# 1 Basic set theory

- We use capital letters to denote sets and lowercase letters to denote their elements.
- We write $A \subseteq B$ whenever every element of $A$ is also an element of $B$.
- The union $A \cup B$ of two sets consists of the elements $x$ with $x \in A$ or $x \in B$.
- The intersection $A \cap B$ of two sets consists of the elements $x$ with $x \in A$ and $x \in B$.
- The difference $A - B$ of two sets consists of the elements $x$ with $x \in A$, but $x \notin B$.

**Theorem 1.1 – De Morgan's laws**

The difference of a union/intersection is the intersection/union of the differences, namely

$$A - (B \cup C) = (A - B) \cap (A - C), \qquad A - (B \cap C) = (A - B) \cup (A - C).$$

**Proof.** To prove the statement about the difference of a union, one argues that

$$\begin{aligned} x \in A - (B \cup C) &\iff x \in A, \text{ but } x \notin B \cup C \\ &\iff x \in A, \text{ but } x \notin B \text{ and } x \notin C \\ &\iff x \in A - B \text{ and } x \in A - C \\ &\iff x \in (A - B) \cap (A - C). \end{aligned}$$

Since the difference of an intersection can be treated similarly, we omit the details. ■

**Definition 1.2 – Image of a set**

Given a function $f\colon A \to B$ and a set $A_1 \subseteq A$, we define $f(A_1) = \{f(x) : x \in A_1\}$.

**Theorem 1.3 – Properties of images**

Let $f\colon A \to B$ be a function and let $A_1, A_2 \subseteq A$ be arbitrary.

(a) If $A_1 \subseteq A_2$, then $f(A_1) \subseteq f(A_2)$.

(b) One has $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

(c) One has $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$ and equality holds when $f$ is injective.

(d) One has $f(A_1 - A_2) \supseteq f(A_1) - f(A_2)$ and equality holds when $f$ is injective.

- Thus, images preserve inclusions and unions, but not intersections and differences.

**Proof.** To prove the first part, suppose that $A_1 \subseteq A_2$. We then have

$$\begin{aligned} y \in f(A_1) &\implies y = f(x) \text{ for some } x \in A_1 \\ &\implies y = f(x) \text{ for some } x \in A_2 \\ &\implies y \in f(A_2). \end{aligned}$$

This implies that $f(A_1) \subseteq f(A_2)$, as needed. To prove the second part, we note that

$$\begin{aligned} y \in f(A_1 \cup A_2) &\iff y = f(x) \text{ for some } x \in A_1 \cup A_2 \\ &\iff y = f(x) \text{ for some } x \in A_1 \text{ or some } x \in A_2 \\ &\iff y \in f(A_1) \text{ or } y \in f(A_2) \\ &\iff y \in f(A_1) \cup f(A_2). \end{aligned}$$

Next, we turn to the third part. To prove the inclusion, one argues that

$$\begin{aligned} y \in f(A_1 \cap A_2) &\implies y = f(x) \text{ for some } x \in A_1 \cap A_2 \\ &\implies y = f(x) \text{ with } x \in A_1 \text{ and } x \in A_2 \\ &\implies y \in f(A_1) \text{ and } y \in f(A_2) \\ &\implies y \in f(A_1) \cap f(A_2). \end{aligned}$$

This shows that $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$, as needed. If it happens that $f$ is injective, then we can also establish the opposite inclusion. In that case, one has

$$\begin{aligned} y \in f(A_1) \cap f(A_2) &\implies y = f(x_1) \text{ for some } x_1 \in A_1 \text{ and } y = f(x_2) \text{ for some } x_2 \in A_2 \\ &\implies y = f(x_1) = f(x_2) \text{ with } x_1 \in A_1 \text{ and } x_2 \in A_2 \\ &\implies y = f(x_1) = f(x_2) \text{ with } x_1 = x_2 \in A_1 \cap A_2 \text{ (by injectivity)} \\ &\implies y \in f(A_1 \cap A_2). \end{aligned}$$

This completes the proof of the third part. The proof of the last part is quite similar. ■

**Example 1.4** Consider the case $f(x) = x^2$. If we take $A_1 = [-1, 0]$ and $A_2 = [0, 1]$, then

$$A_1 \cap A_2 = \{0\}, \qquad f(A_1 \cap A_2) = \{0\}, \qquad f(A_1) = [0, 1] = f(A_2).$$

In particular, $f(A_1 \cap A_2) = \{0\}$ is a proper subset of $f(A_1) \cap f(A_2) = [0, 1]$. Similarly,

$$A_1 - A_2 = [-1, 0), \qquad f(A_1 - A_2) = (0, 1], \qquad f(A_1) - f(A_2) = \varnothing$$

and so $f(A_1) - f(A_2)$ could be a proper subset of $f(A_1 - A_2)$ when $f$ is not injective. □

**Definition 1.5 – Inverse image of a set**

Given a function $f\colon A \to B$ and a set $B_1 \subseteq B$, we define its inverse image by

$$f^{-1}(B_1) = \{x \in A : f(x) \in B_1\}.$$

This set is defined for any function $f$. In particular, $f$ does not need to be bijective.

**Example 1.6** Consider the case $f(x) = x^2$. The inverse image of $B_1 = [-2, -1]$ is then

$$f^{-1}(B_1) = \{x \in \mathbb{R} : x^2 \in B_1\} = \{x \in \mathbb{R} : -2 \leq x^2 \leq -1\} = \varnothing.$$

On the other hand, the inverse image of $B_2 = [1, 4]$ can be computed as

$$f^{-1}(B_2) = \{x \in \mathbb{R} : x^2 \in B_2\} = \{x \in \mathbb{R} : 1 \leq x^2 \leq 4\} = [1, 2] \cup [-2, -1].$$ □

**Theorem 1.7 – Properties of inverse images**

Let $f\colon A \to B$ be a function and let $B_1, B_2 \subseteq B$ be arbitrary.

(a) If $B_1 \subseteq B_2$, then $f^{-1}(B_1) \subseteq f^{-1}(B_2)$.

(b) One has $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.

(c) One has $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

(d) One has $f^{-1}(B_1 - B_2) = f^{-1}(B_1) - f^{-1}(B_2)$.

- Thus, inverse images preserve inclusions, unions, intersections and also differences.

**Proof.** To prove the first part, we assume that $B_1 \subseteq B_2$ and we note that

$$x \in f^{-1}(B_1) \quad \Longrightarrow \quad f(x) \in B_1 \quad \Longrightarrow \quad f(x) \in B_2 \quad \Longrightarrow \quad x \in f^{-1}(B_2).$$

This implies that $f^{-1}(B_1) \subseteq f^{-1}(B_2)$, as needed. For the second part, one has

$$\begin{aligned} x \in f^{-1}(B_1 \cup B_2) \quad &\Longleftrightarrow \quad f(x) \in B_1 \cup B_2 \\ &\Longleftrightarrow \quad f(x) \in B_1 \text{ or } f(x) \in B_2 \\ &\Longleftrightarrow \quad x \in f^{-1}(B_1) \text{ or } x \in f^{-1}(B_2) \\ &\Longleftrightarrow \quad x \in f^{-1}(B_1) \cup f^{-1}(B_2). \end{aligned}$$

This proves the statement in the second part, while the other two parts are similar. ■

**Theorem 1.8 – Images and inverse images**

Let $f\colon A \to B$ be a function. Let $A_1 \subseteq A$ and $B_1 \subseteq B$ be arbitrary.

(a) One has $f^{-1}(f(A_1)) \supseteq A_1$ and equality holds whenever $f$ is injective.

(b) One has $f(f^{-1}(B_1)) \subseteq B_1$ and equality holds whenever $f$ is surjective.

**Proof.** We only establish part (b), as part (a) is similar. First of all, we note that

$$\begin{aligned} y \in f(f^{-1}(B_1)) \quad &\Longrightarrow \quad y = f(x) \text{ for some } x \in f^{-1}(B_1) \\ &\Longrightarrow \quad y = f(x) \text{ and also } f(x) \in B_1 \\ &\Longrightarrow \quad y \in B_1. \end{aligned}$$

This proves the inclusion $f(f^{-1}(B_1)) \subseteq B_1$. If we also assume that $f$ is surjective, then

$$\begin{aligned} y \in B_1 \quad &\Longrightarrow \quad y = f(x) \text{ for some } x \in A \text{ (by surjectivity)} \\ &\Longrightarrow \quad y = f(x) \text{ for some } x \in A \text{ and } f(x) \in B_1 \\ &\Longrightarrow \quad y = f(x) \text{ and } x \in f^{-1}(B_1) \\ &\Longrightarrow \quad y \in f(f^{-1}(B_1)). \end{aligned}$$

Thus, the inclusion $B_1 \subseteq f(f^{-1}(B_1))$ also holds and the two sets are actually equal. ■

**Example 1.9** Consider the case $f(x) = x^2$. If we take $A_1 = [0, 1]$ and $B_1 = [-1, 1]$, then

$$f(A_1) = [0, 1] \quad \Longrightarrow \quad f^{-1}(f(A_1)) = \{x \in \mathbb{R} : 0 \leq x^2 \leq 1\} = [-1, 1].$$

In particular, $A_1$ is a proper subset of $f^{-1}(f(A_1))$ and one similarly has

$$f^{-1}(B_1) = \{x \in \mathbb{R} : -1 \leq x^2 \leq 1\} = [-1, 1] \quad \Longrightarrow \quad f(f^{-1}(B_1)) = [0, 1] \neq B_1. \qquad \square$$

# 2 Infimum and supremum

**Definition 2.1 – Minimum and maximum**

If a set $A \subseteq \mathbb{R}$ has a smallest element, then we call that element the minimum of $A$ and we denote it by $\min A$. If a set $A \subseteq \mathbb{R}$ has a largest element, then we call that element the maximum of $A$ and we denote it by $\max A$.

**Example 2.2** When it comes to the interval $A = [1, 2]$, one has $\min A = 1$ and $\max A = 2$. When it comes to the interval $B = [1, 2)$, however, $\min B = 1$ and $\max B$ does not exist. □

**Example 2.3** Consider the set $A = \{\frac{1}{n} : n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. To show that $\max A = 1$, one checks that 1 is an element of $A$ and that 1 is at least as large as any other element. In this case, it is clear that $1 \in A$, while $1 \geq x$ for all $x \in A$ because $1 \geq \frac{1}{n}$ for all $n \in \mathbb{N}$. □

**Example 2.4** Consider the set $A = \{\frac{1}{n} : n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ as before. To show that $A$ has no minimum, one checks that $A$ has no smallest element. Given any element of $A$, we must thus be able to find another element of $A$ which is smaller. Now, let $x \in A$ be given. Then $x = \frac{1}{n}$ for some $n \in \mathbb{N}$ and $y = \frac{1}{n+1}$ is an element of $A$ such that $y < x$. This shows that the original element $x$ was not the smallest, so $A$ does not have a minimum. □

**Definition 2.5 – Upper bounds and supremum**

We say that $A \subseteq \mathbb{R}$ is bounded from above, if there exists a number $x$ such that $x \geq a$ for all $a \in A$. In that case, we also say that $x$ is an upper bound of $A$. The least upper bound of $A$ is called the supremum of $A$ and it is denoted by $\sup A$.

- Both the maximum and the supremum of $A$ must be at least as large as all elements of $A$. However, $\max A$ must itself be an element of $A$, whereas $\sup A$ need not be.

**Axiom of completeness**

If $A \subseteq \mathbb{R}$ is nonempty and bounded from above, then $A$ has a least upper bound.

**Example 2.6** We show that the interval $A = (-\infty, 1)$ has no maximum. Indeed, let $x \in A$ be given and note that $x < 1$. The number $y = \frac{x+1}{2}$ is the average of $x$ and 1 which is easily seen to satisfy $x < y < 1$. This implies that $y$ is an element of $A$ which is larger than the original element $x$. Thus, $x$ was not the largest element and $A$ has no maximum. □

**Example 2.7** Consider the interval $A = (-\infty, 1)$ once again. Upper bounds of $A$ must be at least as large as every element of $A$, so the least upper bound should be $\sup A = 1$. To prove this, we check (a) that 1 is an upper bound of $A$ and (b) that 1 is the least upper bound. The first part is clear, as $1 \geq a$ for all $a \in A$. To establish the second part, we need to show that no number $x < 1$ is an upper bound of $A$. Given any $x < 1$, we must thus be able to find an element of $A$ which is bigger than $x$. If we let $y = \frac{x+1}{2}$ once again, then we have $x < y < 1$ and so $y$ is an element of $A$ which is bigger than $x$, as needed. □

**Definition 2.8 – Lower bounds and infimum**

We say that $A \subseteq \mathbb{R}$ is bounded from below, if there exists a number $x$ such that $x \leq a$ for all $a \in A$. In that case, we also say that $x$ is a lower bound of $A$. The greatest lower bound of $A$ is called the infimum of $A$ and it is denoted by $\inf A$.

- Both the minimum and the infimum of $A$ must be at least as small as all elements of $A$. However, $\min A$ must itself be an element of $A$, whereas $\inf A$ need not be.

**Example 2.9** It is easy to see that $A = (0, \infty)$ has no minimum. Given any element $x \in A$, one has $x > 0$ and then $y = \frac{x}{2}$ satisfies $0 < y < x$, so it is an element of $A$ which is smaller than $x$. To show that the infimum of $A$ is $\inf A = 0$, one needs to check (a) that 0 is a lower bound of $A$ and (b) that 0 is the greatest lower bound. The first part is clear, as $0 \leq a$ for all $a \in A$. To establish the second part, we need to show that no number $z > 0$ is a lower bound of $A$. Given any $z > 0$, we must thus be able to find an element of $A$ which is smaller than $z$. In fact, $y = \frac{z}{2}$ is such an element because $0 < y < z$, so $y \in A$ and also $y < z$. □

**Theorem 2.10 – Relation between inf/min and sup/max**

Suppose that $A$ is a nonempty subset of $\mathbb{R}$.

(a) If $\min A$ exists, then $\inf A$ also exists and the two are equal. If $\inf A$ exists and it happens to be an element of $A$, then $\min A$ exists and the two are equal.

