Analysis Problem Set #1

Problems 1-4 due by Jan. 31st*

1. Let $f\colon A \to B$ be a function and let $B_1, B_2 \subseteq B$ be arbitrary. Show that

$$f^{-1}(B_1 - B_2) = f^{-1}(B_1) - f^{-1}(B_2).$$

2. Let $f\colon A \to B$ be a function and let $A_1 \subseteq A$ be arbitrary. Show that

$$f^{-1}(f(A_1)) \supseteq A_1$$

and that equality holds whenever the function f is injective.

3. Show that the set $A = \left\{\frac{2n+1}{n+3} : n \in \mathbb{N}\right\}$ has a minimum but no maximum.

4. Let A, B be nonempty subsets of $\mathbb{R}$ such that $\sup A < \sup B$. Show that there exists an element $b \in B$ which is an upper bound of A.

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5. Show that $(A \cap B) \cup (A - B) = A$ for any sets A, B.

6. Let $f\colon A \to B$ and $g\colon B \to C$ be two functions and let $g \circ f\colon A \to C$ denote their composition. Given a set $C_1 \subseteq C$, show that $(g \circ f)^{-1}(C_1) = f^{-1}(g^{-1}(C_1))$.

7. Determine the minimum of the set $A = \{2x^2 - 3x : x \in \mathbb{R}\}$.

8. Determine the maximum of the set $A = \{x \in \mathbb{R} : x^3 \leq 7x - 6\}$.

9. Determine the min, inf, max and sup of the following sets, noting that some of these quantities may fail to exist. You do not need to justify your answers.

(a) $A = \left\{n \in \mathbb{N} : \frac{n}{n+1} < \frac{2019}{2020}\right\}$
(b) $B = \{x \in \mathbb{R} : x > 1 \text{ and } 2x \leq 5\}$
(c) $C = \{x \in \mathbb{Z} : x > 1 \text{ and } 2x \leq 5\}$
(d) $D = \{x \in \mathbb{R} : x < y \text{ for all } y > 0\}$

10. Show that the set $A = \left\{x + \frac{1}{x} : x > 0\right\}$ is such that $\inf A = 2$.

11. Show that the set $B = \{x \in \mathbb{R} : |2x - 3| < 5\}$ is such that $\sup B = 4$.

12. Suppose that A, B are nonempty subsets of $\mathbb{R}$ which are bounded from above. Show that $A \cup B$ is also bounded from above and $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

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*You may submit your solutions Wednesday in class, Thursday in class or Friday 11-12 in my office.

Analysis Problem Set #1

Answers and hints

1. One needs to show that $x \in f^{-1}(B_1 - B_2)$ if and only if $x \in f^{-1}(B_1) - f^{-1}(B_2)$.

2. For the first part, assume that $x \in A_1$ and show that $x \in f^{-1}(f(A_1))$. For the second part, assume that $x \in f^{-1}(f(A_1))$. This gives $f(x) \in f(A_1)$ and so $f(x) = f(z)$ for some $z \in A_1$. You need to conclude that $x \in A_1$.

3. Argue that $a_n = \frac{2n+1}{n+3}$ is strictly increasing, namely that $a_n < a_{n+1}$ for each $n \in \mathbb{N}$.

4. By definition, $\sup B$ is the least upper bound of B. Since $\sup A$ is even smaller, we find that $\sup A$ is not an upper bound of B. What does this imply?

5. If we start with an element $x \in (A \cap B) \cup (A - B)$, then we have either $x \in A \cap B$ or $x \in A - B$. Consider these cases to conclude that $x \in A$. If we start with an element $x \in A$, then we have either $x \in B$ or $x \notin B$. Deduce that $x \in (A \cap B) \cup (A - B)$.

6. To show that the given sets are equal, one needs to argue that

$$\begin{aligned} x \in (g \circ f)^{-1}(C_1) \quad &\iff \quad (g \circ f)(x) \in C_1 \quad \iff \quad g(f(x)) \in C_1 \\ &\iff \quad f(x) \in g^{-1}(C_1) \quad \iff \quad x \in f^{-1}(g^{-1}(C_1)). \end{aligned}$$

7. The derivative of $f(x) = 2x^2 - 3x$ is $f'(x) = 4x - 3$. This is negative when $x < 3/4$ and it is positive when $x > 3/4$, so the minimum value is $f(3/4) = -9/8$.

8. We need to find all numbers x such that $x^3 - 7x + 6 \leq 0$. If we now factor the left hand side, we get $(x-1)(x-2)(x+3) \leq 0$ and this implies $A = (-\infty, -3] \cup [1, 2]$.

9. For the first set, $\frac{n}{n+1} < \frac{2019}{2020}$ if and only if $n < 2019$. This gives $A = \{1, 2, \ldots, 2018\}$. For the second set, we have $1 < x \leq \frac{5}{2}$ and thus $B = (1, \frac{5}{2}]$. The third set is defined similarly, but it consists of integers, so $C = \{2\}$. Finally, the fourth set consists of all numbers x that are smaller than every positive number, so $D = (-\infty, 0]$.

10. Show that $x + \frac{1}{x} \geq 2$ for all $x > 0$ and that equality holds when $x = 1$. This shows that 2 is a lower bound of A and that $2 \in A$. Can there be a larger lower bound?

11. If you simplify the given definition, then you will find that $B = (-1, 4)$.

12. Let $\alpha = \sup A$ and $\beta = \sup B$ for convenience. We may assume that $\alpha \leq \beta$, as the case $\beta \leq \alpha$ is similar. Start by showing that $x \leq \beta$ for all $x \in A \cup B$. This makes β an upper bound of $A \cup B$. To show that it is the least, suppose that $y < \beta$ and try to find an element of $A \cup B$ which is bigger than y.

Analysis Problem Set #2

Problems 1-4 due by Feb. 7th*

1. Show that $A = \left\{\frac{4n+3}{2n-1} : n \in \mathbb{N}\right\}$ is bounded from below and that $\inf A = 2$.

2. Let $A \subseteq \mathbb{R}$ be nonempty and bounded from above. Fix some real number $x < 0$ and consider the set $B = \{ax : a \in A\}$. Show that $\inf B = x \sup A$.

