \max\{N_1, N_2\}$ to conclude that

$$|x_m - x| = |x_m - x_{n_k} + x_{n_k} - x| \le |x_m - x_{n_k}| + |x_{n_k} - x|.$$

Pick some fixed $n_k > N_1$ and then $|x_m - x| < \varepsilon$ for all $m \geq N$. ∎

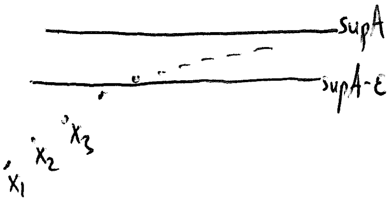
Theorem ($\mathbb{R}$ is complete) Every Cauchy sequence in $\mathbb{R}$ converges.

Proof. Suppose $\{x_n\}$ Cauchy. Then $\{x_n\}$ bounded. Thus $\{x_n\}$ has a convergent subsequence by Bolzano-Weierstrass. By Theorem 2, the whole sequence converges. ∎

We now have the following results:

1. Axiom of completeness: $A \subseteq \mathbb{R}$ nonempty & bounded above $\Rightarrow$ $\sup A$ exists
2. Monotone convergence theorem: $\{x_n\}$ increasing & bounded above $\Rightarrow$ $\{x_n\}$ converges
3. Bolzano-Weierstrass theorem: $\{x_n\}$ bounded $\Rightarrow$ $\{x_n\}$ has converg. subseq.
4. $\mathbb{R}$ is complete: Every Cauchy sequence converges.
1. Axiom of completeness.

(1) implies (2): If $\{x_n\}$ increasing & bounded above, then $A = \{x_1, x_2, x_3, \dots\}$ has a supremum.

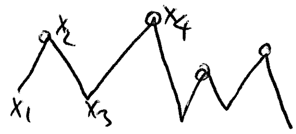


We claim $x_n \to \sup A$. Let $\varepsilon > 0$.

Then there exists $x_N > \sup A - \varepsilon$ so $x_n \geq x_N > \sup A - \varepsilon$ for all $n \geq N$ and so $\sup A - \varepsilon < x_n \leq \sup A$ for all $n \geq N$

$$0 \leq \sup A - x_n < \varepsilon.$$ ∎

(2) implies (3): One looks for peak points.

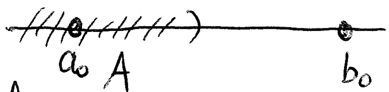


Infinitely many of those $\Rightarrow$ decreasing subseq.

Finitely many of those $\Rightarrow$ increasing subseq. ∎

(3) implies (4): We did this above using Theorems 1 and 2. ∎

(4) implies (1): This is the hard part. We assume A nonempty and bounded from above. There is some $a_0 \in A$ and some $b_0 =$ u.b. of A. We look at the average of those two.



Suppose we have some $a_n \in A$ and $b_n =$ u.b of A.

We consider $\frac{a_n + b_n}{2}$ which satisfies $a_n \leq b_n \Rightarrow \boxed{a_n \leq \frac{a_n+b_n}{2} \leq b_n}$.

Case 1. If $\frac{a_n+b_n}{2}$ is an u.b. of A, we let $a_{n+1}=a_n$, $b_{n+1}=\frac{a_n+b_n}{2}$.
Then $b_{n+1}\le b_n$ and we have a smaller u.b.

Case 2. If $\frac{a_n+b_n}{2}$ is not an u.b. of A, there exists an element of A that's bigger. Pick such an element $a_{n+1}\geq \frac{a_n+b_n}{2}$ and note that $a_{n+1}\geq \frac{a_n+b_n}{2}\geq a_n$. We let $b_{n+1}=b_n$.

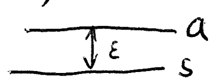
We obtain a sequence $\{a_n\}$ ---- increasing elements of A
$\{b_n\}$ ---- decreasing u.b. of A.

~~Note~~ To show $\{a_n\}$ is Cauchy, we note that

$$b_{n+1}-a_{n+1} = \frac{a_n+b_n}{2}-a_n = \frac{1}{2}(b_n-a_n) \quad \text{---- in Case 1}$$

and
$$b_{n+1}-a_{n+1} \le b_n - \frac{a_n+b_n}{2} = \frac{1}{2}(b_n-a_n) \quad \text{---- in Case 2}$$

This gives $b_n-a_n\le \frac{1}{2^n}(b_0-a_0)$ by induction so the length of $[a_n,b_n]$ ~~is~~ becomes arbitrarily small. But $a_N, a_{N+1}, a_{N+2}, \dots$ are all in $[a_N, b_N]$ because $\{a_n\}$ is increasing and b_N is an upper bound of A. This proves $\{a_n\}$ is Cauchy so (4) implies $\{a_n\}$ converges. Let $s=\lim_{n\to\infty} a_n$. We claim $s=$ least upper bound of A. To check s is an upper bound of A, we note that $a_n\le s\le b_n$ for all n.



~~(c) If s is not an upper bound of A, there is an element $a\in A$ with $a>s$. Let $\varepsilon = a-s>0$. Then the fact that $a_n\to s$ implies $|s-a_n|<\varepsilon$ for large n, say $n\geq N$~~
~~$s-a_n<\varepsilon$ for large n~~
~~$s-\varepsilon<a_n$ for large n~~
~~which implies~~

We know $a_n\to s$ and $b_n-a_n\to 0$, so $b_n\to s$ as well.