(b) If $\max A$ exists, then $\sup A$ also exists and the two are equal. If $\sup A$ exists and it happens to be an element of $A$, then $\max A$ exists and the two are equal.

**Proof.** We only prove the first part, as the second part is similar. If $\min A$ exists, then $\min A \leq x$ for all $x \in A$ and so $\min A$ is a lower bound of $A$. To show that it is the greatest lower bound, suppose $y > \min A$. Then $\min A$ is an element of $A$ which is smaller than $y$, so $y$ is not a lower bound of $A$ and the greatest lower bound is $\min A$.

Similarly, suppose that $\inf A$ exists and that $\inf A \in A$. Then $\inf A \leq x$ for all $x \in A$ and $\inf A$ is itself an element of $A$, so $\inf A$ is the smallest element of $A$. ■

**Theorem 2.11 – Existence of infimum**

If $A \subseteq \mathbb{R}$ is nonempty and bounded from below, then $A$ has a greatest lower bound.

**Proof.** We consider the set $B = \{x \in \mathbb{R} : -x \in A\}$. This consists of the negatives of the elements of $A$, so any lower bound of $A$ should be an upper bound of $B$ and vice versa.

First of all, we show that $B$ is bounded from above. Since $A$ is bounded from below, there exists a real number $z$ such that $z \leq a$ for all $a \in A$. This implies that $-z \geq -a$ for all $a \in A$, so $-z \geq b$ for all $b \in B$. We conclude that $-z$ is an upper bound of $B$.

Since $B$ is bounded from above, $\sup B$ exists by the axiom of completeness. We now show that $-\sup B$ is the greatest lower bound of $A$. In fact, we have

$$\begin{aligned} \sup B \geq b \text{ for all } b \in B \quad &\Longrightarrow \quad -\sup B \leq -b \text{ for all } b \in B \\ &\Longrightarrow \quad -\sup B \leq a \text{ for all } a \in A \end{aligned}$$

and this means that $-\sup B$ is a lower bound of $A$. To show that it is the greatest one, suppose $z > -\sup B$ and note that $-z < \sup B$. Then $-z$ is not an upper bound of $B$, so there exists some $b \in B$ such that $-z < b$. This gives $z > -b$, so $-b$ is an element of $A$ which is smaller than $z$. In particular, $z$ is not a lower bound of $A$, as needed. ■

**Theorem 2.12 – Inf/Sup of a subset**

(a) Suppose that $A \subseteq \mathbb{R}$ is nonempty and bounded from below. If $B \subseteq A$, then $B$ is bounded from below as well and one has $\inf B \geq \inf A$.

(b) Suppose that $A \subseteq \mathbb{R}$ is nonempty and bounded from above. If $B \subseteq A$, then $B$ is bounded from above as well and one has $\sup B \leq \sup A$.

- Plainly stated, larger sets must have a larger supremum, but a smaller infimum.

**Proof.** We only prove the first part, as the second part is similar. Since $A$ has a lower bound by assumption, its infimum $\inf A$ exists and one has

$$\inf A \leq x \text{ for all } x \in A \quad \Longrightarrow \quad \inf A \leq x \text{ for all } x \in B.$$

Thus, $\inf A$ is a lower bound of $B$, so $B$ is bounded from below and $\inf B$ exists. As $\inf A$ is a lower bound of $B$ and $\inf B$ is the greatest lower bound of $B$, one has $\inf A \leq \inf B$. ■

**Theorem 2.13 – Archimedean property**

The set $\mathbb{N}$ of natural numbers is not bounded from above. Given any real number $x$, that is, there exists a natural number $n$ such that $n > x$.

**Proof.** To prove the first statement, suppose $\mathbb{N}$ is bounded from above and let $\alpha = \sup \mathbb{N}$ be its least upper bound. Since $\alpha - 1$ is smaller, it is not an upper bound of $\mathbb{N}$, so there exists some $x \in \mathbb{N}$ such that $\alpha - 1 < x$. This gives $x + 1 > \alpha$ which means that $x + 1$ is a natural number that is actually larger than $\alpha = \sup \mathbb{N}$, a contradiction.

To prove the second statement, suppose $n \leq x$ for all $n \in \mathbb{N}$. Then $x$ is an upper bound of $\mathbb{N}$ and this contradicts the first statement. Thus, there exists $n \in \mathbb{N}$ such that $n > x$. ■

**Example 2.14** Consider the set $A = \left\{ \frac{2n+1}{n+3} : n \in \mathbb{N} \right\}$. To show that $\sup A = 2$, we check that 2 is an upper bound and that it is the least upper bound. The first part is clear, as

$$2 \geq \frac{2n+1}{n+3} \quad \Longleftrightarrow \quad 2n + 6 \geq 2n + 1 \quad \Longleftrightarrow \quad 6 \geq 1.$$

To check the second part, suppose that $x < 2$. We need to find an element of $A$ which is larger than $x$ and this amounts to ensuring that $\frac{2n+1}{n+3} > x$. On the other hand, one has

$$\begin{aligned} \frac{2n+1}{n+3} > x \quad &\Longleftrightarrow \quad 2n + 1 > nx + 3x \\ &\Longleftrightarrow \quad (2 - x)n > 3x - 1 \quad \Longleftrightarrow \quad n > \frac{3x - 1}{2 - x}. \end{aligned}$$

Pick a natural number $n$ that satisfies the rightmost inequality. Then $\frac{2n+1}{n+3} > x$, so there is an element of $A$ which is larger than $x$. This shows that $x$ is not an upper bound of $A$. □

**Theorem 2.15 – Nonempty subsets of $\mathbb{N}$**

Every nonempty subset of $\mathbb{N}$ must have a minimum.

**Proof.** Suppose that $A \subseteq \mathbb{N}$ is nonempty. Since $x \geq 1$ for all $x \in A$, the set $A$ is then bounded from below and $\inf A$ exists. If we can show that $\inf A \in A$, then $\min A$ also exists and the two are equal. Thus, it suffices to show that $\inf A \in A$.

Since $\inf A + 1 > \inf A$, there exists an element $x \in A$ such that $\inf A \leq x < \inf A + 1$. If it happens that $\inf A = x$, then $\inf A \in A$ and the proof is complete. Otherwise, we must have $\inf A < x$ and we may proceed as before to find some element $y \in A$ such that

$$\inf A \leq y < x \quad \Longrightarrow \quad \inf A \leq y < x < \inf A + 1.$$

This is impossible because two integers $x, y$ cannot lie in an interval of length 1. ■

**Theorem 2.16 – Principle of mathematical induction**

Consider a statement $P(n)$ involving the natural numbers $n \in \mathbb{N}$. Suppose that $P(1)$ holds and that $P(n)$ implies $P(n+1)$ for each $n \in \mathbb{N}$. Then $P(n)$ holds for all $n \in \mathbb{N}$.

**Proof.** We study the set $A = \{n \in \mathbb{N} : P(n) \text{ does not hold}\}$. If we show that $A$ is empty, then $P(n)$ holds for all $n \in \mathbb{N}$ and the result follows. Suppose then that $A$ is nonempty. According to the previous theorem, it must have a least element $m = \min A$.

Since $P(1)$ holds by assumption, $1 \notin A$ and so $m > 1$. In particular, $m - 1$ is a natural number which is smaller than the least element of $A$, so $m - 1 \notin A$ and $P(m-1)$ holds. It follows by assumption that $P(m)$ also holds and this gives $m \notin A$, a contradiction. ■

# 3 Open sets and convergence

**Definition 3.1 – Open set**

We say that a set $A \subseteq \mathbb{R}$ is open in $\mathbb{R}$ if, given any point $x \in A$, there exists some $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq A$.

**Example 3.2** Consider the interval $A = [a, b)$ which contains its endpoint $x = a$. If $A$ was actually open in $\mathbb{R}$, then we would have $(a - \varepsilon, a + \varepsilon) \subseteq A$ for some $\varepsilon > 0$. This is not the case, however, because points such as $a + \frac{\varepsilon}{2}$ lie in $(a - \varepsilon, a + \varepsilon)$ but not in $A$. □

**Theorem 3.3 – Unions and intersections of open sets**

Every union of open sets is open and every finite intersection of open sets is open.

- Infinite intersections of open sets need not be open. For instance, $U_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ is open in $\mathbb{R}$ for each $n \in \mathbb{N}$, but one has $\bigcap_{n=1}^{\infty} U_n = \{0\}$ and this is not open in $\mathbb{R}$.

**Proof.** Let us worry about unions first. We assume that the sets $U_i$ are open in $\mathbb{R}$ and we look at their union $A = \bigcup_i U_i$. To show that $A$ is open in $\mathbb{R}$, let $x \in A$ be given. Since $x$ belongs to the union of the sets $U_i$, we have $x \in U_i$ for some $i$. We can thus find some $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq U_i$ and this implies that $(x - \varepsilon, x + \varepsilon) \subseteq \bigcup_i U_i = A$.

Next, we prove the statement for intersections. Assume that the sets $U_i$ are open in $\mathbb{R}$ and let $B = \bigcap_{i=1}^{n} U_i$. To show that $B$ is open in $\mathbb{R}$, let $x \in B$ be given. Then $x \in U_i$ for each $i$ and there exist $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n > 0$ such that $(x - \varepsilon_i, x + \varepsilon_i) \subseteq U_i$ for each $i$. If we now take $\varepsilon > 0$ to be the smallest of the numbers $\varepsilon_i$, then $\varepsilon \leq \varepsilon_i$ for each $i$ and so

$$(x - \varepsilon, x + \varepsilon) \subseteq (x - \varepsilon_i, x + \varepsilon_i) \subseteq U_i$$

for each $i$. It easily follows that $(x - \varepsilon, x + \varepsilon) \subseteq \bigcap_{i=1}^{n} U_i = B$, as needed. ■

**Theorem 3.4 – Examples of open sets**

(a) The intervals $(a,\infty)$, $(-\infty,b)$ and $(a,b)$ are open in $\mathbb{R}$ for all $a,b$.

(b) A set $A\subseteq\mathbb{R}$ is open in $\mathbb{R}$ if and only if it is a union of open intervals.

**Proof.** First, consider the interval $A=(a,\infty)$. Given a point $x\in A$, we have $x>a$ and we need to find some $\varepsilon>0$ such that $(x-\varepsilon,x+\varepsilon)\subseteq A$. Letting $\varepsilon=x-a$, we get

$$y\in(x-\varepsilon,x+\varepsilon) \implies y>x-\varepsilon=a \implies y\in A.$$

This shows that $A=(a,\infty)$ is open in $\mathbb{R}$. A similar argument shows that $B=(-\infty,b)$ is also open in $\mathbb{R}$, so their intersection $A\cap B=(a,b)$ is open in $\mathbb{R}$ as well.

Let us now turn to part (b). If a set is a union of open intervals, then it is a union of open sets, so it is open. Conversely, suppose $A\subseteq\mathbb{R}$ is open. Given any $x\in A$, we can find some $\varepsilon_x>0$ such that $(x-\varepsilon_x,x+\varepsilon_x)\subseteq A$. Since $A$ is the union of its elements, we get

$$A=\bigcup_{x\in A}\{x\}\subseteq\bigcup_{x\in A}(x-\varepsilon_x,x+\varepsilon_x)\subseteq A.$$

Thus, the above sets are all equal and $A$ itself is a union of open intervals. ■

**Example 3.5** Consider the set $A=\{x\in\mathbb{R}:x^3>x\}$. To show that $A$ is open in $\mathbb{R}$, we first find the values of $x$ such that $x^3>x$. Note that $x^3-x$ can be factored as

$$x^3-x=x(x^2-1)=x(x-1)(x+1).$$

When $x<-1$, all three factors are negative, so the product is negative. When $-1<x<0$, only two factors are negative, so the product is positive. Arguing in this manner, one finds that $A=(-1,0)\cup(1,\infty)$. Thus, $A$ is open in $\mathbb{R}$ by the previous theorem. □

**Definition 3.6 – Convergence of sequences**

A sequence $\{x_n\}$ of real numbers converges to $x$ as $n\to\infty$ if, given any $\varepsilon>0$, there exists a natural number $N$ such that $x_n\in(x-\varepsilon,x+\varepsilon)$ for all $n\geq N$. In that case, we call $x$ the limit of the sequence and we write $x_n\to x$ as $n\to\infty$.

**Theorem 3.7 – Monotone convergence theorem**

(a) If a sequence $\{x_n\}$ is increasing and bounded from above, then $\{x_n\}$ converges.

(b) If a sequence $\{x_n\}$ is decreasing and bounded from below, then $\{x_n\}$ converges.

**Proof.** We only prove the first part, as the second part is similar. Our goal is to show that the sequence converges to $\sup A$, where $A=\{x_1,x_2,\ldots\}$. Let $\varepsilon>0$ be given. As $\sup A-\varepsilon$ is smaller than the least upper bound of $A$, there exists $x_N\in A$ such that $x_N>\sup A-\varepsilon$. Since the sequence is increasing, this actually gives $x_n\geq x_N>\sup A-\varepsilon$ for all $n\geq N$. On the other hand, $\sup A$ is an upper bound of $A$, so $\sup A\geq x_n$ for all $n$. We thus have

$$\sup A-\varepsilon<x_N\leq x_n\leq\sup A<\sup A+\varepsilon$$

for all $n\geq N$. In other words, $x_n\in(\sup A-\varepsilon,\sup A+\varepsilon)$ for all $n\geq N$, as needed. ■

**Theorem 3.8 – Squeeze theorem**

If $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ and $x_n, z_n \to \alpha$ as $n \to \infty$, then $y_n \to \alpha$ as $n \to \infty$.