3. Let $A \subseteq \mathbb{R}$ be nonempty, open and bounded from above. Show that $\sup A \notin A$.

4. Let $\{x_n\}$ be a sequence of real numbers such that $x_n \to x$ as $n \to \infty$ and consider the sequence $\{y_n\}$ defined by $y_n = \frac{1}{2}(3x_n + x_{n+1})$ for each $n \geq 1$. Use the definition of convergence to show that $y_n \to 2x$ as $n \to \infty$.

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5. Show that $A = \left\{\frac{m}{m+n} : m, n \in \mathbb{N}\right\}$ is bounded from above and that $\sup A = 1$.

6. Suppose that A, B are subsets of $\mathbb{R}$ such that $\inf A < \sup B$. Show that there exist an element $a \in A$ and an element $b \in B$ such that $\inf A \leq a < b \leq \sup B$.

7. Let $A \subseteq \mathbb{Z}$ be nonempty and bounded from below. Show that A has a minimum.

8. Show that each of the following sets is open in $\mathbb{R}$.

$$A = \left\{x \in \mathbb{R} : x^3 > 13x - 12\right\}, \qquad B = \left\{0 < x < 1 : \frac{1}{x} \notin \mathbb{N}\right\}.$$

9. Do there exist sets $A, B \subseteq \mathbb{R}$ such that $A, B, A - B$ are all nonempty and open?

10. Suppose that $A \subseteq \mathbb{R}$ is nonempty and bounded from above. Show that there exists a sequence of points $x_n \in A$ such that $x_n \to \sup A$ as $n \to \infty$.

11. Define a sequence $\{a_n\}$ by setting $a_1 = 1$ and $a_{n+1} = \sqrt{2a_n + 1}$ for each $n \geq 1$. Show that $a_n < a_{n+1} < 3$ for all $n \in \mathbb{N}$ and that the sequence $\{a_n\}$ converges.

12. Suppose that $A, B \subseteq \mathbb{R}$ are nonempty and bounded from above. Show that the set

$$C = \{x \in \mathbb{R} : x = a + b \text{ for some } a \in A \text{ and } b \in B\}$$

is also bounded from above and that $\sup C = \sup A + \sup B$.

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*You may submit your solutions Wednesday/Thursday in class or else Friday by 1pm in my office.

Analysis Problem Set #2
Answers and hints

1. It is easy to check that 2 is a lower bound of A. To show that it is the greatest lower bound, suppose $x > 2$ and try to find an element of A which is smaller than x. This amounts to solving the inequality $\frac{4n+3}{2n-1} < x$ in terms of n.

2. First, you need to check that $x \sup A$ is a lower bound of B. Then, you need to show that it is the greatest lower bound. Suppose $y > x \sup A$, in which case $y/x < \sup A$. Since y/x is smaller than $\sup A$, there exists an element $a \in A$ such that $y/x < a$.

3. If $\sup A \in A$, then $(\sup A - \varepsilon, \sup A + \varepsilon) \subseteq A$ for some $\varepsilon > 0$. Why is this impossible?

4. Let $\varepsilon > 0$ be given. Then there exists some $N \in \mathbb{N}$ such that $x - \varepsilon < x_n < x + \varepsilon$ for all $n \geq N$. This also implies that $x - \varepsilon < x_{n+1} < x + \varepsilon$ for all $n \geq N$.

5. It is easy to check that 1 is an upper bound. To show that it is the least upper bound, suppose $x < 1$ and try to find an element of A which is bigger. It suffices to look for an element of the form $\frac{m}{m+1}$, so you need to ensure that $\frac{m}{m+1} > x$ for some $m \in \mathbb{N}$.

6. Since $\inf A$ is smaller than $\sup B$, it is not an upper bound of B, so there exists an element $b \in B$ such that $\inf A < b$. Since b is larger than $\inf A$, it is not a lower bound of A, so there exists an element $a \in A$ such that $a < b$.

7. Since A is bounded from below, $\inf A$ exists. Since $\inf A + 1$ is larger than $\inf A$, there exists an element $x \in A$ such that $\inf A \leq x < \inf A + 1$. If equality holds, then $\inf A = x \in A$ and we have $\inf A = \min A$. Otherwise, we have $\inf A < x$ and we can proceed as before to find another element $y \in A$ such that $\inf A \leq y < x < \inf A + 1$.

8. Try to express the given sets as unions of open intervals. For the first set, one checks that $x^3 - 13x + 12 = (x-1)(x-3)(x+4)$ which implies $A = (-4, 1) \cup (3, \infty)$. The second set is $(0, 1)$ with $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ removed. It is the union of the intervals $\left(\frac{1}{n+1}, \frac{1}{n}\right)$.

9. Yes. Is it possible that A, B and $A - B$ are all unions of open intervals?

10. Since $\sup A - \frac{1}{n} < \sup A$ for each $n \in \mathbb{N}$, there exists an element $x_n \in A$ such that $\sup A - \frac{1}{n} < x_n$ for each $n \in \mathbb{N}$. This gives $\sup A - \frac{1}{n} < x_n \leq \sup A$ for each $n \in \mathbb{N}$.

11. Use induction on n to show that $a_n < a_{n+1} < 3$ for all $n \in \mathbb{N}$.

12. It is easy to check that $\sup A + \sup B$ is an upper bound of C. To show that it is the least upper bound, suppose $x < \sup A + \sup B$. Then $x - \sup B < \sup A$, so there exists an element $a \in A$ such that $x - \sup B < a$. Rearrange this inequality in a way that will allow you to proceed with the argument.

Analysis Problem Set #3
Problems 1-4 due by Feb. 14th*

1. Let $A \subseteq \mathbb{R}$ be nonempty, closed and bounded from above. Show that $\max A$ exists.

2. Show that $(A \cap B)^\circ = A^\circ \cap B^\circ$ for any sets $A, B \subseteq \mathbb{R}$.

3. Let $A, B \subseteq \mathbb{R}$ be arbitrary. Show that $(A \cup B)^\circ$ and $A^\circ \cup B^\circ$ are not necessarily equal, but one of these sets is always contained in the other.