(1) if $a\in A$, then $a\le b_n$ for all n, so $a\le \lim b_n$ and $a\le s \Rightarrow$ s is an upper bound.

(2) if t is an u.b of A, $t\geq a_n$ for all n, so $t\geq s$ and $s=\sup A$. ▨

$\boxed{\text{Complete subsets of } \mathbb{R}}$

Suppose $A \subseteq \mathbb{R}$.

① If A is complete, then A is closed in $\mathbb{R}$.

② If A is closed in $\mathbb{R}$, then A is complete.

For instance, $A = [0,1]$ is complete but $A = (0,1)$ is not. The problem in the second case is that $1/2, 1/3, 1/4, \dots$ converges to a point $0 \notin A$.

Proof of ②: Suppose $\{x_n\}$ Cauchy with $x_n \in A$.
By completeness of $\mathbb{R}$, we know $x = \lim x_n$ exists.
Since A is closed, and $x_n \in A$, we get $x \in A$ as well.
This proves A is complete. ▨

Proof of ①: Suppose A is complete. To show A is closed, we need A^c to be open. Given any $x \in A^c$, we thus need $(x-\varepsilon, x+\varepsilon) \subseteq A^c$ for some $\varepsilon > 0$. We look at $(x-1/n, x+1/n)$ .... for each $n \in \mathbb{N}$.

If $(x-1/n, x+1/n) \subseteq A^c$ for some $n \in \mathbb{N}$, we are done.
Otherwise, there exists $x_n \in (x-1/n, x+1/n)$ but $x_n \notin A^c$.
This gives a sequence $\{x_n\}$ of elements of A with

$$x - 1/n < x_n < x + 1/n \quad \text{for all } n.$$

Then $x_n \to x$ by the Squeeze Theorem.
Since A is complete and $x_n \in A$, we get $x \in A$.
This contradicts our assumption $x \in A^c$. ▨

Remark. We know A closed $\Leftrightarrow$ A complete.
If $A \subseteq \mathbb{R}$ is arbitrary, then $\bar{A}$ is the smallest closed set $\bar{A} \supseteq A$, hence $\bar{A}$ is the smallest complete set containing A.

$\boxed{\text{Definition}}$ $\boxed{\text{Dense subsets of } \mathbb{R}}$

We say that $A \subseteq \mathbb{R}$ is dense in $\mathbb{R}$, if the closure of A is $\bar{A} = \mathbb{R}$.
Recall that $\bar{A} = A \cup A'$ so we are assuming that every $x \in \mathbb{R}$ is either an element of A or the limit of a sequence of points in A. The standard example is: the set $\mathbb{Q}$ of all rationals.

Theorem ($\mathbb{Q}$ is dense in $\mathbb{R}$) Given any reals $x<y$ there exists a rational number z: $x<z<y$.

Proof. We need to ensure $x < \frac{m}{n} < y$ for some $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. This means $nx < m < ny$ for some $m \in \mathbb{Z}$, $n \in \mathbb{N}$.

① We ensure (nx, ny) has length bigger than 1. That means
$$ny - nx > 1 \iff n(y-x) > 1 \iff n > \frac{1}{y-x}.$$
By the Archimedean property, such an n exists.

② We look at $A = \{ m \in \mathbb{Z} : m > nx \}$.
This set is bounded from below by nx and $A \neq \emptyset$ because ~~$ny - nx > 1$ and $ny-1 > nx$, namely $ny-1 \in A$~~ of the Ar. Prop.
It follows by Problem Set #2 that $\min A$ exists, say m.
Then ~~$m > nx$~~ $m \in A$ and $m-1 \notin A$
$$m > nx \quad \text{and} \quad m-1 \leq nx$$
so $nx < m \leq nx + 1 < ny$, as needed. ∎

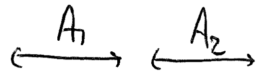
Corollary. ($\overline{\mathbb{Q}} = \mathbb{R}$) The set $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof. Consider $\overline{\mathbb{Q}} = \mathbb{Q} \cup \mathbb{Q}'$. If $x \in \mathbb{R}$ is arbitrary, we claim x is a limit point of $\mathbb{Q}$. Consider a neighbourhood $\mathcal{U}$ of x. This is an open set containing x, so $(x-\varepsilon, x+\varepsilon) \subseteq \mathcal{U}$. Thus $(x-\varepsilon, x) \subseteq \mathcal{U}$ and $(x-\varepsilon, x)$ contains a rational by above, so $\mathcal{U}$ contains a rational other than x. This proves that x is a limit point of $\mathbb{Q}$. ∎

Homework problem. If $f(x) = g(x)$ for all $x \in \mathbb{Q}$ and f, g continuous, then $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Standard application: Consider linear transformations $T: \mathbb{R} \to \mathbb{R}$ with $T(x+y) = T(x) + T(y)$. We check $T(2) = T(1) + T(1) = 2T(1)$, $T(3) = T(2) + T(1) = 3T(1)$ and $T(n) = nT(1)$ for all $n \in \mathbb{N}$. We also have $T(0+0) = T(0) + T(0) \Rightarrow T(0) = 0$ and $n=0$ also works. It also works for negative integers $T(n-n) = T(n) + T(-n) \Rightarrow T(-n) = -T(n) = -nT(1)$.
Then $T(m/n) = m/n\, T(1)$ because $nT(m/n) = T(m/n) + T(m/n) + \dots + T(m/n) = T(n \cdot m/n) = T(m)$, so $T(m/n) = \frac{m}{n} T(1)$.