**Proof.** Let $\varepsilon > 0$ be given. Since $x_n \to \alpha$ as $n \to \infty$, there exists a natural number $N_1$ such that $x_n \in (\alpha - \varepsilon, \alpha + \varepsilon)$ for all $n \geq N_1$. Since $z_n \to \alpha$ as $n \to \infty$, there also exists a natural number $N_2$ such that $z_n \in (\alpha - \varepsilon, \alpha + \varepsilon)$ for all $n \geq N_2$. We must thus have

$$\alpha - \varepsilon < x_n, z_n < \alpha + \varepsilon$$

for all $n \geq \max\{N_1, N_2\}$. Since $x_n \leq y_n \leq z_n$ by assumption, this implies that

$$\alpha - \varepsilon < x_n \leq y_n \leq z_n < \alpha + \varepsilon$$

for all $n \geq \max\{N_1, N_2\}$. In other words, it implies that $y_n \to \alpha$ as $n \to \infty$. ■

**Theorem 3.9 – Convergence in terms of open intervals/sets**

The following statements are equivalent whenever $\{x_n\}$ is a sequence and $x \in \mathbb{R}$.

(a) One has $x_n \to x$ as $n \to \infty$.

(b) Given any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $x_n \in (x - \varepsilon, x + \varepsilon)$ for all $n \geq N$.

(c) Given any open $U$ with $x \in U$, there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.

**Proof.** The first two parts are equivalent by definition.

To show that (b) implies (c), suppose $U$ is open and $x \in U$. Then $(x - \varepsilon, x + \varepsilon) \subseteq U$ for some $\varepsilon > 0$ and one may use part (b) to find some $N \in \mathbb{N}$ such that $x_n \in (x - \varepsilon, x + \varepsilon)$ for all $n \geq N$. This implies that $x_n \in (x - \varepsilon, x + \varepsilon) \subseteq U$ for all $n \geq N$, so part (c) follows.

To prove that (c) implies (b), let $\varepsilon > 0$ be given and take $U = (x - \varepsilon, x + \varepsilon)$. Then $U$ is an open set that contains $x$, so one may use part (c) to find some $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. This gives $x_n \in (x - \varepsilon, x + \varepsilon)$ for all $n \geq N$, so part (b) follows. ■

# 4 Closure and interior

**Definition 4.1 – Closed set**

We say that a set $A \subseteq \mathbb{R}$ is closed in $\mathbb{R}$, if its complement $A^c = \mathbb{R} - A$ is open in $\mathbb{R}$.

- **Remark.** When it comes to sets, being closed is not the opposite of being open.

**Example 4.2** The interval $A = (-\infty, b)$ is open in $\mathbb{R}$, so its complement $A^c = [b, \infty)$ is closed in $\mathbb{R}$. Similarly, $B = (a, \infty)$ is open in $\mathbb{R}$ and so $B^c = (-\infty, a]$ is closed in $\mathbb{R}$. □

**Theorem 4.3 – Unions and intersections of closed sets**

Every intersection of closed sets is closed and every finite union of closed sets is closed.

**Proof.** We only prove the statement about intersections, as the statement about unions is similar. Suppose that the sets $U_i$ are closed in $\mathbb{R}$ and let $A = \bigcap_i U_i$. To show that $A$ is also closed, we need to show that its complement is open. Using De Morgan's laws, we get

$$A^c = \mathbb{R} - A = \mathbb{R} - \bigcap_i U_i = \bigcup_i (\mathbb{R} - U_i) = \bigcup_i U_i^c.$$

Since each $U_i$ is closed by assumption, each complement $U_i^c$ is open. This makes $A^c$ a union of open sets, so $A^c$ is itself open and its complement $A$ is closed. ■

**Example 4.4** If a set $A = \{x\}$ consists of a single element, then $A$ is closed in $\mathbb{R}$ because its complement $A^c = (-\infty, x) \cup (x, \infty)$ is a union of open intervals and thus open. □

**Example 4.5** If a set $A \subseteq \mathbb{R}$ consists of finitely many elements, then $A$ can be expressed as the finite union of its elements $\{x\}$, so it easily follows that $A$ is closed. □

**Theorem 4.6 – Closed sets and convergence**

Suppose that $A \subseteq \mathbb{R}$ is closed in $\mathbb{R}$ and let $\{x_n\}$ be a sequence of elements of $A$ which converges to the point $x$ as $n \to \infty$. Then the limit $x$ must also be an element of $A$.

**Proof.** Suppose that $x \notin A$. Then $x \in A^c$, while $A^c$ is open because $A$ is closed. Thus, there exists some $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq A^c$. Since $x_n \to x$ as $n \to \infty$, there also exists some natural number $N$ such that $x_n \in (x - \varepsilon, x + \varepsilon)$ for all $n \geq N$. This actually gives $x_n \in A^c$ for all $n \geq N$ and thus $x_n \notin A$ for all $n \geq N$, a contradiction. ■

**Example 4.7** Consider the set $A = (0, 1]$ which is not closed in $\mathbb{R}$. Letting $x_n = 1/n$, one obtains a sequence of elements of $A$ whose limit $x = 0$ is not an element of $A$. □

**Theorem 4.8 – Nested interval property**

Consider a sequence of closed intervals $I_n = [a_n, b_n]$ such that $I_{n+1} \subseteq I_n$ for all $n \in \mathbb{N}$. Then the intersection of the intervals $\bigcap_{n=1}^{\infty} I_n$ must be nonempty.

**Proof.** First of all, we study the behaviour of the left and right endpoints

$$a_n = \min I_n = \inf I_n, \qquad b_n = \max I_n = \sup I_n.$$

Since $I_{n+1} \subseteq I_n$ for each $n \in \mathbb{N}$, one easily finds that

$$I_{n+1} \subseteq I_n \implies \inf I_{n+1} \geq \inf I_n \implies a_{n+1} \geq a_n$$

for each $n \in \mathbb{N}$. This means that the sequence $\{a_n\}$ is increasing, while a similar argument shows that the sequence $\{b_n\}$ is decreasing. Now, consider the set $A = \{a_1, a_2, \ldots\}$. Since

$$I_n \subseteq I_1 \quad \Longrightarrow \quad [a_n, b_n] \subseteq [a_1, b_1] \quad \Longrightarrow \quad a_n \leq b_1$$

for each $n \in \mathbb{N}$, the set $A$ is bounded from above and $\sup A$ exists. Moreover, one has

$$a_m \leq a_{m+n} \leq b_{m+n} \leq b_n$$

for all $m, n \in \mathbb{N}$ since $\{a_n\}$ is increasing and $\{b_n\}$ is decreasing. Thus, each $b_n$ is an upper bound of $A$ and we must have $\sup A \leq b_n$ for all $n \in \mathbb{N}$. Since $a_n \leq \sup A$ by definition, this actually gives $a_n \leq \sup A \leq b_n$ for all $n \in \mathbb{N}$ and thus $\sup A \in I_n$ for all $n \in \mathbb{N}$. ■

**Definition 4.9 – Interior and closure**

(a) The interior of a set $A \subseteq \mathbb{R}$ is the union of all open sets that are contained in $A$. It is the largest open set that is contained in $A$ and it is usually denoted by $A^\circ$.

(b) The closure of a set $A \subseteq \mathbb{R}$ is the intersection of all closed sets that contain $A$. It is the smallest closed set that contains $A$ and it is usually denoted by $\overline{A}$.

- The following table lists the interiors and closures of some typical sets.

| **Set** | **Interior** | **Closure** |
|---|---|---|
| $\{0, 1\}$ | $\varnothing$ | $\{0, 1\}$ |
| $[0, 1)$ | $(0, 1)$ | $[0, 1]$ |
| $[0, 1] \cup \{2\}$ | $(0, 1)$ | $[0, 1] \cup \{2\}$ |
| $(0, 1) \cup [2, \infty)$ | $(0, 1) \cup (2, \infty)$ | $[0, 1] \cup [2, \infty)$ |

**Theorem 4.10 – Properties of interior and closure**

(I1) One has $A^\circ \subseteq A$ for each $A \subseteq \mathbb{R}$.

(I2) If $A \subseteq B$, then $A^\circ \subseteq B^\circ$.

(I3) $A \subseteq \mathbb{R}$ is open if and only if $A^\circ = A$.

(I4) One has $(A^\circ)^\circ = A^\circ$ for each $A \subseteq \mathbb{R}$.

(C1) One has $A \subseteq \overline{A}$ for each $A \subseteq \mathbb{R}$.

(C2) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.

(C3) $A \subseteq \mathbb{R}$ is closed if and only if $\overline{A} = A$.

(C4) One has $\overline{\overline{A}} = \overline{A}$ for each $A \subseteq \mathbb{R}$.

**Proof.** We only prove the statements about the interior, as the ones about the closure are similar. First of all, $A^\circ$ is contained in $A$ by definition, so (I1) is clear. To prove (I2), we note that $A \subseteq B$ implies $A^\circ \subseteq A \subseteq B$. Since $A^\circ$ is an open set which is contained in $B$, while $B^\circ$ is the largest open set which is contained in $B$, one has $A^\circ \subseteq B^\circ$. To prove (I3), we recall that $A^\circ$ is open by definition. If $A^\circ = A$, then $A$ is certainly open. If $A$ is open, then $A$ is the largest open set which is contained in $A$, so $A^\circ = A$. Finally, (I4) is a direct consequence of (I3) because $A^\circ$ is open, so $A^\circ$ must be equal to its own interior. ■

**Definition 4.11 – Neighbourhood**

A neighbourhood of the point $x \in \mathbb{R}$ is an open set $U \subseteq \mathbb{R}$ that contains $x$.

**Theorem 4.12 – Interior/Closure in terms of open sets**

(a) One has $x \in A^\circ$ if and only if some neighbourhood of $x$ is contained in $A$.

(b) One has $x \in \overline{A}$ if and only if every neighbourhood of $x$ intersects $A$.

**Proof.** Let us first focus on part (a). If $x \in A^\circ$, then $x$ lies in the open set $A^\circ$ and there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq A^\circ$. Since $A^\circ \subseteq A$, this gives $(x - \varepsilon, x + \varepsilon) \subseteq A$ and so $x$ has a neighbourhood that is contained in $A$. Conversely, suppose that $x$ has a neighbourhood $U \subseteq A$. Since $U$ is an open set that is contained in $A$ and $A^\circ$ is the largest open set that is contained in $A$, one has $U \subseteq A^\circ$ and so $x \in U \subseteq A^\circ$.

Next, we turn to part (b). To establish this part, we prove the equivalent statement

$$x \notin \overline{A} \quad \Longleftrightarrow \quad \text{there exists a neighbourhood of } x \text{ that does not intersect } A.$$

If $x \notin \overline{A}$, then $x$ is not in the intersection of all closed sets that contain $A$, so there exists a closed set $K \supseteq A$ such that $x \notin K$. The complement $U = K^c$ is then open, it contains $x$ and it does not intersect $A$, as every element of $A$ is in $K$. Thus, $x$ has a neighbourhood $U$ that does not intersect $A$. Conversely, suppose $U$ is a neighbourhood of $x$ such that $U \cap A = \varnothing$. Then $x$ is not in $K = U^c$ and this is a closed set which contains $A$. We conclude that $x$ is not in the intersection of all closed sets that contain $A$ and thus $x \notin \overline{A}$. ∎

**Definition 4.13 – Limit point**

We say that a real number $x$ is a limit point of the set $A \subseteq \mathbb{R}$, if every neighbourhood of $x$ intersects $A$ at a point other than $x$.

**Example 4.14** It is easy to see that both $A = [0, 1]$ and $A = (0, 1)$ have $x = 0$ as a limit point. Thus, a limit point of $A$ might be an element of $A$, but it does not have to be. □

**Theorem 4.15 – Limit points and sequences**

Let $A \subseteq \mathbb{R}$ and suppose $x \in \mathbb{R}$ is a limit point of $A$. Then there exists a sequence $\{x_n\}$ of elements of $A$ such that $x_n \to x$ as $n \to \infty$.

**Proof.** The interval $(x - \frac{1}{n}, x + \frac{1}{n})$ is a neighbourhood of $x$ for each $n \in \mathbb{N}$. It must thus intersect $A$ at some point $x_n \neq x$. This gives a sequence $\{x_n\}$ of elements of $A$ such that

$$x - \frac{1}{n} < x_n < x + \frac{1}{n} \quad \text{for all } n \in \mathbb{N}.$$

Since $x \pm \frac{1}{n} \to x$ as $n \to \infty$, it follows by the Squeeze Theorem that $x_n \to x$ as $n \to \infty$. ∎

**Theorem 4.16 – Limit points and closure**

Given any set $A \subseteq \mathbb{R}$, one has $\overline{A} = A \cup A'$, where $A'$ consists of the limit points of $A$.

**Proof.** First, we show that $A \cup A' \subseteq \overline{A}$. If $x \in A'$, every neighbourhood of $x$ intersects $A$ at a point other than $x$, so every neighbourhood of $x$ intersects $A$ and $x \in \overline{A}$. This proves the inclusion $A' \subseteq \overline{A}$ and the inclusion $A \subseteq \overline{A}$ holds by definition, so $A \cup A' \subseteq \overline{A}$.

Next, we show that $\overline{A} \subseteq A \cup A'$. If $x \in \overline{A}$, all neighbourhoods of $x$ intersect $A$. If they all intersect $A$ at a point other than $x$, then $x \in A'$. Otherwise, there is a neighbourhood that intersects $A$ only at $x$, and this means that $x \in A$. Thus, $x \in A \cup A'$ in any case. ■

# 5 Continuity

**Definition 5.1 – Continuity at a point**

Let $A, B \subseteq \mathbb{R}$. We say that a function $f\colon A \to B$ is continuous at the point $x \in A$ if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$y \in (x-\delta, x+\delta) \cap A \quad \Longrightarrow \quad f(y) \in (f(x)-\varepsilon, f(x)+\varepsilon).$$

**Theorem 5.2 – Composition of continuous functions**

Let $A, B, C \subseteq \mathbb{R}$. Suppose that $f\colon A \to B$ is continuous at $x \in A$ and that $g\colon B \to C$ is continuous at $f(x)$. Then the composition $g \circ f\colon A \to C$ is continuous at $x$.