4. Show that the closure of the complement is the complement of the interior. In other words, show that $\overline{A^c} = (A^\circ)^c$ for any set $A \subseteq \mathbb{R}$.

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5. Suppose $A \subseteq \mathbb{R}$ is open in $\mathbb{R}$ and $B \subseteq \mathbb{R}$ is closed. Show that $A - B$ is open in $\mathbb{R}$.

6. Show that each of the following sets is closed in $\mathbb{R}$.

$$A = \{x \in \mathbb{R} : x^4 \leq 5x^2 - 4\}, \qquad B = \{x \in \mathbb{R} : x^3 \leq 3x - 2\}.$$

7. Find a sequence of nested intervals I_n such that their intersection $\bigcap_{n=1}^\infty I_n$ is empty.

8. Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$ for any sets $A, B \subseteq \mathbb{R}$.

9. Let $A, B \subseteq \mathbb{R}$ be arbitrary. Show that $\overline{A \cap B}$ and $\overline{A} \cap \overline{B}$ are not necessarily equal, but one of these sets is always contained in the other.

10. Show that a set $A \subseteq \mathbb{R}$ is closed in $\mathbb{R}$ if and only if A contains its limit points.

11. Suppose that $A \subseteq \mathbb{R}$ is nonempty and $x \in \mathbb{R}$ is a limit point of A. Show that every neighbourhood of x must contain infinitely many points of A.

12. Suppose that $A \subseteq \mathbb{R}$ is open in $\mathbb{R}$. Show that the set of limit points A' is equal to the closure $\overline{A}$. Is this statement true for an arbitrary subset of $\mathbb{R}$?

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Analysis Problem Set #3
Answers and hints

1. It suffices to show that $\sup A$ is an element of A, as this implies $\max A = \sup A$. If it is not an element of A, then it is an element of A^c. Since this set is open, we must then have $(\sup A - \varepsilon, \sup A + \varepsilon) \subseteq A^c$ for some $\varepsilon > 0$. Why is that a contradiction?

2. To prove one of the inclusions, note that $A^\circ \subseteq A$ and $B^\circ \subseteq B$. Then $A^\circ \cap B^\circ \subseteq A \cap B$, so $A^\circ \cap B^\circ$ is an open set that is contained in $A \cap B$. To prove the opposite inclusion, note that $A \cap B \subseteq A$ implies $(A \cap B)^\circ \subseteq A^\circ$ and $A \cap B \subseteq B$ implies $(A \cap B)^\circ \subseteq B^\circ$.

3. If we let $A = [0, 1]$ and $B = [1, 2]$, then $(A \cup B)^\circ = (0, 2)$ and $A^\circ \cup B^\circ = (0, 1) \cup (1, 2)$. On the other hand, one always has $A^\circ \cup B^\circ \subseteq A \cup B$ and thus $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$.

4. If $x \in \overline{A^c}$, then every neighbourhood of x intersects A^c. Can x have a neighbourhood that is contained in A? If $x \in (A^\circ)^c$, then $x \notin A^\circ$ and there is no neighbourhood of x that is contained in A. Conclude that every neighbourhood of x intersects A^c.

5. The set $A - B$ consists of all points $x \in A$ with $x \notin B$. That is, $A - B = A \cap B^c$.

6. Solve the given inequalities to find that $A = [-2, -1] \cup [1, 2]$ and $B = (-\infty, -2] \cup \{1\}$.

7. Two simple examples are provided by the intervals $I_n = \left(0, \frac{1}{n}\right]$ and $I_n = \left(0, \frac{1}{n}\right)$.

8. To prove one of the inclusions, recall that $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$. Then $A \cup B \subseteq \overline{A} \cup \overline{B}$ and this gives a closed set that contains $A \cup B$. To prove the opposite inclusion, note that $A \subseteq A \cup B$ implies $\overline{A} \subseteq \overline{A \cup B}$ and $B \subseteq A \cup B$ implies $\overline{B} \subseteq \overline{A \cup B}$.

9. If we let $A = (0, 1)$ and $B = (1, 2)$, then $\overline{A \cap B} = \varnothing$ and $\overline{A} \cap \overline{B} = \{1\}$. On the other hand, one always has $A \cap B \subseteq \overline{A} \cap \overline{B}$ and thus $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

10. Recall that $\overline{A} = A \cup A'$, while A is closed if and only if $\overline{A} = A$. If A contains its limit points, then $\overline{A} = A \cup A' = A$, so A is closed. If A is closed, then $A' \subseteq A \cup A' = \overline{A} = A$.

11. Suppose there is a neighbourhood U of x which intersects A at finitely many points other than x and let $B = \{x_1, x_2, \ldots, x_n\}$ consist of these points. Show that $U - B$ is a neighbourhood of x which does not intersect A at a point other than x.

12. Since $\overline{A} = A \cup A'$, it is always true that $A' \subseteq A \cup A' = \overline{A}$. To show that the opposite inclusion does not hold in general, note that $A = \{0\}$ satisfies $A' = \varnothing$ and $\overline{A} = A$. If it happens that A is open and $x \in A$, then $(x - \varepsilon, x + \varepsilon) \subseteq A$ for some $\varepsilon > 0$. You need to show that every neighbourhood of x intersects A at a point other than x.

Analysis Problem Set #4

Problems 1-4 due by Feb. 21st*

1. Show that the function $f\colon \mathbb{R} \to \mathbb{R}$ is not continuous at all points when

$$f(x) = \left\{ \begin{array}{ll} x^2 & \text{if } x \leq 1 \\ 2x & \text{if } x > 1 \end{array} \right\}.$$

2. Suppose that $B \subseteq \mathbb{R}$ is open in $\mathbb{R}$ and let $A \subseteq B \subseteq \mathbb{R}$. Show that A is open in B if and only if A is open in $\mathbb{R}$.

3. Let $A, B \subseteq \mathbb{R}$. Show that a function $f\colon A \to B$ is continuous at all points if and only if the inverse image $f^{-1}(K)$ is closed in A whenever K is closed in B.