Connected sets



Definition. We say that $A \subseteq \mathbb{R}$ is not connected, if it can be written as a union $A = A_1 \cup A_2$ where A_1, A_2 are nonempty, disjoint and open in A.

Example. Let $A = (0,1) \cup [2,3)$, for instance.

Then $A_1 = (0,1)$ is open in $\mathbb{R}$, so A_1 is open in A as well

because A_1 open in $\mathbb{R} \Rightarrow A_1 \cap A = A_1$ open in A.

In fact, $A_2 = [2,3)$ is open in A as well. It is not open in $\mathbb{R}$, but it is open in A since $(3/2, 3)$ is open in $\mathbb{R}$

$$\Rightarrow (3/2, 3) \cap A = [2,3) \text{ is open in } A$$

Thus, $A = (0,1) \cup [2,3)$ is not connected.

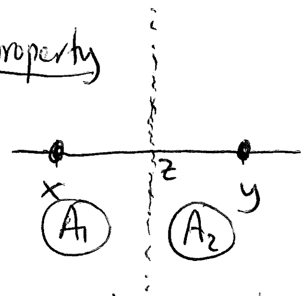
Example. Let $A = [0,1] \cup \{2\}$. This is not connected, either.

Then $A_1 = [0,1]$ is closed in $\mathbb{R} \Rightarrow A_1 \cap A = [0,1]$ is closed in A

and $A_2 = \{2\}$ is closed in $\mathbb{R} \Rightarrow A_2 \cap A = \{2\}$ is closed in A.

The complements of these are then open in A, so A_1, A_2 are open in A.

Definition. We say that $A \subseteq \mathbb{R}$ has the intermediate point property if $x, y \in A$ and $x < z < y$ implies $z \in A$ as well.



Theorem (Criterion for being connected)

$A \subseteq \mathbb{R}$ is connected $\Leftrightarrow$ A has the inter. point property.

Proof. Suppose A does not have the IPP. Then there exist $x, y \in A$ and $x < z < y$ with $z \notin A$.

Define $A_1 = (-\infty, z) \cap A$ and $A_2 = (z, \infty) \cap A$.

Nonempty ----- $x < z$ and $x \in A$ so $x \in A_1$

$z < y$ and $y \in A$ so $y \in A_2$.

Disjoint ----- A_1 contains elements $< z$, A_2 contains elements $> z$.

Open in A ----- Since $(-\infty, z)$ is open in $\mathbb{R}$, A_1 is open in A.

Similarly, A_2 is open in A.

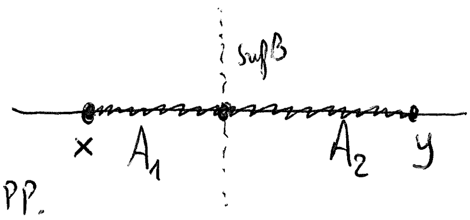
This implies $A_1 \cup A_2$ is not connected, so A is not connected.

We proved A not IPP $\Rightarrow$ A not connected.

Suppose now A does have the IPP. We need to show A connected. Assume it is not connected. Then $A = A_1 \cup A_2$ for some A_1, A_2 nonempty, disjoint and open in A. Pick $x \in A_1$ and $y \in A_2$. We may assume $x < y$. Since $x \in A_1 \subseteq A$

and $y \in A_2 \subseteq A$,

we get $[x,y] \subseteq A$ by the IPP.



Consider the set $B = A_1 \cap [x,y]$. This set contains x and it has y as an upper bound, so $\sup B$ exists and

$$x \le \sup B \le y.$$

Note that $\sup B \in [x,y] \subseteq A$ so $\sup B \in A_1$ or $\sup B \in A_2$.

Case 1. Suppose $\sup B \in A_1$. Since A_1 is open in A, so

$$(\sup B - \varepsilon, \sup B + \varepsilon) \cap A \subseteq A_1 \quad \text{for some } \varepsilon > 0.$$

Note that $y \notin A_1$ so $\sup B \neq y$. This gives $x \le \sup B < y$.

Let $\delta = \min\{\varepsilon/2, y - \sup B\} > 0$. Then

$$\sup B + \delta \in [\sup B, \sup B + \varepsilon) \cap [x,y]$$
$$\subseteq [\sup B, \sup B + \varepsilon) \cap A \subseteq A_1 \text{ by above.}$$

This is a contradiction since $\sup B + \delta \in A_1 \cap [x,y] = B$.

Case 2. Suppose $\sup B \in A_2$. Since A_2 is open in A, we get

$$(\sup B - \varepsilon, \sup B + \varepsilon) \cap A \subseteq A_2 \quad \text{for some } \varepsilon > 0.$$

Since $\sup B - \varepsilon < \sup B$, there exists some $z \in B$ such that

$$\sup B - \varepsilon < z \le \sup B.$$

Being an element of $B = A_1 \cap [x,y]$, z is in A_1 and thus A, but

$$z \in (\sup B - \varepsilon, \sup B] \cap A \subseteq A_2,$$

a contradiction since $A_1 \cap A_2 = \emptyset$. $\blacksquare$

Theorem (Connected subsets of $\mathbb{R}$) The only connected subsets are:

- finite intervals (a,b), $(a,b]$, $[a,b)$, $[a,b]$ with $a<b$
- infinite intervals (a,∞), $(-\infty,b)$, $[a,\infty)$, $(-\infty,b]$ with $a,b\in\mathbb{R}$
- generic intervals $(a,a)=\emptyset$, $[a,a]=\{a\}$, $(-\infty,\infty)=\mathbb{R}$.