**Proof.** Let $\varepsilon > 0$ be given. Since $g$ is continuous at $f(x)$, there exists $\delta_1 > 0$ such that

$$y \in (f(x)-\delta_1, f(x)+\delta_1) \cap B \quad \Longrightarrow \quad g(y) \in (g(f(x))-\varepsilon, g(f(x))+\varepsilon).$$

Since $f$ is continuous at $x$, there similarly exists $\delta_2 > 0$ such that

$$z \in (x-\delta_2, x+\delta_2) \cap A \quad \Longrightarrow \quad f(z) \in (f(x)-\delta_1, f(x)+\delta_1).$$

Once we now combine the last two equations, we arrive at

$$\begin{aligned} z \in (x-\delta_2, x+\delta_2) \cap A \quad &\Longrightarrow \quad f(z) \in (f(x)-\delta_1, f(x)+\delta_1) \cap B \\ &\Longrightarrow \quad g(f(z)) \in (g(f(x))-\varepsilon, g(f(x))+\varepsilon). \end{aligned}$$

This verifies the definition of continuity for the composition $g \circ f$ at the point $x$. ■

**Theorem 5.3 – Continuity and sequences**

Let $A, B \subseteq \mathbb{R}$ and suppose $f\colon A \to B$ is continuous at the point $x \in A$. If $\{x_n\}$ is a sequence of elements of $A$ such that $x_n \to x$ as $n \to \infty$, then $f(x_n) \to f(x)$ as $n \to \infty$.

**Proof.** Let $\varepsilon > 0$ be given. Since $f$ is continuous at $x$, there exists $\delta > 0$ such that

$$y \in (x-\delta, x+\delta) \cap A \quad \Longrightarrow \quad f(y) \in (f(x) - \varepsilon, f(x) + \varepsilon).$$

Since $x_n \to x$ as $n \to \infty$, there also exists a natural number $N$ such that

$$x_n \in (x-\delta, x+\delta) \quad \text{for all } n \geq N.$$

Once we now combine the last two equations, we find that

$$x_n \in (x-\delta, x+\delta) \cap A \quad \Longrightarrow \quad f(x_n) \in (f(x) - \varepsilon, f(x) + \varepsilon)$$

for all $n \geq N$. Thus, the definition of convergence holds and $f(x_n) \to f(x)$ as $n \to \infty$. ■

**Definition 5.4 – Relatively open/closed**

Let $A \subseteq B \subseteq \mathbb{R}$. We say that $A$ is open in $B$ if, given any $x \in A$, there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \cap B \subseteq A$. We say that $A$ is closed in $B$, if $B - A$ is open in $B$.

**Theorem 5.5 – Relatively open/closed**

Let $A \subseteq B \subseteq \mathbb{R}$. Then $A$ is open/closed in $B$ if and only if $A$ has the form $A = U \cap B$ for some set $U \subseteq \mathbb{R}$ which is open/closed in $\mathbb{R}$, respectively.

**Proof.** First, suppose that $A = U \cap B$ and $U$ is open in $\mathbb{R}$. To show that $A$ is open in $B$, let $x \in A$ be given. Since $x \in U$ and $U$ is open in $\mathbb{R}$, there exists $\varepsilon > 0$ such that

$$(x-\varepsilon, x+\varepsilon) \subseteq U \quad \Longrightarrow \quad (x-\varepsilon, x+\varepsilon) \cap B \subseteq U \cap B = A.$$

This means that $A$ is open in $B$. Conversely, suppose $A$ is open in $B$. Given any $x \in A$, we can then find some $\varepsilon_x > 0$ such that $(x-\varepsilon_x, x+\varepsilon_x) \cap B \subseteq A$. Consider the set

$$U = \bigcup_{x \in A} (x-\varepsilon_x, x+\varepsilon_x).$$

This is a union of open intervals and thus open in $\mathbb{R}$. Moreover, one can easily check that

$$A = \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} (x-\varepsilon_x, x+\varepsilon_x) \cap B \subseteq A,$$

so these sets are all equal and $A = \bigcup_{x \in A} (x-\varepsilon_x, x+\varepsilon_x) \cap B = U \cap B$, as needed.

It remains to prove the statement about closed sets. If $A = K \cap B$ and $K$ is closed in $\mathbb{R}$, then one may use De Morgan's laws to find that

$$B - A = B - (K \cap B) = (B - K) \cup (B - B) = B - K = B \cap K^c.$$

Since $K^c$ is open in $\mathbb{R}$, this implies that $B - A = B \cap K^c$ is open in $B$, so $A$ is closed in $B$. Conversely, if $A$ is closed in $B$, then $B - A$ is open in $B$ and $B - A = U \cap B$ for some set $U$ which is open in $\mathbb{R}$. Since $A \subseteq B$ by assumption, we conclude that

$$A = B - (B - A) = B - (U \cap B) = B - U = B \cap U^c.$$

In other words, $A$ has the form $A = U^c \cap B$ for some set $U^c$ which is closed in $\mathbb{R}$. ■

**Example 5.6** Since $(0,2)$ is open in $\mathbb{R}$, the intersection $(0,2) \cap [0,1]$ is open in $[0,1]$ and this means that $(0,1]$ is open in $[0,1]$. Similarly, the fact that $[0,2]$ is closed in $\mathbb{R}$ implies that the intersection $[0,2] \cap (1,3)$ is closed in $(1,3)$ and thus $(1,2]$ is closed in $(1,3)$. □

**Theorem 5.7 – Continuity in terms of open sets**

Let $A, B \subseteq \mathbb{R}$. To say that a function $f\colon A \to B$ is continuous at all points is to say that the inverse image $f^{-1}(U)$ is open in $A$ whenever $U$ is open in $B$.

**Proof.** First, suppose $f$ is continuous at all points and let $U$ be a set which is open in $B$. To show that the inverse image $f^{-1}(U)$ is open in $A$, we let $x \in f^{-1}(U)$. Then $f(x) \in U$ and the set $U$ is open in $B$, so there exists some $\varepsilon > 0$ such that

$$(f(x) - \varepsilon, f(x) + \varepsilon) \cap B \subseteq U.$$

Since $f$ is continuous at the point $x$, there also exists $\delta > 0$ such that

$$y \in (x - \delta, x + \delta) \cap A \quad \Longrightarrow \quad f(y) \in (f(x) - \varepsilon, f(x) + \varepsilon).$$

Once we now combine the last two equations, we may conclude that

$$y \in (x - \delta, x + \delta) \cap A \quad \Longrightarrow \quad f(y) \in U \quad \Longrightarrow \quad y \in f^{-1}(U).$$

This proves the inclusion $(x - \delta, x + \delta) \cap A \subseteq f^{-1}(U)$, so the set $f^{-1}(U)$ is open in $A$.

Conversely, suppose $f^{-1}(U)$ is open in $A$ whenever $U$ is open in $B$. To show that $f$ is continuous at any point $x \in A$, let $\varepsilon > 0$ be given. Then $U = (f(x) - \varepsilon, f(x) + \varepsilon)$ is open in $\mathbb{R}$, so the intersection $U \cap B$ is open in $B$ and $f^{-1}(U \cap B)$ is open in $A$. Note that this set contains $x$ because $f(x) \in U \cap B$. Since the inverse image is open in $A$, we must have

$$(x - \delta, x + \delta) \cap A \subseteq f^{-1}(U \cap B)$$

for some $\delta > 0$. Given any point $y \in (x - \delta, x + \delta) \cap A$, we must thus have

$$y \in f^{-1}(U \cap B) \quad \Longrightarrow \quad f(y) \in U \cap B \quad \Longrightarrow \quad f(y) \in (f(x) - \varepsilon, f(x) + \varepsilon).$$

This verifies the definition of continuity at the point $x$ and also completes the proof. ■

**Example 5.8** Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ which is defined by

$$f(x) = \left\{ \begin{array}{cl} 2x & \text{if } x < 0 \\ x + 1 & \text{if } x \geq 0 \end{array} \right\}.$$

To show that $f$ is not continuous, we let $U = (0,2)$ and we compute the inverse image

$$f^{-1}(U) = \{x \in \mathbb{R} : f(x) \in U\} = \{x \in \mathbb{R} : 0 < f(x) < 2\}.$$

In view of the piecewise definition of the function $f$, this set can be expressed as

$$f^{-1}(U) = \{x < 0 : 0 < 2x < 2\} \cup \{x \geq 0 : 0 < x + 1 < 2\} = [0,1).$$

Noting that $U$ is open in $\mathbb{R}$, while $f^{-1}(U)$ is not, we conclude that $f$ is not continuous. □

**Example 5.9** Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is any continuous function and consider the set

$$A = \{x \in \mathbb{R} : f(x) > 0\}.$$

To show that this set is necessarily open in $\mathbb{R}$, we note that $A$ can be expressed as

$$A = \{x \in \mathbb{R} : f(x) \in (0, \infty)\} = \{x \in \mathbb{R} : f(x) \in U\} = f^{-1}(U),$$

where $U = (0, \infty)$. Since $U$ is open and $f$ is continuous, $A = f^{-1}(U)$ is open as well. □

**Theorem 5.10 – Inclusions are always continuous**

Let $A \subseteq \mathbb{R}$ and consider the inclusion map $i\colon A \to \mathbb{R}$ which is defined by $i(x) = x$ for all $x \in A$. This function $i$ is then continuous at all points.

**Proof.** We assume that $U$ is open in $\mathbb{R}$ and we compute the inverse image

$$i^{-1}(U) = \{x \in A : i(x) \in U\} = \{x \in A : x \in U\} = U \cap A.$$

Since $U$ is open in $\mathbb{R}$, the intersection $i^{-1}(U) = U \cap A$ is open in $A$, as needed. ■

**Theorem 5.11 – Restrictions of continuous functions**

Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function and $A \subseteq \mathbb{R}$. If the function $g\colon A \to \mathbb{R}$ is defined by $g(x) = f(x)$ for all $x \in A$, then $g$ is continuous as well.

**Proof.** We assume that $U$ is open in $\mathbb{R}$ and we compute the inverse image

$$\begin{aligned} g^{-1}(U) = \{x \in A : g(x) \in U\} &= \{x \in A : f(x) \in U\} \\ &= \{x \in A : x \in f^{-1}(U)\} = f^{-1}(U) \cap A. \end{aligned}$$

Since $U$ is open in $\mathbb{R}$ by assumption, $f^{-1}(U)$ is open in $\mathbb{R}$ by continuity. This means that the intersection $f^{-1}(U) \cap A$ is open in $A$, so $g^{-1}(U)$ is open in $A$, as needed. ■

**Theorem 5.12 – Bolzano's theorem**

Suppose that $f\colon [a, b] \to \mathbb{R}$ is continuous at all points and the values $f(a), f(b)$ have opposite sign. Then there exists a point $x \in (a, b)$ such that $f(x) = 0$.

**Proof.** Consider the case $f(a) < 0 < f(b)$, as the other case is similar. Letting

$$A = \{a \leq x \leq b : f(x) < 0\},$$

we see that $a \in A$ and $b$ is an upper bound of $A$. Thus, $\sup A$ exists and $a \leq \sup A \leq b$. Our goal is to show that $\sup A$ is actually a root of $f$, namely that $f(\sup A) = 0$.

Step 1. Suppose $f(\sup A) < 0$. Then $a \leq \sup A < b$ and $\sup A$ is an element of

$$A = \{a \leq x \leq b : f(x) < 0\} = \{a \leq x \leq b : f(x) \in U\} = f^{-1}(U),$$

where $U = (-\infty, 0)$. Since $U$ is open in $\mathbb{R}$, we see that $A = f^{-1}(U)$ is open in $[a, b]$ by continuity. Since $\sup A$ is an element of $A$, we can thus find some $\varepsilon > 0$ such that

$$(\sup A - \varepsilon, \sup A + \varepsilon) \cap [a, b] \subseteq A.$$

Let $\delta = \min\{\frac{\varepsilon}{2}, b - \sup A\}$ for convenience. Then $\sup A < \sup A + \delta \leq b$ and one has

$$\sup A + \delta \in (\sup A, \sup A + \varepsilon) \cap [a, b] \subseteq A.$$

This makes $\sup A + \delta$ an element of $A$ which is larger than $\sup A$, a contradiction.

Step 2. Suppose $f(\sup A) > 0$. Then $a < \sup A \leq b$ and $\sup A$ is an element of

$$B = \{a \leq x \leq b : f(x) > 0\} = \{a \leq x \leq b : f(x) \in V\} = f^{-1}(V),$$

where $V = (0, \infty)$. Arguing as before, we find that $B$ is open in $[a, b]$ and that

$$(\sup A - \varepsilon, \sup A + \varepsilon) \cap [a, b] \subseteq B$$

for some $\varepsilon > 0$. Since $\sup A - \varepsilon$ is smaller than $\sup A$, there exists some $x \in A$ such that

$$\sup A - \varepsilon < x \leq \sup A.$$

This implies that $x \in B$ as well, contrary to the fact that $A \cap B = \varnothing$ by definition. ■

**Definition 5.13 – Uniform continuity**

Let $A, B \subseteq \mathbb{R}$. We say that a function $f \colon A \to B$ is uniformly continuous on $A$ if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon \quad \text{for all } x, y \in A.$$

- The definition of uniform continuity is similar to the definition of continuity, but it is more stringent. When it comes to uniform continuity, one needs a single $\delta > 0$ that satisfies the definition at all points. When it comes to continuity at the point $x$, the point $x$ is fixed in advance and the choice of $\delta > 0$ may generally depend on $x$.

**Theorem 5.14 – Uniform continuity implies continuity**

If a function $f$ is uniformly continuous, then $f$ is continuous at all points.