4. Show that $f\colon [0,1] \to \mathbb{R}$ is uniformly continuous when $f(x) = x^3$ for all x.

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5. Show that $A = \{x \in \mathbb{R} : f(x) \neq 0\}$ is open in $\mathbb{R}$ whenever $f\colon \mathbb{R} \to \mathbb{R}$ is continuous.

6. Suppose that $f\colon [0,1] \to [0,1]$ is continuous. Show that $f(x) = x$ for some $x \in [0,1]$.

7. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and $|f(x)| \leq 3$ for all $x \in \mathbb{R}$. Show that there exists some real number x such that $f(x) = x$.

8. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and f has a root in every open interval (a, b). Show that f is the zero function, namely that $f(x) = 0$ for all $x \in \mathbb{R}$.

9. Show that every subset of A is open in A when $A \subseteq \mathbb{R}$ has finitely many elements.

10. Suppose $A \subseteq \mathbb{R}$ and $f\colon A \to \mathbb{R}$ is uniformly continuous with $|f(x)| \geq 2$ for all x. Show that $g\colon A \to \mathbb{R}$ is also uniformly continuous when $g(x) = 1/f(x)$ for all x.

11. Show that $f\colon (0,1) \to \mathbb{R}$ is not uniformly continuous when $f(x) = 1/x$ for all x.

12. Let $A, B \subseteq \mathbb{R}$ and let $i\colon B \to \mathbb{R}$ be the inclusion map which is defined by $i(x) = x$ for all $x \in B$. Show that a function $f\colon A \to B$ is continuous at all points if and only if the composition $i \circ f\colon A \to \mathbb{R}$ is continuous at all points.

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Analysis Problem Set #4
Answers and hints

1. It suffices to find an open set U whose inverse image $f^{-1}(U)$ is not open. Consider an open interval such as $U = (\frac{1}{4}, 2)$. Its inverse image is the union of two intervals.

2. To say that A is open in B is to say that $A = U \cap B$ for some set U which is open in $\mathbb{R}$. Use this fact along with the assumption that B itself is open in $\mathbb{R}$.

3. To say that K is closed in B is to say that $B - K$ is open in B. Use this fact along with the identity $f^{-1}(B-K) = f^{-1}(B) - f^{-1}(K) = A - f^{-1}(K)$.

4. Try to verify that $|f(x) - f(y)| \leq k|x-y|$ for some fixed constant $k > 0$.

5. Express A as the union of $A^+ = \{x \in \mathbb{R} : f(x) > 0\}$ and $A^- = \{x \in \mathbb{R} : f(x) < 0\}$.

6. The result is clear, if $f(0) = 0$ or $f(1) = 1$. Suppose $f(0) > 0$ and $f(1) < 1$. To show that $f(x) = x$ for some $x \in (0,1)$, one needs $g(x) = f(x) - x$ to have a root in $(0,1)$.

7. The function $g(x) = f(x) - x$ is the difference of two continuous functions and thus continuous. Show that $g(4)$ must be negative, while $g(-4)$ must be positive.

8. Let $x \in \mathbb{R}$ be arbitrary. By assumption, f has a root $x_n \in (x, x + \frac{1}{n})$ for each $n \in \mathbb{N}$. Note that $x_n \to x$ by the Squeeze Theorem and that $f(x_n) \to f(x)$ by continuity.

9. Every set that has finitely many elements is closed in $\mathbb{R}$. If we assume that $B \subseteq A$, then B is closed in $\mathbb{R}$, so $B \cap A = B$ is closed in A. Why is it also open in A?

10. Try to verify that $|g(x) - g(y)| \leq \frac{1}{4}|f(x) - f(y)|$ for all $x, y \in A$.

11. If the definition of uniform continuity holds when $\varepsilon = 1$, there exists $\delta > 0$ such that

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < 1$$

for all $0 < x, y < 1$. Consider the points $x = \frac{1}{n}$ and $y = \frac{1}{n+1}$ for any integer $n > \frac{2}{\delta}$.

12. Inclusions are always continuous. If f is continuous, then $i \circ f$ is the composition of continuous functions and thus continuous. Conversely, suppose $i \circ f$ is continuous. To show that f is continuous, one needs to check that $f^{-1}(U)$ is open in A for each set U which is open in B. Start by writing $U = V \cap B$ for some set V which is open in $\mathbb{R}$.

Analysis Problem Set #5

Problems 1-4 due by Feb. 28th*

1. Suppose that $\{x_n\}$ is a sequence of real numbers such that $x_n \leq \alpha$ for all $n \in \mathbb{N}$. If it happens that $\{x_n\}$ converges to some number x, show that $x \leq \alpha$ as well.

2. What can you say about a Cauchy sequence which consists entirely of integers?

3. Let $A, B \subseteq \mathbb{R}$ and suppose that $f\colon A \to B$ is uniformly continuous. Given a Cauchy sequence $\{x_n\}$ of elements of A, show that $\{f(x_n)\}$ is a Cauchy sequence as well.

4. Let $A \subseteq \mathbb{R}$. Show that A is a dense subset of $\mathbb{R}$, if and only if every nonempty open subset of $\mathbb{R}$ intersects A at some point.

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5. Show that the sequence $\{x_n\}$ is Cauchy, and thus convergent, when

$$x_n = \frac{\sin 1}{2} + \frac{\sin 2}{4} + \ldots + \frac{\sin n}{2^n} \quad \text{for each } n \geq 1.$$

6. Which of the following subsets of $\mathbb{R}$ are complete? Explain.

$$A = [0,1), \qquad B = \mathbb{Z}, \qquad C = \mathbb{Q}, \qquad D = \{x \in \mathbb{R} : x^2 \geq \sin x\}.$$

7. Show that the Bolzano-Weierstrass theorem implies the nested interval property.

8. Let $0 < \alpha < 1$. Show that a sequence $\{x_n\}$ of real numbers is Cauchy, if it satisfies

$$|x_{n+1} - x_n| \leq \alpha \cdot |x_n - x_{n-1}| \quad \text{for each } n \geq 2.$$

9. Suppose that $\{x_n\}$ is an increasing sequence of real numbers which has a convergent subsequence. Show that the whole sequence $\{x_n\}$ converges as well.