Proof. Those have the IPP $\Rightarrow$ they are connected.

Conversely, suppose $A\subseteq\mathbb{R}$ connected and nonempty.

(1) Suppose A is bounded from above and below. Then $\inf A$, $\sup A$ exist and $\inf A\le x\le \sup A$ for all $x\in A$ so $A\subseteq[\inf A, \sup A]$.

We let $a=\inf A$, $b=\sup A$ and conclude that $A\subseteq[a,b]$.

It remains to show $(a,b)\subseteq A$.... so that $(a,b)\subseteq A\subseteq[a,b]$.

- Indeed, if $x\in(a,b)$, then $a<x<b$ so $\inf A<x<\sup A$.

Since $x<\sup A$, there exists $a_1\in A$ such that $x<a_1$

Since $x>\inf A$, there exists $a_2\in A$ such that $x>a_2$.

Since $a_1\in A$, $a_2\in A$ and $a_2<x<a_1$, the IPP gives $x\in A$ as well.

This proves the inclusion $(a,b)\subset A$ and we also have $A\in[a,b]$.

(2) Suppose A is bounded from above but not bounded from below.

In this case, we expect A to be either $(-\infty,\sup A]$ or $(-\infty,\sup A)$.

We proceed as before to let $a=\sup A$ and show

$(-\infty,a)\subseteq A\subseteq(-\infty,a]$. One inclusion is trivial since

$$x\in A\Rightarrow x\le\sup A\Rightarrow x\le a\Rightarrow x\in(-\infty,a].$$

The other inclusion follows by noting that

$$x\in(-\infty,a)\Rightarrow x<a\Rightarrow x<\sup A\Rightarrow x<a_1 \text{ for some } a_1\in A.$$

Since A is not bounded from below, x is not a lower bound of A, so there exists $a_2<x$ ~~and~~ with $a_2\in A$ $\Rightarrow x\in A$ as before.

(3) If A is bounded from below, but not above, we get (a,∞) or $[a,\infty)$.

(4) If A is not bounded from either above or below, then $A=\mathbb{R}$. ▨

<u>Theorem</u> (Sets both open and closed)

If $A \subseteq \mathbb{R}$ is connected, then the only subsets of A that are both open and closed in A are $\emptyset, A$. If $A \subseteq \mathbb{R}$ is not connected, then $A = A_1 \cup A_2$ for some sets A_1, A_2 nonempty that are open & closed in A.

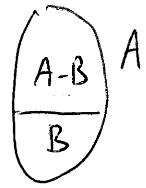
<u>Proof.</u> A not connected $\Rightarrow$ $A = A_1 \cup A_2$ with A_1, A_2 nonempty, open in A, disjoint

$\Rightarrow$ $A = A_1 \cup A_2$ with A_1 open in A and A_1 closed in A but A_1 nonempty.

For instance, $A = (0,1) \cup [2,3)$ contains $(0,1)$ which is both open and closed in A.

⊙ If A is ~~not~~ connected, and if $B \subseteq A$ is both open and closed in A, then B is open in A and $A - B$ is open in A, while

$A = B \cup (A-B)$ is the union of open disjoint sets.



Since A connected, either B is empty or $A-B$ is empty

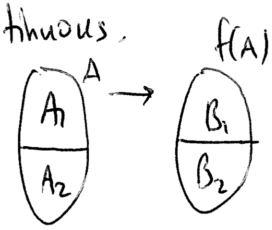
so $B = \emptyset$ or $B = A$. ▨

<u>Theorem</u> (Images of connected sets) Suppose $f : A \to \mathbb{R}$ is continuous and A is connected. Then $f(A)$ must be connected. In particular, the image of an interval can only be $\mathbb{R}$, an interval or just a point $\{x\}$.

⊙ This implies the Intermediate Value Theorem (Bolzano's theorem).

<u>Proof.</u> Suppose $f(A)$ is not connected and $f : A \to f(A)$ continuous.

Then $f(A) = B_1 \cup B_2$ with $B_1, B_2 \neq \emptyset$ open in $f(A)$ and disjoint.



Take $A_1 = f^{-1}(B_1)$, $A_2 = f^{-1}(B_2)$. These are open in A by continuity.

They are nonempty since $y_i \in B_i \Rightarrow y_i \in f(A) \Rightarrow y_i = f(x_i)$ for some $x_i \in A$

$\Rightarrow x_i \in f^{-1}(B_i)$.

They are also disjoint since

$$x \in A_1 \cap A_2 \Rightarrow x \in f^{-1}(B_1) \cap f^{-1}(B_2) \Rightarrow f(x) \in B_1 \cap B_2 = \emptyset .$$

Finally, $A_1 \cup A_2 = f^{-1}(B_1) \cup f^{-1}(B_2) = f^{-1}(B_1 \cup B_2) = f^{-1}(f(A)) \supseteq A$

and thus $A_1 \cup A_2 = A$, contradicting the fact that A connected. ▨

Example. There is no continuous map $f:(0,1)\to\mathbb{R}$ whose image is $\mathbb{Q}$.

Example. There is no continuous surjection $f:(0,1)\to(0,1)\cup(2,3)$.

Example. Suppose $f:[a,b]\to\mathbb{R}$ is continuous. Then f attains all values between $f(a)$ and $f(b)$. This is because $f([a,b])$ is connected, so it has the intermediate point property. We'll eventually show the image is a closed interval $[x,y]$.