**Proof.** Let $A, B \subseteq \mathbb{R}$ and suppose $f\colon A \to B$ is uniformly continuous. To show that $f$ is continuous at each point $x \in A$, let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon \quad \text{for all } x, y \in A.$$

Since $x \in A$ by assumption, one may rewrite the last equation in the form

$$y \in (x - \delta, x + \delta) \cap A \quad \Longrightarrow \quad f(y) \in (f(x) - \varepsilon, f(x) + \varepsilon).$$

This verifies the definition of continuity at the point $x$, so the proof is complete. ■

**Example 5.15** Consider $f(x) = x^2$ as a function $f\colon [0, a] \to \mathbb{R}$, where $a > 0$ is fixed. To show that $f$ is uniformly continuous on the given interval, we note that

$$|f(x) - f(y)| = |x^2 - y^2| = |x + y| \cdot |x - y|.$$

Let $\varepsilon > 0$ be given. As long as $x, y \in [0, a]$, we then have $|x + y| = x + y \leq 2a$ and

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| = |x + y| \cdot |x - y| \leq 2a|x - y| < 2a\delta.$$

If we now take $\delta = \frac{\varepsilon}{2a}$, we find that $|f(x) - f(y)| < \varepsilon$, so $f$ is uniformly continuous. □

**Example 5.16** Consider $f(x) = x^2$ as a function $f\colon \mathbb{R} \to \mathbb{R}$. In this case, we show that $f$ is not uniformly continuous. Indeed, suppose there exists some $\delta > 0$ such that

$$|x - y| < \delta \quad \Longrightarrow \quad |x^2 - y^2| < 1.$$

We can then look at the special case $x = y + \frac{\delta}{2}$. Since $|x - y| < \delta$, we find that

$$1 > |x^2 - y^2| = |x - y| \cdot |x + y| = \frac{\delta}{2} \cdot \left| 2y + \frac{\delta}{2} \right| > \delta y$$

for any $y > 0$. This is obviously false when $y \geq \frac{1}{\delta}$, so $f$ is not uniformly continuous. □

**Theorem 5.17 – Differentiable functions with bounded derivative**

Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a differentiable function and suppose that there exists $k > 0$ such that $|f'(x)| \leq k$ for all $x \in \mathbb{R}$. Then $f$ is uniformly continuous.

**Proof.** Let $\varepsilon > 0$ be given and let $x, y \in \mathbb{R}$. In view of the Mean Value Theorem, one has

$$|f(x) - f(y)| = |f'(z)| \cdot |x - y|$$

for some point $z$ between $x$ and $y$. Since $|f'(z)| \leq k$ by assumption, this implies that

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| \leq k|x - y| < k\delta.$$

Taking $\delta = \varepsilon/k$ now gives $|f(x) - f(y)| < \varepsilon$. In particular, $f$ is uniformly continuous. ■

# 6 Completeness

**Definition 6.1 – Cauchy sequence**

We say that a sequence $\{x_n\}$ of real numbers is a Cauchy sequence if, given any $\varepsilon > 0$, there exists some natural number $N$ such that

$$|x_m - x_n| < \varepsilon \quad \text{for all } m, n \geq N.$$

**Theorem 6.2 – Convergent implies Cauchy implies bounded**

Every convergent sequence is Cauchy and every Cauchy sequence is bounded.

**Proof.** To prove the first part, suppose that the sequence $\{x_n\}$ converges to $x$ as $n \to \infty$. Given any $\varepsilon > 0$, we can then find some natural number $N$ such that

$$|x_n - x| < \frac{\varepsilon}{2} \quad \text{for all } n \geq N.$$

Using this fact along with the triangle inequality, we conclude that

$$|x_m - x_n| \leq |x_m - x| + |x - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $m, n \geq N$. This shows that the given sequence is also Cauchy.

To prove the second part, suppose that $\{x_n\}$ is a Cauchy sequence. Using the definition of a Cauchy sequence with $\varepsilon = 1$, we may then find some natural number $N$ such that

$$|x_m - x_n| < 1 \quad \text{for all } m, n \geq N.$$

This inequality yields a precise bound for the terms $x_m$ with $m \geq N$ because

$$|x_m| \leq |x_m - x_N| + |x_N| < 1 + |x_N| \quad \text{for all } m \geq N.$$

On the other hand, it is clear that the remaining terms can be trivially bounded by

$$|x_m| \leq \max\{|x_1|, |x_2|, \ldots, |x_N|\} \quad \text{for all } 1 \leq m \leq N.$$

We conclude that $|x_m| \leq \max\{|x_1|, |x_2|, \ldots, |x_N|, 1 + |x_N|\}$ for any $m \geq 1$. ■

**Example 6.3** Consider the sequence defined by $x_n = (-1)^n$. Since its terms are oscillating between $-1$ and $1$, this sequence is bounded, but it is neither Cauchy nor convergent. □

**Definition 6.4 – Complete set**

We say that a set $A \subseteq \mathbb{R}$ is complete, if every Cauchy sequence $\{x_n\}$ which consists of elements of $A$ must actually converge to an element of $A$.

**Example 6.5** We show that $A = (0, 1]$ is not complete. Note that $x = 0$ is the limit of the sequence defined by $x_n = \frac{1}{n}$. This sequence consists of elements of $A$ and it is convergent, hence also Cauchy. Since the limit is not an element of $A$, however, $A$ is not complete. □

**Theorem 6.6 – Bolzano-Weierstrass theorem**

Every bounded sequence of real numbers has a convergent subsequence.

**Proof.** Suppose that $\{x_n\}$ is a bounded sequence of real numbers. If we can show that it has a monotonic subsequence $\{x_{n_k}\}$, then that subsequence will converge by the monotonic convergence theorem and the proof will be complete.

Let us say that $x_N$ is a peak point, if all subsequent terms are smaller than $x_N$, namely if $x_n < x_N$ for all $n > N$. There are two possible cases. If there are infinitely many peak points, then these form a decreasing subsequence $x_{N_1}, x_{N_2}, x_{N_3}, \ldots$ because

$$x_{N_1} > x_{N_2} > x_{N_3} > \ldots$$

by definition. Otherwise, there are finitely many peak points and we may assume that there are no peak points $x_n$ with $n \geq n_1$. Since $x_{n_1}$ is not a peak point, there exists $n_2 > n_1$ such that $x_{n_2} \geq x_{n_1}$. Since $x_{n_2}$ is not a peak point, there exists $n_3 > n_2$ such that $x_{n_3} \geq x_{n_2}$. One may thus proceed in this manner to obtain an increasing subsequence. ■

**Theorem 6.7 – Cauchy sequence with convergent subsequence**

Suppose that $\{x_n\}$ is a Cauchy sequence of real numbers that has a subsequence $\{x_{n_k}\}$ which converges. Then the original sequence $\{x_n\}$ converges as well.

**Proof.** Let us denote by $x$ the limit of the subsequence $\{x_{n_k}\}$. To show that the original sequence also converges to $x$, let $\varepsilon > 0$ be given. Since the sequence $\{x_n\}$ is Cauchy, there exists a natural number $N_1 \in \mathbb{N}$ such that

$$|x_m - x_n| < \varepsilon/2 \quad \text{for all } m, n \geq N_1.$$

Since the subsequence $\{x_{n_k}\}$ converges to $x$, there also exists some $N_2 \in \mathbb{N}$ such that

$$|x_{n_k} - x| < \varepsilon/2 \quad \text{for all } n_k \geq N_2.$$

Let $N = \max\{N_1, N_2\}$ and fix some $n_k \geq N$. Using the inequalities above, we now get

$$|x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n \geq N$. This shows that the original sequence $\{x_n\}$ converges to $x$ as $n \to \infty$. ■

**Theorem 6.8 – Completeness of $\mathbb{R}$**

Every Cauchy sequence in $\mathbb{R}$ converges. Thus, the set of real numbers is complete.

**Proof.** Suppose that $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}$. Then $\{x_n\}$ is bounded, so it has a convergent subsequence by the Bolzano-Weierstrass theorem. Being a Cauchy sequence with a convergent subsequence, $\{x_n\}$ must then converge by the previous theorem. ■

**Theorem 6.9 – Equivalence of various results**

The following results are equivalent: (1) Axiom of completeness, (2) Monotone convergence theorem, (3) Bolzano-Weierstrass theorem and (4) Completeness of $\mathbb{R}$.

**Proof.** We have already seen that (1) implies (2) in the proof of Theorem 3.7, that (2) implies (3) in the proof of Theorem 6.6 and that (3) implies (4) in the proof of Theorem 6.8. If we now show that (4) implies (1), then the equivalence of these results will follow.

Suppose $A$ is a nonempty subset of $\mathbb{R}$ that has an upper bound. One may then fix an element of $A$ and an upper bound of $A$. Looking at the average of these two, we try to find either a larger element of $A$ or a smaller upper bound of $A$. Suppose that we already have one element $a_n \in A$ and one upper bound $b_n$. Then $a_n \leq b_n$ and we consider two cases.

Case 1. If the average $\frac{a_n+b_n}{2}$ is an upper bound of $A$, we let $a_{n+1} = a_n$ and $b_{n+1} = \frac{a_n+b_n}{2}$. In this case, we have $a_n \leq b_{n+1} \leq b_n$ and also $b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n)$.

Case 2. If the average $\frac{a_n+b_n}{2}$ is not an upper bound of $A$, then there exists an element of $A$ which is larger. We choose $a_{n+1} \in A$ such that $a_{n+1} > \frac{a_n+b_n}{2} \geq a_n$ and let $b_{n+1} = b_n$. Then

$$b_{n+1} - a_{n+1} < b_n - \frac{a_n + b_n}{2} = \frac{b_n - a_n}{2}.$$

This process gives rise to an increasing sequence $\{a_n\}$ of elements of $A$ and a decreasing sequence $\{b_n\}$ of upper bounds of $A$. Note that $a_n \leq b_n$ for all $n \in \mathbb{N}$ and that

$$b_{n+1} - a_{n+1} \leq \frac{b_n - a_n}{2} \quad \Longrightarrow \quad b_{n+1} - a_{n+1} \leq \frac{b_1 - a_1}{2^n}$$

by induction. Thus, the length of the interval $[a_N, b_N]$ becomes arbitrarily small. On the other hand, this interval contains all terms $a_n, b_n$ with $n \geq N$ because $a_N \leq a_n \leq b_n \leq b_N$ for all such $n$. In particular, $\{a_n\}, \{b_n\}$ are Cauchy, so they converge by completeness.

Let $s$ be the limit that $a_n$ approaches as $n \to \infty$. Then $b_n$ approaches the same limit, as the difference $b_n - a_n$ approaches zero. We claim that $s$ is the least upper bound of $A$. Given any element $a \in A$, we have $a \leq b_n$ for all $n \in \mathbb{N}$ and thus $a \leq s$. This shows that $s$ is an upper bound of $A$. Given any other upper bound $t$, we have $t \geq a_n$ for all $n \in \mathbb{N}$ and thus $t \geq s$. This shows that $s$ is the least upper bound of $A$, as needed. ■

**Theorem 6.10 – Complete subsets of $\mathbb{R}$**

A set $A \subseteq \mathbb{R}$ is complete, if and only if $A$ is closed in $\mathbb{R}$.

**Proof.** First, suppose that $A \subseteq \mathbb{R}$ is closed and let $\{x_n\}$ be a Cauchy sequence of elements of $A$. Such a sequence converges because $\mathbb{R}$ is complete, while its limit is an element of $A$ because of Theorem 4.6. This means that $A$ itself is complete.

Conversely, suppose that $A \subseteq \mathbb{R}$ is complete. To show that $A$ is closed in $\mathbb{R}$, we need to show that $A^c$ is open in $\mathbb{R}$. Given any $x \in A^c$, we should thus be able to find some $\varepsilon > 0$

such that $(x-\varepsilon, x+\varepsilon) \subseteq A^c$. If it happens that $(x-\frac{1}{n}, x+\frac{1}{n}) \subseteq A^c$ for some $n \in \mathbb{N}$, then we are done. Otherwise, this inclusion fails for each $n \in \mathbb{N}$, so there exist points

$$x - \frac{1}{n} < x_n < x + \frac{1}{n}, \qquad x_n \in A.$$

It follows by the Squeeze Theorem that $x_n \to x$ as $n \to \infty$. Since $\{x_n\}$ is convergent, it is also Cauchy and its limit $x$ is an element of $A$ by completeness. This contradicts our initial assumption that $x \in A^c$. Thus, we do have $(x-\frac{1}{n}, x+\frac{1}{n}) \subseteq A^c$ for some $n \in \mathbb{N}$. ■

**Definition 6.11 – Dense subset**

We say that a set $A \subseteq \mathbb{R}$ is dense in $\mathbb{R}$, if the closure of $A$ is equal to $\overline{A} = \mathbb{R}$.

**Theorem 6.12 – Every open interval contains a rational**

Given any two real numbers $x < y$, there exists a rational number $x < z < y$.

**Proof.** Suppose first that $x, y$ are both positive. We need to ensure that $x < m/n < y$ for some natural numbers $m, n$ and one may express this inequality as $nx < m < ny$. First of all, we choose a natural number $n$ so that $ny - nx > 1$. Since

$$ny - nx > 1 \quad \Longleftrightarrow \quad n(y-x) > 1 \quad \Longleftrightarrow \quad n > 1/(y-x),$$

such a natural number exists by the Archimedean property. Consider the set

$$A = \{m \in \mathbb{N} : m > nx\}.$$

Since $A$ is a nonempty subset of $\mathbb{N}$, it has a minimum by Theorem 2.15. Let us denote its minimum by $m$. Then $m \in A$ and $m - 1 \notin A$, so it easily follows that

$$m - 1 \leq nx < m \quad \Longrightarrow \quad nx < m \leq nx + 1 < ny \quad \Longrightarrow \quad x < \frac{m}{n} < y.$$

This completes the proof in the case that $x, y$ are both positive. If $x, y$ are both negative, then the above argument gives a rational number $z$ that lies between $-x$ and $-y$, so $-z$ is a rational number that lies between $x$ and $y$. Suppose, finally, that $x$ is negative and $y$ is positive. In that case, the result is trivial because $z = 0$ is rational and $x < z < y$. ■

**Theorem 6.13 – $\mathbb{Q}$ is dense in $\mathbb{R}$**

The set of rational numbers $\mathbb{Q}$ is a dense subset of $\mathbb{R}$.

**Proof.** We need to show that $\overline{\mathbb{Q}} = \mathbb{Q} \cup \mathbb{Q}'$ contains all real numbers. In fact, we show that every real number $x$ is a limit point of $\mathbb{Q}$. Suppose $U$ is a neighbourhood of $x$. Since $U$ is open, we must have $(x-\varepsilon, x+\varepsilon) \subseteq U$ for some $\varepsilon > 0$. This gives $(x-\varepsilon, x) \subseteq U$ and the last interval contains a rational number, so $U$ intersects $\mathbb{Q}$ at a point other than $x$. ■

# 7 Connectedness

**Definition 7.1 – Connected set**

We say that $A \subseteq \mathbb{R}$ is connected, if $A$ cannot be expressed as the union $A = A_1 \cup A_2$ of two nonempty disjoint sets which are both open in $A$.