10. Show that there exists an irrational number between any two real numbers.

11. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is both continuous and surjective. Given a set $A \subseteq \mathbb{R}$ which is a dense subset of $\mathbb{R}$, show that its image $f(A)$ is also a dense subset of $\mathbb{R}$.

12. Suppose that $f, g\colon \mathbb{R} \to \mathbb{R}$ are continuous and let $A \subseteq \mathbb{R}$ be a dense subset of $\mathbb{R}$ such that $f(x) = g(x)$ for all $x \in A$. Show that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

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Analysis Problem Set #5
Answers and hints

1. Suppose that $x > \alpha$ and use the definition of convergence with $\varepsilon = x - \alpha$.

2. There must exist a natural number N such that $x_n = x_N$ for all $n \geq N$. To show this, use the definition of a Cauchy sequence with $\varepsilon = 1$.

3. Let $\varepsilon > 0$ be given. Since f is uniformly continuous, there exists $\delta > 0$ such that

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon$$

for all $x, y \in A$. Combine this fact with the definition of a Cauchy sequence.

4. To say that A is dense in $\mathbb{R}$ is to say that $\overline{A} = \mathbb{R}$. Use Theorem 4.12 in the notes.

5. One needs to show that $|x_m - x_n|$ becomes arbitrarily small for large enough m, n. Assume that $m > n$ without loss of generality and show that $|x_m - x_n| < 1/2^n$.

6. A subset of $\mathbb{R}$ is complete if and only if it is closed in $\mathbb{R}$. In this case, A is not closed because it fails to contain a limit point and C is not closed because $\overline{C} = \mathbb{R} \neq C$. On the other hand, it is easy to check that B, D are closed in $\mathbb{R}$ and thus complete.

7. Consider a nested sequence of closed intervals $I_n = [a_n, b_n]$. Since $\{a_n\}$ is a bounded sequence, it contains a convergent subsequence. Show that its limit is in I_n for all n.

8. One needs to show that $|x_m - x_n|$ becomes arbitrarily small for large enough m, n. Assume that $m > n$ without loss of generality and argue that

$$|x_m - x_n| \leq \sum_{k=n}^{m-1} |x_{k+1} - x_k| \leq \sum_{k=n}^{m-1} \alpha^{k-1} \cdot |x_2 - x_1| \leq \sum_{k=n}^{\infty} \alpha^{k-1} \cdot |x_2 - x_1|.$$

9. The subsequence is convergent and thus bounded. Let M be an upper bound for the subsequence and show that M is actually an upper bound for the whole sequence.

10. Consider two real numbers $x < y$ and pick a rational number z such that $x < z < y$. Then $w_n = z + \frac{1}{n}\sqrt{2}$ is irrational for any $n \in \mathbb{N}$, while $w_n < y$ for large enough n.

11. Use the result of Problem 4. If U is a nonempty open subset of $\mathbb{R}$, then U contains an element y and $y = f(x)$ for some $x \in \mathbb{R}$ by surjectivity. Then $f^{-1}(U)$ is a nonempty open subset of $\mathbb{R}$ by continuity. Use this fact to conclude that U intersects $f(A)$.

12. The difference $h(x) = f(x) - g(x)$ is continuous and it satisfies $h(x) = 0$ for all $x \in A$. Suppose $h(x_0) > 0$ for some $x_0 \in \mathbb{R}$. Then $U = (0, 2h(x_0))$ is open and $h^{-1}(U)$ is a neighbourhood of x_0 that does not intersect A. The case $h(x_0) < 0$ is similar.

Analysis Problem Set #6
Practice problems

1. Show that a set $A \subseteq \mathbb{R}$ is connected if and only if there is no function $f\colon A \to \{0,1\}$ which is both continuous and surjective.

2. Suppose that $A \subseteq \mathbb{R}$ is connected and $f\colon A \to \mathbb{R}$ is continuous with $f(x) \neq 1$ for all $x \in A$. Show that either $f(x) > 1$ for all $x \in A$ or else $f(x) < 1$ for all $x \in A$.

3. Show that the union of two countable sets is countable.

4. Show that the set A consisting of all subsets of $\mathbb{N}$ is uncountable.

5. Is the set $A = \{x \in \mathbb{R} : x^4 - 12x^2 + 16x \leq 0\}$ complete? Is it connected?

6. Suppose that the sets $A, B \subseteq \mathbb{R}$ are nonempty, disjoint and open in $\mathbb{R}$. If there is a connected set U such that $U \subseteq A \cup B$, show that either $U \subseteq A$ or else $U \subseteq B$.

7. Consider two functions $f\colon A \to B$ and $g\colon B \to C$. If f, g are both surjective, then show that $g \circ f$ is surjective. If $g \circ f$ is surjective, then show that g is surjective.

8. Find a bijective function $f\colon (0,1] \to (0,1] \cup (2,3]$. Is such a function continuous?

9. Find a bijective function $f\colon A \to A - \{x_0\}$ when A is an infinite set and $x_0 \in A$.

10. Show that every subset of a countable set is countable.

11. Suppose A is a countable set. Show that there is no surjective map $f\colon A \to (0,1)$.

12. A set $A \subseteq \mathbb{R}$ is called path connected if, given any two points $x, y \in A$, there exists a continuous function $f\colon [0,1] \to A$ such that $f(0) = x$ and $f(1) = y$. Show that every path connected subset of $\mathbb{R}$ is connected.

Analysis Problem Set #6

Answers and hints

1. Show that $\{0\}, \{1\}$ are both open in $\{0,1\}$. If such a function exists, then the inverse images of these sets are open in A and their union is equal to A.

2. Note that A can be expressed as the union of the sets

$$A_1 = \{x \in A : f(x) < 1\}, \qquad A_2 = \{x \in A : f(x) > 1\}.$$

These sets are disjoint, they are both open in A and their union is equal to A.

3. Suppose A, B are countable. Then there exist surjections $f\colon \mathbb{N} \to A$ and $g\colon \mathbb{N} \to B$. To obtain a surjection $h\colon \mathbb{N} \to A \cup B$, one may associate the even integers with the elements of A and the odd integers with the elements of B.