Example. There is a continuous map $f:(0,1)\to\mathbb{R}$ whose image is all of $\mathbb{R}$. An example is $f(x)=\tan\left(\pi x-\frac{\pi}{2}\right)$ which maps $(0,1)$ to $(-\pi/2,\pi/2)$ and takes the tangent. □

Definition. (Countable) We say $A\subseteq\mathbb{R}$ is countably infinite, if there is a bijection $f:\mathbb{N}\to A$. We say $A\subseteq\mathbb{R}$ is countable, if A is either finite or countably infinite.

Examples: countable sets include $\mathbb{N}$, any subset of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$.
Uncountable sets include the irrationals, $(0,1)$ and $\mathbb{R}$.

Theorem 1. (Cantor's diagonal argument) The interval $(0,1)$ is uncountable.

Proof. Suppose $f:\mathbb{N}\to(0,1)$ is a bijection. List the elements in decimal expansion.

$$f(1)=0.a_{11}a_{12}a_{13}a_{14}\dots$$
$$f(2)=0.a_{21}a_{22}a_{23}a_{24}\dots$$
$$f(3)=0.a_{31}a_{32}a_{33}a_{34}\dots$$

We replace the nth decimal digit of the nth number. Take
$$b_{ii}=\begin{cases}1 & \text{if } a_{ii}\neq 1\\ 2 & \text{if } a_{ii}=1\end{cases},\quad\text{for instance.}$$

We get a number $0.b_{11}b_{22}b_{33}\dots$ such that $b_{ii}\neq a_{ii}$. Thus, we get a number in $(0,1)$ which is not in the list. Thus, f is not surjective. ▨

Theorem 2. (Subsets of $\mathbb{N}$). Every subset of $\mathbb{N}$ is countable.

Proof. Suppose $A\subseteq\mathbb{N}$. If A is finite, then A is countable.
If $A\subseteq\mathbb{N}$ is infinite, we show A is countably infinite.
We need to find a bijection $f:\mathbb{N}\to A$.

Since A is a nonempty subset of $\mathbb{N}$, $\min A$ exists by Thm 2.15.
Let's define $f(1) = \min A$

$f(2) = \min A - \{f(1)\}$

and similarly $f(n) = \min \underline{A - \{f(1), f(2), \dots, f(n-1)\}}$, for all $n \geq 2$.

This gives a function $f: \mathbb{N} \to A$.

Injectivity: if $m > n$, the set $\{f(1), f(2), \dots, f(m-1)\}$ contains $f(n)$ so the set $A - \{f(1), f(2), \dots, f(m-1)\}$ does not. Since $f(m)$ is the min of this set, $f(m)$ is in the set and $f(m) \neq f(n)$.

Surjectivity: We know $f: \mathbb{N} \to A$ is injective, so $f(\mathbb{N})$ is infinite as well.
Let $x \in A \subseteq \mathbb{N}$ be arbitrary. There is some $n \in \mathbb{N}$: $f(n) \geq x$.
Consider the set $B = \{n \in \mathbb{N} : f(n) \geq x\}$. This is nonempty,
so $m = \min B$ exists. Then

ⓐ $m \in B$ and thus $f(m) \geq x$

ⓑ $1, 2, \dots, m-1 \notin B$ and thus $f(1), f(2), \dots, f(m-1) < x$.

The set $A - \{f(1), f(2), \dots, f(m-1)\}$ contains $f(m)$ as its least element and $f(m) \geq x$ by above, so $f(m) = x$. This proves surjectivity. ▨

Theorem 3. (Criteria for countability) If $A \subseteq \mathbb{R}$ is nonempty, then
① A is countable $\Leftrightarrow$ ② there is a ~~injective~~ surjective map $f: \mathbb{N} \to A$ $\Leftrightarrow$ ③ there is an injective map $g: A \to \mathbb{N}$.

Proof. ① $\Rightarrow$ ②: If A is countable $\Rightarrow$ A is finite or countably infinite.
In the latter case, we have a bijection $f: \mathbb{N} \to A$, thus a surject.
In the former case, $A = \{a_1, \dots a_n\}$, let $f(k) = \begin{cases} a_k & \text{if } k \leq n \\ a_1 & \text{if } k > n \end{cases}$.
Then f is surjective.

② $\Rightarrow$ ③: Suppose $f: \mathbb{N} \to A$ is surjective.
We define $g: A \to \mathbb{N}$ as follows. If $x \in A$, then $x \in f(\mathbb{N})$ by surjectivity, so $f^{-1}(\{x\})$ is a nonempty subset of $\mathbb{N}$ and we can define $g(x) = \min f^{-1}(\{x\})$.
To show g is injective, note that $f^{-1}(\{x\})$

and $f^{-1}(\{y\})$ are disjoint when $x \neq y$ since

$z \in f^{-1}(\{x\}) \cap f^{-1}(\{y\}) \implies x = f(z) = y \implies x = y$.

Thus, $g(x) \neq g(y)$ because these arefer to disjoint sets.

③ $\Rightarrow$ ①: if A is finite, then A is countable.

If A is infinite and $g: A \to \mathbb{N}$ is injective

then $g: A \to g(A)$ is bijective

and $\underline{g(A) \subseteq \mathbb{N} \text{ is countabley infinite}}$ by Theorem 2.

Thus we have bijections $g: A \to g(A)$ and $h: g(A) \to \mathbb{N}$.