**Example 7.2** Consider the set $A = [0,1] \cup [2,3)$. If we let $A_1 = [0,1]$, then $A_1$ is not open in $\mathbb{R}$, but it is easy to check that $A_1$ is open in $A$. For instance, $(-1, 3/2)$ is open in $\mathbb{R}$ and its intersection with $A$ is open in $A$, so $[0,1]$ is open in $A$. Using the same reasoning, one finds that $A_2 = [2,3)$ is also open in $A$. This means that $A = A_1 \cup A_2$ is not connected. $\square$

**Example 7.3** Consider the set $A = [0,1] \cup \{2\}$. If we let $A_1 = [0,1]$ and $A_2 = \{2\}$, then neither of these sets is open in $\mathbb{R}$, but they are both open in $A$. This is because $A_1$ is the intersection of $(-1, 3/2)$ with $A$ and $A_2$ is the intersection of $(3/2, 3)$ with $A$. Since $A_1, A_2$ are also nonempty and disjoint, we conclude that $A = A_1 \cup A_2$ is not connected. $\square$

**Definition 7.4 – Intermediate point property**

We say that $A \subseteq \mathbb{R}$ has the intermediate point property, if any point that lies between two elements of $A$ is an element of $A$, namely if $x, y \in A$ and $x < z < y$ implies $z \in A$.

**Theorem 7.5 – Criterion for being connected**

To say that $A \subseteq \mathbb{R}$ is connected is to say that $A$ has the intermediate point property.

**Proof.** Suppose first that $A$ does not have the intermediate point property. Then there exist some numbers $x < z < y$ such that $x, y \in A$ and $z \notin A$. Consider the sets

$$A_1 = (-\infty, z) \cap A, \qquad A_2 = (z, \infty) \cap A.$$

These are open in $A$ and they are also nonempty because $A_1$ contains $x$ and $A_2$ contains $y$. Since $z \notin A$, it is clear that $A_1 \cup A_2 = A$. This implies that $A$ is not connected.

Suppose now that $A$ does have the intermediate point property. To show that $A$ must be connected, assume $A = A_1 \cup A_2$ for some nonempty disjoint sets $A_1, A_2$ which are both open in $A$. We can then pick some elements $x \in A_1$ and $y \in A_2$. Assume $x < y$ without loss of generality. Then $[x, y] \subseteq A$ by the intermediate point property. Let us now define

$$B = [x, y] \cap A_1.$$

Since $x \in B$ and $y$ is an upper bound of $B$, we see that $\sup B$ exists and $x \leq \sup B \leq y$. In particular, $\sup B$ is an element of $[x, y] \subseteq A$, so it is an element of either $A_1$ or $A_2$.

<u>Case 1.</u> Suppose that $\sup B \in A_1$. Since $A_1$ is open in $A$, we must then have

$$(\sup B - \varepsilon, \sup B + \varepsilon) \cap A \subseteq A_1$$

for some $\varepsilon > 0$. Since $\sup B \notin A_2$ in this case, one has $\sup B \neq y$ and so $x \leq \sup B < y$. Once we now let $\delta = \min\{\frac{\varepsilon}{2}, y - \sup B\}$ for convenience, we get $\sup B + \delta \leq y$ and

$$\sup B + \delta \in (\sup B, \sup B + \varepsilon) \cap [x, y] \subseteq (\sup B, \sup B + \varepsilon) \cap A \subseteq A_1.$$

This makes $\sup B + \delta$ an element of $[x, y] \cap A_1 = B$, which is obviously a contradiction.

Case 2. Suppose that $\sup B \in A_2$. Since $A_2$ is open in $A$, we must then have

$$(\sup B - \varepsilon, \sup B + \varepsilon) \cap A \subseteq A_2$$

for some $\varepsilon > 0$. Since $\sup B - \varepsilon < \sup B$, there also exists some element $z \in B$ such that

$$\sup B - \varepsilon < z \leq \sup B.$$

We note that $B = [x, y] \cap A_1$ by definition, while $[x, y] \subseteq A$ by above. This gives

$$z \in (\sup B - \varepsilon, \sup B] \cap [x, y] \subseteq (\sup B - \varepsilon, \sup B] \cap A \subseteq A_2,$$

which is a contradiction because $z \in B \subseteq A_1$ and $A_1, A_2$ have no element in common. ■

**Theorem 7.6 – Connected subsets of $\mathbb{R}$**

The only connected subsets of $\mathbb{R}$ are $\varnothing$, $\mathbb{R}$, sets with one element, and also intervals.

**Proof.** It is easy to see that each of these sets has the intermediate point property, so each of these sets is connected. Conversely, suppose that $A \subseteq \mathbb{R}$ is nonempty and connected.

Case 1. If $A$ is bounded from above and below, we let $a = \inf A$ and $b = \sup A$. Then

$$a \leq x \leq b \quad \text{for all } x \in A$$

and this means that $A \subseteq [a, b]$. If we can also show that $(a, b) \subseteq A$, then this will leave

$$A = (a, b), \qquad A = (a, b], \qquad A = [a, b), \qquad A = [a, b]$$

as the only possibilities. Suppose now that $z \in (a, b)$. Since $z > a$, we have $z > \inf A$ and there exists some $x \in A$ such that $z > x$. Since $z < b$, we have $z < \sup A$ and there exists some $y \in A$ such that $z < y$. This gives $x, y \in A$ and $x < z < y$. On the other hand, $A$ is connected, so it has the intermediate point property. We conclude that $z \in A$, as needed.

Case 2. If $A$ is bounded from below but not from above, we let $a = \inf A$. Then

$$a \leq x \quad \text{for all } x \in A$$

and this means that $A \subseteq [a, \infty)$. If we can also show that $(a, \infty) \subseteq A$, then this will leave

$$A = (a, \infty), \qquad A = [a, \infty)$$

as the only possibilities. Once again, suppose that $z \in (a, \infty)$. Since $z > a = \inf A$, there exists some $x \in A$ such that $z > x$. Since $A$ is not bounded from above, $z$ is not an upper bound of $A$, so we can find some $y \in A$ such that $z < y$. This gives $x, y \in A$ and $x < z < y$. Using the intermediate point property once again, we conclude that $z \in A$, as needed.

Case 3. If $A$ is bounded from above but not from below, we let $b = \sup A$ and we proceed as in the previous case to find that either $A = (-\infty, b)$ or else $A = (-\infty, b]$.

Case 4. If $A$ is not bounded from either above or below, we may use our previous approach to conclude that $A = \mathbb{R}$. In fact, let $z \in \mathbb{R}$ be given. Since $z$ is not a lower bound of $A$, there exists some $x \in A$ such that $x < z$. Since $z$ is not an upper bound of $A$, there also exists some $y \in A$ such that $y > z$. This gives $x < z < y$ and thus $z \in A$ as before. ■

**Theorem 7.7 – Subsets both open and closed**

If a set $A \subseteq \mathbb{R}$ is connected, then the only subsets of $A$ which are both open and closed in $A$ are the trivial subsets $\varnothing, A$. If a set $A \subseteq \mathbb{R}$ is not connected, however, then $A$ has a nontrivial subset other than $\varnothing, A$ which is both open and closed in $A$.

**Proof.** First, suppose that $A$ is connected and $B \subseteq A$ is both open and closed in $A$. Then the disjoint sets $B, A - B$ are both open in $A$ and their union is $A$. Since $A$ is connected, one of these sets must be empty. This means that either $B = \varnothing$ or else $B = A$.

Next, suppose that $A$ is not connected. Then $A = A_1 \cup A_2$ for some nonempty disjoint sets $A_1, A_2$ which are both open in $A$. Since the complement of $A_1$ is $A_2$, we see that $A_1$ is both open and closed in $A$. This is a nontrivial subset of $A$ other than $\varnothing, A$. ■

**Theorem 7.8 – Continuous image of connected sets**

If $f\colon A \to \mathbb{R}$ is continuous and $A \subseteq \mathbb{R}$ is connected, then $f(A)$ is connected as well.

**Proof.** Suppose that $f(A)$ is not connected and write $f(A) = B_1 \cup B_2$ for some nonempty disjoint sets $B_1, B_2$ which are both open in $f(A)$. Let $A_1 = f^{-1}(B_1)$ and $A_2 = f^{-1}(B_2)$ be the inverse images. To show that the sets $A_1, A_2$ are also disjoint, we note that

$$x \in A_1 \cap A_2 \quad \Longrightarrow \quad x \in f^{-1}(B_1) \cap f^{-1}(B_2) \quad \Longrightarrow \quad f(x) \in B_1 \cap B_2.$$

There is no such element $x$ because $B_1 \cap B_2$ is empty. To show that $A_1, A_2$ are nonempty, pick some $y_i \in B_i$ for each $i$. Then $y_i \in f(A)$, so $y_i = f(x_i)$ for some $x_i \in A$ and

$$f(x_i) = y_i \in B_i \quad \Longrightarrow \quad x_i \in f^{-1}(B_i) = A_i$$

for each $i$. Finally, the sets $A_1, A_2$ are both open in $A$ by continuity, while their union is

$$A_1 \cup A_2 = f^{-1}(B_1) \cup f^{-1}(B_2) = f^{-1}(B_1 \cup B_2) = f^{-1}(f(A)) \supseteq A.$$

This implies that $A_1 \cup A_2 = A$, which contradicts our assumption that $A$ is connected. ■

**Example 7.9** Consider a continuous function $f\colon (0,1) \to \mathbb{R}$. Since $(0,1)$ is connected, its image is connected as well. Thus, the image cannot be $\mathbb{Q}$ or $(0,1) \cup (2,3)$, for instance. □

**Example 7.10** Consider a continuous function $f\colon [a,b] \to \mathbb{R}$. Since $[a,b]$ is connected, its image is also connected, so it has the intermediate point property. As the image includes the values $f(a)$ and $f(b)$, it actually includes every value that lies between them. □

# 8 Countability

**Definition 8.1 – Countable**

We say that a set $A$ is countably infinite, if there is a bijection $f\colon \mathbb{N} \to A$. We say that a set $A$ is countable, if $A$ is either finite or else countably infinite.

**Theorem 8.2 – Cantor's diagonal argument**

The unit interval $(0,1)$ is uncountable.

**Proof.** Suppose that there is a bijection $f\colon \mathbb{N} \to (0,1)$. Then the numbers $f(1), f(2), \ldots$ are all positive numbers which are less than 1, so their decimal expansion has the form

$$\begin{aligned} f(1) &= 0.a_{11}a_{12}a_{13}a_{14}\ldots \\ f(2) &= 0.a_{21}a_{22}a_{23}a_{24}\ldots \\ f(3) &= 0.a_{31}a_{32}a_{33}a_{34}\ldots \end{aligned}$$

and so on. We now proceed to change one of the digits in each case. Define

$$b_{nn} = \left\{ \begin{array}{ll} 1 & \text{if } a_{nn} \neq 1 \\ 2 & \text{if } a_{nn} = 1 \end{array} \right\}$$

and note that $b_{nn} \neq a_{nn}$ for each $n$. The number $x = 0.b_{11}b_{22}b_{33}b_{44}\ldots$ is then in $(0,1)$ and it differs from $f(1)$ in the first decimal digit, from $f(2)$ in the second decimal digit, and so on. This means that $x$ is not in the image and that $f$ is not bijective, a contradiction. ■

**Theorem 8.3 – Subsets of $\mathbb{N}$ are countable**

Every subset of $\mathbb{N}$ is countable.

**Proof.** If a set $A \subseteq \mathbb{N}$ is finite, then it is certainly countable. If a set $A \subseteq \mathbb{N}$ is infinite, then we need to show that $A$ is countably infinite. We thus need to find a bijection $f\colon \mathbb{N} \to A$. Since $A$ is infinite, it is nonempty and $\min A$ exists by Theorem 2.15, so one may let

$$f(1) = \min A.$$

Assuming that $f(1), f(2), \ldots, f(n-1)$ have already been defined, we may then take

$$f(n) = \min A - \{f(1), f(2), \ldots, f(n-1)\} \quad \text{for each } n \geq 2.$$

To show that the resulting function $f$ is injective, suppose that $m > n$. Then $f(m)$ is the least element of the set $A - \{f(1), f(2), \ldots, f(m-1)\}$, while $f(n)$ is not an element of this set, so the two are not equal and $f$ is injective.

To show that $f\colon \mathbb{N} \to A$ is surjective, let $x \in A \subseteq \mathbb{N}$ be given. Since $f$ is known to be injective, the image $f(\mathbb{N})$ is infinite, so it contains arbitrarily large numbers. Thus,

$$B = \{n \in \mathbb{N} : f(n) \geq x\}$$

is nonempty and $m = \min B$ exists by Theorem 2.15. If it happens that $m = 1$, then

$$1 \in B \implies f(1) \geq x \implies \min A \geq x \geq \min A \implies x = \min A = f(1).$$

If it happens that $m > 1$, then $m \in B$ and $1, 2, \ldots, m-1 \notin B$. This implies that

$$f(1), f(2), \ldots, f(m-1) < x \leq f(m).$$

Thus, the set $A - \{f(1), f(2), \ldots, f(m-1)\}$ contains $x$ and the minimum of this set is at most $x$. Since the minimum is $f(m) \geq x$, we conclude that $f(m) = x$, as needed. ■

**Theorem 8.4 – Criteria for being countable**

The following statements are equivalent for every nonempty set $A$.

(a) The set $A$ is countable.

(b) There exists a surjective function $f : \mathbb{N} \to A$.

(c) There exists an injective function $g : A \to \mathbb{N}$.