4. Use Cantor's diagonal argument. Suppose A is countable and $A_1, A_2, A_3, \ldots$ are the only subsets of $\mathbb{N}$. Construct another subset by changing one element in each A_n.

5. Start by showing that $A = [-4, 0] \cup \{2\}$. This set is complete but not connected.

6. Consider the sets $U \cap A$ and $U \cap B$. These are disjoint and open in U, while their union is equal to U. Since U is connected, either $U \cap A$ or $U \cap B$ must be empty.

7. Let $c \in C$ be given. If the functions f, g are surjective, then there exists $b \in B$ such that $g(b) = c$ and there also exists $a \in A$ such that $f(a) = b$. If $g \circ f$ is surjective, on the other hand, then there exists $a \in A$ such that $g(f(a)) = c$.

8. Since $(0, 1]$ is connected and its image is not, the function f is not continuous. Look for a piecewise linear function which maps $(0, 1/2]$ to $(0, 1]$ and $(1/2, 1]$ to $(2, 3]$.

9. Since the set A is infinite, it contains a sequence $\{x_n\}$ of distinct elements. Define f so that $x_0, x_1, x_2, \ldots$ map to $x_1, x_2, x_3, \ldots$ and all other points are fixed.

10. Suppose A is countable and $B \subseteq A$. Then there exists an injective map $g\colon A \to \mathbb{N}$ and this gives a bijective map $g\colon B \to g(B)$. Use the fact that $g(B)$ is a subset of $\mathbb{N}$.

11. Since A is countable, there is a surjective map $g\colon \mathbb{N} \to A$. Were $f\colon A \to (0, 1)$ also surjective, $f \circ g\colon \mathbb{N} \to (0, 1)$ would be surjective and $(0, 1)$ would be countable.

12. Suppose that $A = A_1 \cup A_2$ for some nonempty disjoint sets A_1, A_2 which are open in A. Let $x \in A_1$ and $y \in A_2$. Then there exists a continuous function $f\colon [0, 1] \to A$ such that $f(0) = x$ and $f(1) = y$. Consider the inverse images $f^{-1}(A_1)$ and $f^{-1}(A_2)$.

Analysis Problem Set #7
Practice problems

1. Show that the set A consisting of all functions $f\colon \{0,1\} \to \mathbb{N}$ is countable.

2. Show that the set B consisting of all functions $f\colon \mathbb{N} \to \{0,1\}$ is uncountable.

3. Show that the union of two compact subsets of $\mathbb{R}$ is compact.

4. Are the following subsets of $\mathbb{R}$ compact? Why or why not?

$$A = \{x \in \mathbb{R} : x^4 - 2x^2 - 8 \leq 0\}, \qquad B = \{x \in \mathbb{R} : x + \sin x \geq 0\}.$$

5. Let $\{x_n\}$ be a sequence of real numbers such that x_n converges to x as $n \to \infty$ and consider the set $A = \{x, x_1, x_2, x_3, \ldots\}$. Show that A is a compact subset of $\mathbb{R}$.

6. Show that none of the following sets are compact.

$$A = (0,\infty), \qquad B = (1,3), \qquad C = \{x \in \mathbb{R} : x^2 \geq x\}, \qquad D = \{1/n : n \in \mathbb{N}\}\,.$$

7. Show that there exists no continuous surjective function $f\colon [0,1] \to A$ when

$$A = [0,1] \cup [2,3], \qquad A = [0,1), \qquad A = \mathbb{Q} \cap [0,1], \qquad A = (0,\infty).$$

8. Find a bijective function $f\colon [0,1] \to [0,1)$. Is such a function continuous?

9. Show that every open subset of $\mathbb{R}$ can be written as the union of open intervals (r,s) whose endpoints r,s are rational numbers with $r < s$.

10. Show that a set $A \subseteq \mathbb{Z}$ is compact if and only if it is finite.

11. Suppose that $A \subseteq \mathbb{R}$ is nonempty and compact. Show that $\max A$ exists.

12. Suppose that $A \subseteq \mathbb{R}$ is compact and $f\colon A \to A$ is continuous with

$$|f(x) - f(y)| < |x - y| \quad \text{for all } x \neq y.$$

Show that there exists a point $x_0 \in A$ such that $f(x_0) = x_0$.

Analysis Problem Set #7

Answers and hints

1. Such a function is uniquely determined by its values $f(0)$ and $f(1)$. One may thus associate each element of A with a pair $(f(0), f(1))$ of natural numbers. Use this fact to obtain a bijection $g\colon A \to \mathbb{N} \times \mathbb{N}$ and then recall that $\mathbb{N} \times \mathbb{N}$ is countable.

2. Use Cantor's diagonal argument. Suppose that B is countable and $f_1, f_2, \ldots$ are the only functions $f\colon \mathbb{N} \to \{0,1\}$. Construct another function which is not in this list.

3. Suppose that A, B are compact subsets of $\mathbb{R}$. If some sets U_i form an open cover of their union $A \cup B$, then the sets U_i must cover both A and B.

4. Since $x^4 - 2x^2 - 8 = (x^2 - 4)(x^2 + 2)$, the first set is $A = [-2, 2]$ and this is compact. On the other hand, the second set is unbounded, so it is not compact.

5. Suppose that the sets U_i form an open cover of A. Then one of these sets, say U_{i_0}, must contain x. It follows by Theorem 3.9(c) that U_{i_0} contains $x_N, x_{N+1}, \ldots$ for some natural number N. Thus, only finitely many terms are not contained in U_{i_0}.

6. The sets A, C are not compact because they are not bounded. To show that B is not compact, consider the function $f\colon B \to \mathbb{R}$ defined by $f(x) = 1/(x-1)$. Since this is continuous, but not bounded, B is not compact. A similar argument applies for D.

7. Suppose that such a function exists. Since $[0, 1]$ is both connected and compact, its image A must be both connected and compact. Examine the four possibilities.

8. Since $[0, 1]$ is compact and its image $[0, 1)$ is not, the function f is not continuous. To find a specific bijection $f\colon [0, 1] \to [0, 1)$, one may consider the function

$$f(x) = \left\{ \begin{array}{cc} 1/(n+1) & \text{if } x = 1/n \text{ for some } n \in \mathbb{N} \\ x & \text{otherwise} \end{array} \right\}.$$

This function is meant to map $1, \frac{1}{2}, \frac{1}{3}, \ldots$ to $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ keeping all other points fixed.