The composition $h \circ g : A \to \mathbb{N}$ is a bijection. $\blacksquare$

Example 1. (Integers $\mathbb{Z}$). We define a surjection $f: \mathbb{N} \longrightarrow \mathbb{Z}$

We'll define $f(1), f(2), f(3)$ etc by ordering them as $0, 1, -1, 2, -2$, etc.

This gives $f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -\frac{n-1}{2} & \text{if } n \text{ odd} \end{cases}$ which is surjective.

Countability Recall that A is countable, if there exists a surjection $f: \mathbb{N} \to A$. Some typical examples are $\mathbb{N}$, subsets of $\mathbb{N}$, the set of integers $\mathbb{Z}$, the set of rationals $\mathbb{Q}$ and so on.

<u>Example 1.</u> (Integers $\mathbb{Z}$) To define a surjection $f: \mathbb{N} \to \mathbb{Z}$, one needs to define $f(1), f(2), f(3), \ldots$ in such a way that all integers are listed. This can be achieved by listing the integers in the order $0, 1, -1, 2, -2, \ldots$ alternating between positive and negative numbers. More precisely, one may define $f: \mathbb{N} \to \mathbb{Z}$ by

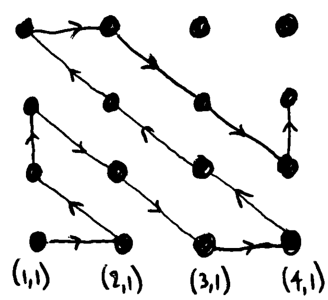
$$f(n) = \left\{ \begin{array}{ll} n/2 & \text{if } n \text{ is even} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd} \end{array} \right\}.$$

This is easily seen to be a surjection because the list

$f(2) = 1,\ f(4) = 2,\ f(6) = 3, \ldots$ includes all positive integers

and the list $f(1) = 0,\ f(3) = -1,\ f(5) = -2, \ldots$ includes the non-positive ones.

<u>Example 2.</u> The set $\mathbb{N} \times \mathbb{N}$ consisting of all pairs (m,n) of natural numbers is also countable. To define a surjection $f: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, one needs to define $f(1), f(2), f(3), \ldots$ in such a way that all elements of $\mathbb{N} \times \mathbb{N}$ are listed. Intuitively speaking, elements of $\mathbb{N} \times \mathbb{N}$ can be represented as points in $\mathbb{R}^2$ with (positive) integer coordinates. One may thus list them one by one following a zig zag pattern as in the figure. This is the intuitive way of checking that $\mathbb{N} \times \mathbb{N}$ is countable.



A more formal approach would be to define an injective function $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. As we have already shown, A is countable $\iff$ there exists a surjection $f: \mathbb{N} \to A$

$\iff$ there exists an injection $f: A \to \mathbb{N}$.

In this case, the latter condition is somewhat easier to check. Define $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by letting $f(m,n) = 2^m \cdot 3^n$. We need to show that f is injective. Indeed, suppose $f(m,n) = f(x,y)$ for some $m,n,x,y \in \mathbb{N}$. Then

$2^m \cdot 3^n = 2^x \cdot 3^y$ and we need to show $m=x$, $n=y$.

If $m>x$, then $2^{m-x} \cdot 3^n = 3^y$ and the left hand side is even, while the right hand side is odd. If $m<x$, then $3^n = 2^{x-m} \cdot 3^y$ and a similar contradiction arises because 3^n is odd and $2^{x-m} \cdot 3^y$ is even. We must thus have $m=x$. This gives $2^m \cdot 3^n = 2^x \cdot 3^y \Rightarrow 3^n = 3^y \Rightarrow n=y$ as well.

Example 3. More generally, the product $A \times B$ is defined as the set of all pairs (x,y) with $x \in A$ and $y \in B$. If the sets A, B are both countable, then one can show that $A \times B$ is countable. More precisely, there exist injective functions $f: A \to \mathbb{N}$ and $g: B \to \mathbb{N}$, so one may define a function $h: A \times B \to \mathbb{N} \times \mathbb{N}$ by letting $h(x,y) = (f(x), g(y))$. This is easily seen to be injective. Moreover, $\mathbb{N} \times \mathbb{N}$ is countable by the previous example, so there is a bijection $\phi: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Taking the composition with $h: A \times B \to \mathbb{N} \times \mathbb{N}$, we get an injection $\phi \circ h: A \times B \to \mathbb{N}$.

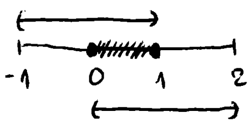
Example 4. The set $\mathbb{Q}$ of all rational numbers is countable. More precisely, $\mathbb{Q}$ consists of quotients m/n, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. We can thus define a surjection $f : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$ using the formula $f(m,n) = m/n$. Since each of $\mathbb{Z}, \mathbb{N}$ is countable, their product $\mathbb{Z} \times \mathbb{N}$ is countable as well. One can thus find a bijection $\phi : \mathbb{N} \to \mathbb{Z} \times \mathbb{N}$. Taking the composition with f, we obtain a function $f \circ \phi : \mathbb{N} \to \mathbb{Q}$ which is surjective because f, ϕ are. Thus, $\mathbb{Q}$ is countable.

Compactness

This is a difficult topic that we will study both this week and next week. The overall idea is that we have a set $A \subseteq \mathbb{R}$ which is contained in the union of some open sets $\mathcal{U}_i$ and we wish to conclude that A is contained in finitely many of these open sets. If that is the case, we say A is compact.