**Proof.** First, we show that (a) implies (b). If $A$ is countably infinite, then there exists a bijective function $f : \mathbb{N} \to A$ and this is surjective. Otherwise, $A = \{a_1, a_2, \ldots, a_n\}$ is finite and one may define a surjective function $f : \mathbb{N} \to A$ by setting

$$f(k) = \left\{ \begin{array}{ll} a_k & \text{if } k \leq n \\ a_n & \text{if } k > n \end{array} \right\}.$$

Next, we show that (b) implies (c). Suppose that $f : \mathbb{N} \to A$ is surjective. To define an injective function $g : A \to \mathbb{N}$, let $x \in A$ be given. Then $f^{-1}(\{x\})$ is a nonempty subset of $\mathbb{N}$ by surjectivity, so we may let $g(x) = \min f^{-1}(\{x\})$. It is easy to check that

$$z \in f^{-1}(\{x\}) \cap f^{-1}(\{y\}) \implies x = f(z) = y \implies x = y.$$

Thus, the sets $f^{-1}(\{x\})$ and $f^{-1}(\{y\})$ have no element in common whenever $x \neq y$, so their minimum elements are distinct whenever $x \neq y$. This means that $g$ is injective.

Finally, we show that (c) implies (a). If $A$ is finite, then $A$ is certainly countable. If it is infinite and $g : A \to \mathbb{N}$ is injective, then $g : A \to g(A)$ is bijective. However, $g(A) \subseteq \mathbb{N}$ is countable by the previous theorem, so it must be countably infinite. In other words, there exists a bijection $h : g(A) \to \mathbb{N}$. We conclude that the composition $h \circ g : A \to \mathbb{N}$ is also a bijection. This implies that $A$ is countably infinite and thus countable. ■

**Example 8.5** To show that $\mathbb{Z}$ is countable, it suffices to find a surjection $f : \mathbb{N} \to \mathbb{Z}$. This amounts to defining $f(1), f(2), f(3), \ldots$ in such a way that all integers are listed. A simple way to achieve this is to order the integers as $0, 1, -1, 2, -2, \ldots$ alternating between the positive and the non-positive ones. More precisely, one may define $f : \mathbb{N} \to \mathbb{Z}$ by letting

$$f(n) = \left\{ \begin{array}{ll} n/2 & \text{if } n \text{ is even} \\ -(n-1)/2 & \text{if } n \text{ is odd} \end{array} \right\}.$$

If $m \geq 1$ is a positive integer, then $f(2m) = m$ and $m$ lies in the image of $f$. If $x \leq 0$ is a non-positive integer, then $1 - 2x \in \mathbb{N}$ and $f(1 - 2x) = x$ also lies in the image of $f$. □

**Example 8.6** We show that $\mathbb{N}\times\mathbb{N}$ is countable by finding an injection $f\colon \mathbb{N}\times\mathbb{N}\to\mathbb{N}$. A typical example is $f(m,n)=2^m3^n$. Suppose that $2^m3^n=2^x3^y$ for some $m,n,x,y\in\mathbb{N}$. If it happens that $m>x$, then $2^{m-x}3^n=3^y$, and this is not possible because $2^{m-x}3^n$ is even, whereas $3^y$ is odd. If it happens that $m<x$, then $3^n=2^{x-m}3^y$ and a similar contradiction arises. We conclude that $m=x$ and $3^n=3^y$, so $n=y$ and the function $f$ is injective. □

**Theorem 8.7 – Product of countable sets**

The product $A\times B$ of two sets consists of all pairs $(x,y)$ with $x\in A$ and $y\in B$. If the sets $A,B$ are both countable, then their product $A\times B$ is countable as well.

**Proof.** Since $A,B$ are countable, there exist injective functions $f\colon A\to\mathbb{N}$ and $g\colon B\to\mathbb{N}$. One may thus define a function $h\colon A\times B\to\mathbb{N}\times\mathbb{N}$ by letting $h(x,y)=(f(x),g(y))$. It is easy to check that this function is injective, as

$$\begin{aligned} h(x_1,y_1)=h(x_2,y_2) \quad &\Longrightarrow \quad (f(x_1),g(y_1))=(f(x_2),g(y_2)) \\ &\Longrightarrow \quad f(x_1)=f(x_2) \text{ and } g(y_1)=g(y_2) \\ &\Longrightarrow \quad x_1=x_2 \text{ and } y_1=y_2. \end{aligned}$$

Since $\mathbb{N}\times\mathbb{N}$ is countable by the previous example, it is countably infinite and there is a bijection $\varphi\colon\mathbb{N}\times\mathbb{N}\to\mathbb{N}$. It easily follows that $\varphi\circ h\colon A\times B\to\mathbb{N}$ is injective. ■

**Theorem 8.8 – $\mathbb{Q}$ is countable**

The set $\mathbb{Q}$ of all rational numbers is countable.

**Proof.** The set $\mathbb{Q}$ consists of all quotients $m/n$, where $m\in\mathbb{Z}$ and $n\in\mathbb{N}$. One may thus define a surjection $f\colon\mathbb{Z}\times\mathbb{N}\to\mathbb{Q}$ using the formula $f(m,n)=m/n$. Since $\mathbb{Z},\mathbb{N}$ are both countable, the same is true for their product $\mathbb{Z}\times\mathbb{N}$. This gives a bijection $\varphi\colon\mathbb{N}\to\mathbb{Z}\times\mathbb{N}$. Since the composition $f\circ\varphi\colon\mathbb{N}\to\mathbb{Q}$ is surjective, we conclude that $\mathbb{Q}$ is countable. ■

# 9 Compactness

**Definition 9.1 – Compact set**

An open cover of $A\subseteq\mathbb{R}$ is a collection of sets $U_i$ which are open in $\mathbb{R}$ and their union contains $A$. We say that $A$ is compact, if every open cover of $A$ has a finite subcover, namely, if there is always a finite subcollection of open sets which still cover $A$.

**Example 9.2** We show that every finite set is compact. Suppose that the sets $U_i$ form an open cover of the set $A=\{a_1,a_2,\ldots,a_n\}$. Then $A$ is contained in the union of the sets $U_i$, so each $a_k$ is contained in some $U_{i_k}$. This implies that $A$ is contained in $U_{i_1}\cup\cdots\cup U_{i_n}$. □

**Example 9.3** We show that $\mathbb{R}$ is not compact. Consider the open intervals $U_n = (-n, n)$ for each $n \in \mathbb{N}$. These are increasing in the sense that $U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots$ and their union is all of $\mathbb{R}$, so they form an open cover of $\mathbb{R}$. If $\mathbb{R}$ was compact, then $\mathbb{R}$ would be covered by finitely many of these sets, so it would be contained in the union of the first $N$, say. This is not the case, however, because $U_1 \cup U_2 \cup \cdots \cup U_N = U_N = (-N, N)$ for each $N \in \mathbb{N}$. □

**Theorem 9.4 – Compact implies bounded**

If a set $A \subseteq \mathbb{R}$ is compact, then $A$ must be bounded.

**Proof.** Consider the open intervals $U_n = (-n, n)$ for each $n \in \mathbb{N}$. These intervals form an open cover of $\mathbb{R}$, so they certainly cover $A$ as well. Since $A$ is compact, it is covered by finitely many of these sets, so it is contained in the union of the first $N$, say. This gives

$$A \subseteq U_1 \cup U_2 \cup \cdots \cup U_N \quad \Longrightarrow \quad A \subseteq U_N.$$

In other words, $A$ must be contained in $U_N = (-N, N)$, so $A$ is certainly bounded. ■

**Example 9.5** Bounded sets are not necessarily compact. For instance, let $A = (0, 2)$ and consider the open intervals $U_n = (\frac{1}{n}, 2)$ for each $n \in \mathbb{N}$. Since the left endpoint decreases to zero, one has $U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots$ and the union of these sets is equal to $A$. In particular, these sets form an open cover of $A$. If $A$ was compact, then $A$ would be covered by finitely many of the sets, say $U_1, U_2, \ldots, U_N$. Since the union of these sets is $U_N$, we would then have $A \subseteq U_N$. This is not the case because $U_N = (\frac{1}{N}, 2)$ is a proper subset of $A = (0, 2)$. □

**Theorem 9.6 – Continuous image of compact sets**

If $f\colon A \to \mathbb{R}$ is continuous and $A \subseteq \mathbb{R}$ is compact, then $f(A)$ is compact as well.

**Proof.** Suppose that the sets $U_i$ form an open cover of $f(A)$. Since each $f^{-1}(U_i)$ is open in $A$ by continuity, one may write $f^{-1}(U_i) = A \cap W_i$ for some set $W_i$ which is open in $\mathbb{R}$. To see that the sets $W_i$ form an open cover of $A$, we note that

$$A \subseteq f^{-1}(f(A)) \subseteq f^{-1}\left(\bigcup_i U_i\right) = \bigcup_i f^{-1}(U_i) \subseteq \bigcup_i W_i.$$

Since $A$ is compact by assumption, it is covered by finitely many of these sets, say

$$A \subseteq W_{i_1} \cup W_{i_2} \cup \cdots \cup W_{i_n}.$$

To finish the proof, it remains to show that the corresponding sets $U_{i_k}$ cover $f(A)$. Indeed, suppose that $y \in f(A)$ and write $y = f(x)$ for some $x \in A$. Then $x \in W_{i_k}$ for some $k$, so

$$x \in A \cap W_{i_k} \quad \Longrightarrow \quad x \in f^{-1}(U_{i_k}) \quad \Longrightarrow \quad y = f(x) \in U_{i_k}.$$

In particular, $f(A)$ is covered by finitely many of the sets $U_i$ and $f(A)$ is compact. ■

**Example 9.7** The last two theorems provide an easy method for showing that a set is not compact. For instance, let $A = (0, 2)$. If this set is compact and $f : A \to \mathbb{R}$ is continuous, then the image $f(A)$ is compact, hence also bounded. On the other hand, $f(x) = 1/x$ is continuous on $A$, but it is not bounded. This implies that $A$ is not compact. □

**Theorem 9.8 – Extreme value theorem**

If $f : A \to \mathbb{R}$ is continuous and $A \subseteq \mathbb{R}$ is compact, then $f$ attains both a minimum and a maximum value. In other words, there exist points $x_1, x_2 \in A$ such that

$$f(x_1) \leq f(x) \leq f(x_2) \quad \text{for all } x \in A.$$

**Proof.** We only show that $f$ attains a maximum value, as the argument is similar in the case of a minimum value. Since $f(A)$ is compact by Theorem 9.6, it must also be bounded by Theorem 9.4. Let $M$ denote the least upper bound of $f(A) = \{f(x) : x \in A\}$. Then

$$f(x) \leq M \quad \text{for all } x \in A$$

and we need to show that equality holds at some point $x \in A$. If that is not the case, then we have $M - f(x) > 0$ for all $x \in A$ and one may define $g : A \to \mathbb{R}$ using the formula

$$g(x) = \frac{1}{M - f(x)}.$$

This is a composition of continuous functions, so it is itself continuous, and thus bounded by Theorems 9.4 and 9.6. Let $R > 0$ be an upper bound for $g(x)$ and note that

$$g(x) \leq R \quad \iff \quad M - f(x) \geq R^{-1} \quad \iff \quad f(x) \leq M - R^{-1}.$$

Since the leftmost inequality holds for all $x \in A$, the rightmost inequality holds as well. This makes $M - R^{-1}$ an upper bound of $f(A)$. On the other hand, $M$ is the least upper bound by definition and $M - R^{-1}$ is smaller. We have thus reached a contradiction. ■

**Theorem 9.9 – Finite closed intervals are compact**

The closed interval $[a, b]$ is a compact subset of $\mathbb{R}$ for all real numbers $a < b$.

**Proof.** Suppose that the closed interval $I_0 = [a, b]$ is not compact. Then there exist some open sets $U_i$ that cover $I_0$ in such a way that no finite number of them cover $I_0$. Using the bisection method, we now replace $I_0$ by a closed interval of arbitrarily small length.

Split the original interval $I_0$ into two subintervals of equal length. If both of those are covered by finitely many of the sets $U_i$, then their union $I_0$ is also covered by finitely many sets. This is not the case, however, so one of the two subintervals is not covered by finitely many of the sets $U_i$. Denote that subinterval by $I_1$ and proceed in this manner to obtain a sequence of closed intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ such that no $I_n$ is covered by finitely many of the sets $U_i$ and the length of each interval $I_n$ is half the length of the previous interval.

According to the nested interval property, the intersection $\bigcap_{n=0}^{\infty} I_n$ is nonempty. Let $x$ be a point in this intersection. Since $x \in I_0$ and the sets $U_i$ form an open cover of $I_0$, we must have $x \in U_{i_k}$ for some index $i_k$. Moreover, this set is open, so we actually have

$$(x-\varepsilon, x+\varepsilon) \subseteq U_{i_k} \quad \text{for some } \varepsilon > 0.$$

Note that the interval $I = (x-\varepsilon, x+\varepsilon)$ is centred around $x$ and its length is fixed. On the other hand, $I_n$ is an interval containing $x$ whose length $(b-a)/2^n$ is arbitrarily small. If we thus pick a large enough value of $n$, we can ensure that $I_n$ is contained in $I$ so that

$$I_n \subseteq (x-\varepsilon, x+\varepsilon) \subseteq U_{i_k}.$$

This contradicts the fact that $I_n$ is not covered by finitely many of the sets $U_i$. ■

**Theorem 9.10 – Compact implies closed in $\mathbb{R}$**

If a set $A \subseteq \mathbb{R}$ is compact, then $A$ must be closed in $\mathbb{R}$.

**Proof.** We show that the complement $A^c$ is open in $\mathbb{R}$. Let $y \in A^c$ be given and consider the function $f\colon A \to \mathbb{R}$ which is defined by $f(x) = |x-y|$. This measures the distance from the point $y$. Since $f$ is continuous and $A$ is compact, $f$ must attain a minimum value which is obviously non-negative. If the minimum value is zero, then $x = y$ for some $x \in A$ and this contradicts our assumption that $y \in A^c$. Thus, the minimum value is $\varepsilon > 0$ and one has

$$f(x) = |x-y| \geq \varepsilon \quad \text{for all } x \in A.$$

This is easily seen to imply that $(y-\varepsilon, y+\varepsilon) \subseteq A^c$ because

$$x \in (y-\varepsilon, y+\varepsilon) \quad \Longrightarrow \quad |x-y| < \varepsilon \quad \Longrightarrow \quad x \in A^c.$$

In particular, the interval $(y-\varepsilon, y+\varepsilon)$ is contained in $A^c$ and so $A^c$ is open, indeed. ■

**Theorem 9.11 – Heine-Borel theorem**

A subset of $\mathbb{R}$ is compact if and only if it is bounded and closed in $\mathbb{R}$.