9. We used a similar argument in the proof of Theorem 3.4(b). Given any point $x \in A$, there exists some $\varepsilon_x > 0$ such that $(x - \varepsilon_x, x + \varepsilon_x) \subseteq A$. To replace the endpoints by rational numbers, pick some rational numbers $x - \varepsilon_x < r_x < x$ and $x < s_x < x + \varepsilon_x$.

10. Finite sets are certainly compact. Suppose now that $A \subseteq \mathbb{Z}$ is compact. Then A is bounded, so $A \subseteq [-N, N]$ for some $N \in \mathbb{N}$ and this gives $A \subseteq \{0, \pm 1, \ldots, \pm N\}$.

11. The inclusion map $i\colon A \to \mathbb{R}$ is continuous and it attains a maximum value.

12. Consider the function $g\colon A \to \mathbb{R}$ defined by $g(x) = |f(x) - x|$. Since g is continuous and A is compact, g attains a minimum value $g(x_0)$. If $g(x_0) = 0$, then $f(x_0) = x_0$ and the result follows. Otherwise, $g(x_0) > 0$ and $f(x_0) \neq x_0$. Use this fact and the given inequality to obtain a contradiction.

Analysis Problem Set #8

Practice problems

1. Suppose $A \subseteq \mathbb{R}$ is bounded. Show that its closure $\overline{A}$ is compact.

2. Suppose $A \subseteq \mathbb{R}$ is compact and $B \subseteq A$ is closed in A. Show that B is compact.

3. Are the following subsets of $\mathbb{R}$ compact? Why or why not?

$$A = \{x \in \mathbb{R} : \sin x + \cos x \leq 1\}, \qquad B = \{x \in \mathbb{R} : x^2 + \sin x \leq 1\}.$$

4. Suppose $A \subseteq \mathbb{R}$ is nonempty and $f\colon A \to \mathbb{R}$ is continuous. If the set A is bounded, must $f(A)$ be bounded? If the set A is closed, must $f(A)$ be closed?

5. Suppose $A \subseteq \mathbb{R}$ is compact. Show that every infinite subset of A has a limit point.

6. What can you say about a set $A \subseteq \mathbb{R}$, if every subset of A is compact?

7. Show that the function $f\colon [0, a] \to \mathbb{R}$ is integrable for any $a > 0$ when $f(x) = x^2$.

8. Show that the function $f\colon [0, 1] \to \mathbb{R}$ is integrable for any $a, b \in \mathbb{R}$ when

$$f(x) = \left\{ \begin{array}{ll} a & \text{if } x \neq 0 \\ b & \text{if } x = 0 \end{array} \right\}.$$

9. Suppose that $f\colon [0, 1] \to \mathbb{R}$ is integrable and let $a > 0$. If the function $g\colon [0, a] \to \mathbb{R}$ is defined by $g(x) = f(x/a)$, show that g is integrable and $\int_0^a g(x)\,dx = a \int_0^1 f(x)\,dx$.

10. Suppose $f\colon [0, 1] \to [0, \infty)$ is integrable with $f(x) = 0$ for all $x \in \mathbb{Q}$. Show that

$$\int_0^1 f(x)\,dx = 0.$$

11. Let $a < b$. Find a function $f\colon [a, b] \to \mathbb{R}$ such that f^2 is integrable, but f is not.

12. Let $a < b$ and suppose $f\colon [a, b] \to \mathbb{R}$ is increasing. Show that f is integrable on $[a, b]$.

Analysis Problem Set #8

Answers and hints

1. Use the Heine-Borel theorem. First, $\overline{A}$ is closed by definition. Since A is bounded by assumption, one has $A \subseteq [-N, N]$ for some $N > 0$. This implies that $\overline{A} \subseteq [-N, N]$.

2. Use the Heine-Borel theorem. Since $B \subseteq A$ and A is bounded, B is bounded as well. Since B is closed in A, one has $B = A \cap C$ for some set C which is closed in $\mathbb{R}$.

3. Since A contains $n\pi$ for each integer $n \in \mathbb{N}$, it is neither bounded nor compact. To show that B is compact, one must check that B is bounded and closed in $\mathbb{R}$.

4. For the first part, let $A = (0, 1)$ and $f(x) = 1/x$. The second part is a bit tricky. If you consider a set A that is both bounded and closed, then A is compact, so $f(A)$ is compact and $f(A)$ is closed. However, this is not true for unbounded closed sets such as $A = [1, \infty)$. If we let $f(x) = 1/x$ as before, then $f(A) = (0, 1]$ is not closed.

5. Suppose that $B \subseteq A$ is infinite and B has no limit points. Then every element $x \in A$ has a neighbourhood U_x which does not intersect B at a point other than x. Use the fact that the neighbourhoods U_x form an open cover of A to conclude that B is finite.

6. Finite sets have this property because finite sets are compact and their subsets are finite. Suppose that $A \subseteq \mathbb{R}$ is infinite and compact. Then A contains a sequence $\{x_n\}$ of distinct points. Since this sequence is bounded, it has a convergent subsequence. Denote the limit by x and consider the set $A - \{x\}$. Can this set be compact?

7. Consider a partition $P = \{x_0, x_1, \ldots, x_n\}$ of equally spaced points. Then $x_k = ak/n$ for each k and one may check that $U(f, P) - L(f, P) \leq 2a^3/n$.

8. Consider the partition $P_n = \{0, \frac{1}{n}, 1\}$ for any integer $n \geq 2$. Try to show that
$$U(f, P_n) - L(f, P_n) = \frac{\max\{a, b\}}{n} - \frac{\min\{a, b\}}{n}.$$

9. One may easily relate a partition P of $[0, a]$ to a partition Q of $[0, 1]$. Use this fact to find a relation between the corresponding lower and upper Darboux sums.

10. Consider any partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[0, 1]$. Since $f(x)$ is non-negative and it is equal to zero at all rational numbers, one has $L(f, P) = 0$ and thus $\mathcal{L}(f) = 0$.