Definition (Open cover). We say that the sets $\mathcal{U}_i$ form an open cover of the set $A \subseteq \mathbb{R}$, if each $\mathcal{U}_i$ is open in $\mathbb{R}$ and the set A is contained in $\bigcup_i \mathcal{U}_i$.

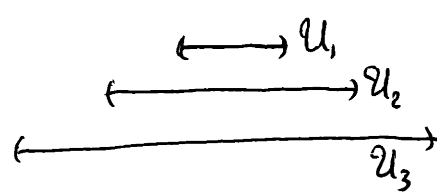
- For instance, the sets $(-1,1)$ and $(0,2)$ form an open cover of the closed interval $[0,1]$.



Definition (Compactness). We say that $A \subseteq \mathbb{R}$ is compact, if every open cover of A has a finite subcover. Given any open sets $\mathcal{U}_i$ that cover A, in other words, there must exist a finite number of these sets that actually cover A.

Example. If a set $A \subseteq \mathbb{R}$ has finitely many elements, then A is compact. To show this, suppose the sets $\mathcal{U}_i$ form an open cover of A and suppose $A = \{x_1, x_2, \dots, x_n\}$. Since A is contained in the union of the sets $\mathcal{U}_i$, we must have $x_1 \in A \Rightarrow x_1 \in \mathcal{U}_{i_1}$ for some i_1. Similarly, $x_2 \in \mathcal{U}_{i_2}$ for some i_2 and so on. This means that $x_1, x_2, \dots, x_n$ are all contained in $\mathcal{U}_{i_1} \cup \mathcal{U}_{i_2} \cup \dots \cup \mathcal{U}_{i_n}$. Thus, A is contained in only finitely many of the open sets.

Example The set $\mathbb{R}$ of all real numbers is not compact. For instance, consider the open intervals $\mathcal{U}_1 = (-1,1)$, $\mathcal{U}_2 = (-2,2)$, $\mathcal{U}_3 = (-3,3)$ and so on. These intervals form an open cover of $\mathbb{R}$ because their union contains every real number.



If the set $\mathbb{R}$ was actually compact, then it would be possible to cover it with finitely many of the sets $\mathcal{U}_i$. On the other hand, it is easy to see that this is not possible. The union of the first n intervals is simply $(-1,1) \cup (-2,2) \cup \dots \cup (-n,n) = (-n,n)$ and this fails to contain $n+1$.

Theorem 1. (Compact implies bounded)

If a set $A \subseteq \mathbb{R}$ is compact, then A must be bounded.

Proof. We proceed as in the previous example. Suppose that $A \subseteq \mathbb{R}$ is compact and consider the open intervals $\mathcal{U}_n = (-n, n)$ for each $n \in \mathbb{N}$. These form an open cover of $\mathbb{R}$, so they certainly cover A. However, A is compact by assumption, so finitely many of the intervals must cover A. Suppose that the first N do. We then have

$$A \subseteq (-1,1) \cup (-2,2) \cup \dots \cup (-N, N) \quad \text{and so} \quad A \subseteq (-N, N).$$

This shows that A must be bounded. ∎

Example. It is not true that every bounded set is compact. In fact, we show that $(0,2)$ is not compact. Consider the open intervals $\mathcal{U}_n = (1/n, 2)$ for each $n \in \mathbb{N}$. This is an increasing sequence of intervals and the leftmost point converges to 0. It is thus easy to see that the sets $\mathcal{U}_n$ form an open cover of $A = (0,2)$. This is because the union of all sets $\mathcal{U}_n$ is precisely A.

Were the set A compact, it would be possible to cover A with only finitely many of the sets $\mathcal{U}_n$. Suppose then that A is covered by $\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_N$. Then A is contained in the union of these sets and $\mathcal{U}_1 \subseteq \mathcal{U}_2 \subseteq \dots \subseteq \mathcal{U}_N$, so the union is equal to $\mathcal{U}_N$. We have thus found that $A \subseteq \mathcal{U}_N$ which means that $(0,2) \subseteq (1/N, 2)$ for some $N \in \mathbb{N}$. This is obviously a contradiction because $\frac{1}{N+1}$ is not in $(1/N, 2)$, but it is certainly in $(0,2)$. We conclude that A is not contained in finitely many of the given sets, so A is not compact.

<u>Theorem 2.</u> (Continuous image of a compact set)
If $f: A \to B$ is continuous and A is compact, then $f(A)$ is compact.

One may use this theorem to give a simple proof that $A = (0,2)$ is not compact. Suppose it is and let $f: A \to \mathbb{R}$ be the function $f(x) = 1/x$. Since f is continuous and A compact, $f(A) = (1/2, \infty)$ must be compact. This is not the case by Theorem 1 because $f(A)$ is not bounded.

<u>Proof of Theorem 2.</u> To show that $f(A)$ is compact, suppose that the sets $\mathcal{U}_i$ form an open cover of $f(A)$. Then their inverse images must be open in A by continuity. Let us write $f^{-1}(\mathcal{U}_i) = A \cap \mathcal{W}_i$ for some sets $\mathcal{W}_i$ that are open in $\mathbb{R}$. Then the sets $\mathcal{W}_i$ form an open cover of A because $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\bigcup_i \mathcal{U}_i) = \bigcup_i f^{-1}(\mathcal{U}_i) \subseteq \bigcup_i \mathcal{W}_i$.
Since A is assumed to be compact, finitely many of

these sets must cover A. Suppose that $\mathcal{W}_{i_1}, \dots, \mathcal{W}_{i_n}$ do. To finish the proof, it remains to show that $\mathcal{U}_{i_1}, \dots, \mathcal{U}_{i_n}$ cover the image $f(A)$, namely that $f(A) \subseteq \mathcal{U}_{i_1} \cup \dots \cup \mathcal{U}_{i_n}$. Recall that $f^{-1}(\mathcal{U}_i) = A \cap \mathcal{W}_i$ for each i. This gives

$$\begin{aligned}
y \in f(A) &\Rightarrow y = f(x) \text{ for some } x \in A \\
&\Rightarrow y = f(x) \text{ for some } x \in \mathcal{W}_{i_1} \cup \dots \cup \mathcal{W}_{i_n} \\
&\Rightarrow y = f(x) \text{ for some } x \in \mathcal{W}_{i_k} \text{ with } 1 \leq k \leq n \\
&\Rightarrow y = f(x) \text{ for some } x \in f^{-1}(\mathcal{U}_{i_k}) \text{ with } 1 \leq k \leq n \\
&\Rightarrow y \in \mathcal{U}_{i_k} \text{ for some } 1 \leq k \leq n.
\end{aligned}$$

Thus, $f(A) \subseteq \mathcal{U}_{i_1} \cup \dots \cup \mathcal{U}_{i_n}$ and the proof is complete. ■

Theorem 3. (Extreme value theorem)

If $f: A \to B$ is continuous and A is compact, then f must attain both a minimum and a maximum value. That is, there exist points $x_{min}, x_{max} \in A$ such that

$$f(x_{min}) \leq f(x) \leq f(x_{max}) \quad \text{for all } x \in A.$$

The overall idea behind this theorem is that the image $f(A)$ is compact by Theorem 2 and thus bounded by Theorem 1. This means that f cannot attain arbitrarily large values. In particular, there should be a minimum and a maximum value. The extreme value theorem holds for continuous functions on compact sets. We will show that each interval $[a,b]$ is compact. Thus, continuous functions on $[a,b]$ attain a min/max value.

Proof of Theorem 3. Suppose $f: A \to B$ is continuous and A is compact. We show that f attains a maximum value. The argument for the minimum value is similar.

Since $f(A)$ is compact by Theorem 2, it is also bounded by Theorem 1. Let M be the least upper bound of this set. Then $f(x) \leq M$ for all $x \in A$. If we show that $f(x) = M$ for some $x \in A$, then the least upper bound of $f(A)$ will be an element of $f(A)$, so it will be the largest element and the result will follow.

Suppose then that $f(x) < M$ for all $x \in A$. We can then define another function $g(x) = \frac{1}{M - f(x)}$ which is both positive and continuous. According to Theorems 1 and 2, this function is also bounded, so there exists some $R > 0$ such that $g(x) \leq R$ for all $x \in A$. This gives

$$\frac{1}{M - f(x)} \leq R \quad \text{for all } x \in A \implies M - f(x) \geq \frac{1}{R} \text{ for all } x \in A$$

$$\implies f(x) \leq M - \frac{1}{R} \text{ for all } x \in A.$$

Thus, $M - 1/R$ is also an upper bound of the set $f(A)$. Since M was assumed to be the least upper bound, we have reached a contradiction. In particular, we do have $f(x) = M$ at some point $x \in A$. ∎

We shall soon show that A is a compact subset of $\mathbb{R}$ if and only if A is bounded and closed in $\mathbb{R}$. The first step in that direction is to show $[a,b]$ is compact. One can prove this fact using the nested interval property.

Theorem 4. The interval $[a,b]$ is compact for all $a<b$.

Proof. We use the bisection method to construct arbitrarily small closed intervals. Suppose $I_0 = [a,b]$ is not compact. Then there exist some open sets $\mathcal{U}_i$ that cover I_0 in such a way that no finite number of them cover I_0.

Split the interval into two intervals of equal length, namely $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$. If both of those can be covered using finitely many of the sets $\mathcal{U}_i$, then their union $[a,b]$ would also be covered by finitely many $\mathcal{U}_i$. This is not the case, however, so one of them is not covered by finitely many $\mathcal{U}_i$. Denote that subinterval by I_1 and proceed in this manner to obtain a nested sequence

$$I_0 \supseteq I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots \quad \text{of closed intervals}$$

such that ~~each~~ no I_n is covered by finitely many $\mathcal{U}_i$ and each I_n has half the length of the previous interval.

According to the nested interval property, the intersection $\bigcap_{n=0}^{\infty} I_n$ is nonempty. Let x be a point in this intersection. Then $x \in I_0 = [a,b]$ and the sets $\mathcal{U}_i$ cover $[a,b]$ by assumption, so x must lie in one of these sets, say $\mathcal{U}_k$. Since $\mathcal{U}_k$ is open, we have

$$(x-\varepsilon,\ x+\varepsilon) \subseteq \mathcal{U}_k \quad \text{for some } \varepsilon > 0.$$

This gives an interval of length 2ε which contains x. On the other hand, I_n is also an interval that contains x and the length of I_n is $\frac{b-a}{2^n}$ which goes to zero as n goes to infinity. If we choose n large enough, I_n will then be contained in $(x-\varepsilon, x+\varepsilon)$ and this implies

$$I_n \subseteq (x-\varepsilon, x+\varepsilon) \subseteq \mathcal{U}_k .$$

In particular, I_n can be covered by finitely many of the sets $\mathcal{U}_i$ and we have reached a contradiction. $\blacksquare$