11. Define $f(x) = 1$ for all $x \in \mathbb{Q}$ and $f(x) = -1$ for all $x \notin \mathbb{Q}$. Example 10.6 is similar.

12. Use the Riemann integrability condition. If the partition $P = \{x_0, x_1, \ldots, x_n\}$ consists of equally spaced points, then $m_k = f(x_k)$ and $M_k = f(x_{k+1})$ for each k, so
$$U(f, P) - L(f, P) = \sum_{k=0}^{n-1} (M_k - m_k)(x_{k+1} - x_k) = (f(b) - f(a)) \cdot \frac{b - a}{n}.$$

Analysis Problem Set #9

Practice problems

1. Suppose that $f\colon [a,b] \to \mathbb{R}$ is integrable and let I be a real number such that

$$L(f,P) \leq I \leq U(f,P)$$

for all partitions P of $[a,b]$. Show that I must be equal to $I = \int_a^b f(x)\,dx$.

2. Suppose f is integrable on the interval $[a,b]$ and let $a < c < b$. Show that f is also integrable on the subintervals $[a,c]$ and $[c,b]$.

3. Suppose $f\colon [a,b] \to \mathbb{R}$ is a bounded function which is zero at all points except for one point. Show that f is integrable on $[a,b]$ and $\int_a^b f(x)\,dx = 0$.

4. Suppose $f\colon [a,b] \to \mathbb{R}$ is bounded and $g\colon [a,b] \to \mathbb{R}$ is integrable. If $f(x) = g(x)$ at all points except for finitely many points, show that f is integrable on $[a,b]$.

5. Find two bounded functions $f, g\colon [a,b] \to \mathbb{R}$ such that $f(x) = g(x)$ at all points except for countably many points and g is integrable on $[a,b]$, while f is not.

6. Show that the function $f\colon [0,2\pi] \to \mathbb{R}$ is integrable when

$$f(x) = \left\{ \begin{array}{ll} \sin x & \text{if } 0 \leq x \leq \pi \\ \cos x & \text{if } \pi < x \leq 2\pi \end{array} \right\}.$$

7. Suppose $f\colon [a,b] \to \mathbb{R}$ is continuous. Show that there exists some $c \in [a,b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

8. Suppose $f\colon [a,b] \to \mathbb{R}$ is continuous and $g\colon [a,b] \to \mathbb{R}$ is a non-negative, integrable function. Show that there exists some $c \in [a,b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(c) \int_a^b g(x)\,dx.$$

9. Given a continuous function $f\colon [a,b] \to \mathbb{R}$, show that $\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$.

10. Suppose $f\colon [a,b] \to \mathbb{R}$ is continuous and let $F(z) = \int_a^z f(x)\,dx$ for each $z \in [a,b]$. Show that the function F is continuous as well.

11. Suppose $f\colon [a,b] \to \mathbb{R}$ is bounded and integrable. Show that f^2 is integrable.

12. Suppose $f, g\colon [a,b] \to \mathbb{R}$ are bounded and integrable. Show that fg is integrable.

Analysis Problem Set #9
Answers and hints

1. Note that I is an upper bound for the lower Darboux sums, while the integral is the least upper bound. This gives $I \geq \int_a^b f(x)\,dx$ and one similarly has $I \leq \int_a^b f(x)\,dx$.

2. Let $\varepsilon > 0$ be given. Since f is integrable on $[a, b]$, there exists a partition P such that $U(f, P) - L(f, P) < \varepsilon$. Note that $Q = P \cup \{c\}$ also satisfies $U(f, Q) - L(f, Q) < \varepsilon$ and that Q can be decomposed into a partition Q_1 of $[a, c]$ and a partition Q_2 of $[c, b]$.

3. Assume $f(c) > 0$ and $f(x) = 0$ for all $x \neq c$. If the partition $P = \{x_0, x_1, \ldots, x_n\}$ consists of equally spaced points, then $U(f, P) - L(f, P) = f(c)(b - a)/n$.

4. Use the previous problem. If f, g only differ at one point, then $f - g$ is integrable by the previous problem, so $f = (f - g) + g$ is integrable as well.

5. Consider the function f defined by $f(x) = 1$ for all $x \in \mathbb{Q}$ and $f(x) = 0$ for all $x \notin \mathbb{Q}$. This is not integrable on $[a, b]$, but $g(x) = 0$ is constant and thus integrable on $[a, b]$.

6. Use Theorem 11.1 to show that f is integrable on $[0, \pi]$ and use Problem 4 to show that f is integrable on $[\pi, 2\pi]$. This implies integrability on $[0, 2\pi]$.

7. Since f is continuous, it attains a minimum value m and a maximum value M. Show that $m \leq \frac{1}{b-a} \int_a^b f(x)\,dx \leq M$ and then use the intermediate value theorem.

8. Since f is continuous, it attains a minimum value m and a maximum value M. Show that $m \int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq M \int_a^b g(x)\,dx$ and consider two cases.

9. The left hand side is $\pm \int_a^b f(x)\,dx$ and this is equal to $\int_a^b (\pm f(x))\,dx$. The right hand side is $\int_a^b |f(x)|\,dx$. Use Theorem 11.5 to compare these expressions.

10. Let $\varepsilon > 0$ be given. To prove continuity, one needs to find some $\delta > 0$ such that

$$|y - z| < \delta \quad \Longrightarrow \quad |F(y) - F(z)| < \varepsilon.$$

When $y \geq z$, one has $|F(y) - F(z)| = \left|\int_z^y f(x)\,dx\right|$ and the previous problem becomes relevant; the case $y \leq z$ is similar. Use the fact that $|f(x)| \leq M$ for some $M > 0$.

11. Suppose that $|f(x)| \leq M$ for some $M > 0$. Given any partition P, try to show that

$$U(f^2, P) - L(f^2, P) \leq 2M \cdot [U(f, P) - L(f, P)].$$

Since the right hand side is arbitrarily small, the same is true for the left hand side.

12. This follows easily from the previous problem because $4fg = (f + g)^2 - (f - g)^2$.