**Proof.** Suppose first that $A \subseteq \mathbb{R}$ is compact. Then $A$ is bounded by Theorem 9.4 and $A$ is closed in $\mathbb{R}$ by Theorem 9.10. Suppose now that $A$ is bounded and closed in $\mathbb{R}$. Then $A$ is contained in $[-N, N]$ for some $N > 0$. We now use this fact to conclude that $A$ is compact. Suppose that the sets $U_i$ form an open cover of $A$. Since $A$ is closed, its complement $A^c$ is open in $\mathbb{R}$. Note that the sets $U_i$ along with $A^c$ form an open cover of $\mathbb{R}$, so they certainly cover $[-N, N]$. Since $[-N, N]$ is compact by Theorem 9.9, it is covered by finitely many of these sets. Thus, $A$ itself is covered by finitely many sets and $A$ is compact. ■

**Theorem 9.12 – Continuity on compact sets**

If $f\colon A \to \mathbb{R}$ is continuous and $A \subseteq \mathbb{R}$ is compact, then $f$ is uniformly continuous.

**Proof.** Let $\varepsilon > 0$ be given. Since $f$ is continuous at each point $x \in A$, one may then find some $\delta_x > 0$ which generally depends on $x$ such that

$$|x - y| < \delta_x \quad\Longrightarrow\quad |f(x) - f(y)| < \varepsilon/2 \tag{9.1}$$

for all $y \in A$. The open intervals $(x - \frac{1}{2}\delta_x, x + \frac{1}{2}\delta_x)$ form an open cover of $A$. Since $A$ is compact, however, finitely many of these intervals must cover $A$. Suppose that

$$A \subseteq \bigcup_{i=1}^{n} \left( x_i - \frac{1}{2}\delta_{x_i}, x_i + \frac{1}{2}\delta_{x_i} \right)$$

and let $\delta = \min\{\frac{1}{2}\delta_{x_1}, \frac{1}{2}\delta_{x_2}, \ldots, \frac{1}{2}\delta_{x_n}\}$. We claim that the definition of uniform continuity holds for this choice of $\delta$. Indeed, suppose that $x, y \in A$ and $|x - y| < \delta$. Then

$$x \in A \quad\Longrightarrow\quad x \in \left( x_i - \frac{1}{2}\delta_{x_i}, x_i + \frac{1}{2}\delta_{x_i} \right) \quad\Longrightarrow\quad |x - x_i| < \frac{1}{2}\delta_{x_i}$$

for some index $1 \leq i \leq n$. Using this fact along with the triangle inequality, we now get

$$|x_i - y| \leq |x_i - x| + |x - y| < \frac{1}{2}\delta_{x_i} + \delta \leq \delta_{x_i}.$$

Since we also have $|x - x_i| < \delta_{x_i}$, it follows by equation (9.1) that

$$|f(x) - f(y)| \leq |f(x) - f(x_i)| + |f(x_i) - f(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This verifies the definition of uniform continuity and it also completes the proof. ■

# 10 Integrability

**Definition 10.1 – Partition and refinement**

We say that $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$, if the elements of $P$ are such that $a = x_0 < x_1 < \cdots < x_n = b$. We say that a partition $Q$ is a refinement of the partition $P$, if $Q$ contains more points than $P$ does, namely if $P \subseteq Q$.

**Definition 10.2 – Lower and upper Darboux sums**

Suppose that $f\colon [a, b] \to \mathbb{R}$ is bounded. Given a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$, we define the lower Darboux sum $L(f, P)$ and the upper Darboux sum $U(f, P)$ by

$$L(f, P) = \sum_{k=0}^{n-1} m_k \cdot (x_{k+1} - x_k), \qquad U(f, P) = \sum_{k=0}^{n-1} M_k \cdot (x_{k+1} - x_k),$$

where $m_k = \inf\{f(x) : x_k \leq x \leq x_{k+1}\}$ and similarly $M_k = \sup\{f(x) : x_k \leq x \leq x_{k+1}\}$.

- If the function $f$ is non-negative on $[a, b]$, then the Darboux sums represent sums of areas of rectangles. These provide approximations for the area under the graph of $f$.

**Theorem 10.3 – Darboux sums and refinements**

Suppose that $f\colon [a,b] \to \mathbb{R}$ is bounded. Given any partition $P$ of the interval $[a,b]$ and any refinement $Q$ of this partition, one must then have

$$L(f,P) \leq L(f,Q), \qquad U(f,Q) \leq U(f,P).$$

Thus, more refined partitions give rise to larger lower sums but smaller upper sums.

**Proof.** We only prove the first inequality, as the second inequality is similar. It suffices to treat the case that $Q$ contains just one more point than $P$ does. If the result holds in that special case, then lower sums increase every time a point is introduced, so the general case follows as well. Let us then concentrate on the special case

$$P = \{x_0, x_1, \ldots, x_n\}, \qquad Q = \{x_0, x_1, \ldots, x_i, y, x_{i+1}, \ldots, x_n\}.$$

According to Definition 10.2, the lower Darboux sum $L(f,P)$ is given by

$$L(f,P) = \sum_{k=0}^{n-1} m_k(x_{k+1} - x_k), \qquad m_k = \inf\{f(x) : x_k \leq x \leq x_{k+1}\}.$$

On the other hand, the corresponding expression for the lower Darboux sum $L(f,Q)$ is

$$L(f,Q) = \sum_{k=0}^{i-1} m_k(x_{k+1} - x_k) + \alpha(y - x_i) + \beta(x_{i+1} - y) + \sum_{k=i+1}^{n-1} m_k(x_{k+1} - x_k),$$

where $\alpha = \inf\{f(x) : x_i \leq x \leq y\}$ and $\beta = \inf\{f(x) : y \leq x \leq x_{i+1}\}$. Recall Theorem 2.12 which asserts that $\inf B \geq \inf A$ whenever $B \subseteq A$. This gives $\alpha, \beta \geq m_i$ and thus

$$\begin{aligned} L(f,Q) &\geq \sum_{k=0}^{i-1} m_k(x_{k+1} - x_k) + m_i(y - x_i) + m_i(x_{i+1} - y) + \sum_{k=i+1}^{n-1} m_k(x_{k+1} - x_k) \\ &= \sum_{k=0}^{i-1} m_k(x_{k+1} - x_k) + m_i(x_{i+1} - x_i) + \sum_{k=i+1}^{n-1} m_k(x_{k+1} - x_k). \end{aligned}$$

Since the last expression is merely $L(f,P)$, we get $L(f,Q) \geq L(f,P)$, as needed. ■

**Definition 10.4 – Riemann integrability**

Suppose that $f\colon [a,b] \to \mathbb{R}$ is bounded and consider the expressions

$$\begin{aligned} \mathcal{L}(f) &= \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}, \\ \mathcal{U}(f) &= \inf\{U(f,P) : P \text{ is a partition of } [a,b]\}. \end{aligned}$$

If it happens that $\mathcal{L}(f) = \mathcal{U}(f)$, we say that $f$ is integrable on $[a,b]$ and we write

$$\int_a^b f(x)\,dx = \mathcal{L}(f) = \mathcal{U}(f).$$

**Example 10.5** Consider a constant function $f\colon [a,b] \to \mathbb{R}$ defined by $f(x) = c$ for all $x$. Given any partition $P = \{x_0, x_1, \ldots, x_n\}$ of the interval $[a,b]$, one has

$$m_k = \inf\{f(x) : x_k \leq x \leq x_{k+1}\} = c,$$
$$M_k = \sup\{f(x) : x_k \leq x \leq x_{k+1}\} = c$$

for each $0 \leq k \leq n-1$. In particular, the lower and upper Darboux sums are equal and

$$L(f,P) = U(f,P) = \sum_{k=0}^{n-1} c(x_{k+1} - x_k) = c(x_n - x_0) = c(b-a).$$

Since this is true for all partitions $P$, we conclude that $\mathcal{L}(f) = \mathcal{U}(f) = c(b-a)$. □

**Example 10.6** We show that the function $f$ is not integrable on $[a,b]$ when $a < b$ and

$$f(x) = \left\{ \begin{array}{ll} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{array} \right\}.$$

Given any partition $P = \{x_0, x_1, \ldots, x_n\}$ of the interval $[a,b]$, it is easy to see that

$$m_k = \inf\{f(x) : x_k \leq x \leq x_{k+1}\} = 0,$$
$$M_k = \sup\{f(x) : x_k \leq x \leq x_{k+1}\} = 1$$

for each $0 \leq k \leq n-1$. This is because each interval $[x_k, x_{k+1}]$ contains both a rational and an irrational number. In particular, $L(f,P) = 0$ for every partition $P$ and $\mathcal{L}(f) = 0$. On the other hand, the upper Darboux sums are given by

$$U(f,P) = \sum_{k=0}^{n-1} (x_{k+1} - x_k) = x_n - x_0 = b - a.$$

This implies that $\mathcal{U}(f) = b - a$ as well. Thus, $\mathcal{L}(f) \neq \mathcal{U}(f)$ and $f$ is not integrable. □

**Theorem 10.7 – Lower and upper Darboux sums**

Given a bounded function $f\colon [a,b] \to \mathbb{R}$, one always has $\mathcal{L}(f) \leq \mathcal{U}(f)$.

**Proof.** Consider any partitions $P, Q$ of $[a,b]$. Since their union $R = P \cup Q$ is a refinement of both partitions, it follows by Theorem 10.3 that

$$L(f,P) \leq L(f,R) \leq U(f,R) \leq U(f,Q).$$

In other words, $L(f,P) \leq U(f,Q)$ for any partitions $P, Q$ of $[a,b]$. This makes $L(f,P)$ a lower bound for the set of all upper Darboux sums which implies that

$$L(f,P) \leq \inf\{U(f,Q) : Q \text{ is a partition of } [a,b]\} = \mathcal{U}(f).$$

Thus, $\mathcal{U}(f)$ is an upper bound for the set of all lower Darboux sums and $\mathcal{L}(f) \leq \mathcal{U}(f)$. ■

**Theorem 10.8 – Riemann integrability condition**

Suppose that $f\colon [a,b] \to \mathbb{R}$ is bounded. To say that $f$ is integrable on $[a,b]$ is to say that given any $\varepsilon > 0$, there exists a partition $P$ of $[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon.$$

**Proof.** Suppose first that $f$ is integrable and let $\varepsilon > 0$ be given. By definition, one has

$$\mathcal{L}(f) = \int_a^b f(x)\,dx = \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}.$$

Since $\mathcal{L}(f) - \frac{\varepsilon}{2}$ is smaller than this supremum, there exists a partition $P$ of $[a,b]$ such that

$$\mathcal{L}(f) - \frac{\varepsilon}{2} < L(f,P).$$

A similar argument applies for upper Darboux sums. More precisely, $f$ is integrable and

$$\mathcal{U}(f) = \int_a^b f(x)\,dx = \inf\{U(f,Q) : Q \text{ is a partition of } [a,b]\}.$$

Since $\mathcal{U}(f) + \frac{\varepsilon}{2}$ is larger than this infimum, there exists a partition $Q$ of $[a,b]$ such that

$$\mathcal{U}(f) + \frac{\varepsilon}{2} > U(f,Q).$$

The refinement $R = P \cup Q$ then satisfies $L(f,P) \leq L(f,R)$ and $U(f,R) \leq U(f,Q)$, so

$$U(f,R) - L(f,R) \leq U(f,Q) - L(f,P) < \mathcal{U}(f) + \varepsilon - \mathcal{L}(f) = \varepsilon.$$

Conversely, suppose that the given condition holds. To show that $f$ is integrable, we need to show that $\mathcal{L}(f) = \mathcal{U}(f)$. If that is not the case, then we must have

$$\varepsilon = \mathcal{U}(f) - \mathcal{L}(f) > 0.$$

According to the given condition, there exists a partition $R$ of $[a,b]$ such that

$$U(f,R) - L(f,R) < \mathcal{U}(f) - \mathcal{L}(f).$$

We now rearrange terms and recall the definition of $\mathcal{L}(f)$ to conclude that

$$\begin{aligned} U(f,R) + L(f,R) &\leq U(f,R) + \sup\{L(f,P) : P \text{ is a partition of } [a,b]\} \\ &= U(f,R) + \mathcal{L}(f) \\ &< \mathcal{U}(f) + L(f,R). \end{aligned}$$

This shows that $U(f,R) < \mathcal{U}(f)$. On the other hand, $\mathcal{U}(f)$ is defined as the infimum of all possible upper sums $U(f,R)$. We must thus have $U(f,R) \geq \mathcal{U}(f)$, a contradiction. ∎

**Example 10.9** Consider the function $f\colon [0,1] \to \mathbb{R}$ defined by $f(x) = x$. To show that $f$ is integrable on $[0,1]$, we introduce the partition $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$. This consists of $n+1$ equally spaced points. Letting $x_k = \frac{k}{n}$ for each $0 \leq k \leq n$, one finds that

$$m_k = \inf\{f(x) : x_k \leq x \leq x_{k+1}\} = x_k,$$
$$M_k = \sup\{f(x) : x_k \leq x \leq x_{k+1}\} = x_{k+1}$$

for each $0 \leq k \leq n-1$. In view of the definition of Darboux sums, we thus have

$$\begin{aligned} U(f,P_n) - L(f,P_n) &= \sum_{k=0}^{n-1} (M_k - m_k) \cdot (x_{k+1} - x_k) \\ &= \sum_{k=0}^{n-1} (x_{k+1} - x_k) \cdot (x_{k+1} - x_k) = \frac{n}{n^2} = \frac{1}{n}. \end{aligned}$$

Since this expression approaches zero as $n \to \infty$, we conclude that $f$ is integrable. □

# 11 Properties of integrals

**Theorem 11.1 – Continuous implies integrable**

Suppose that $f\colon [a,b] \to \mathbb{R}$ is continuous. Then $f$ is integrable on $[a,b]$.

**Proof.** The interval $[a,b]$ is compact by Theorem 9.9 and $f\colon [a,b] \to \mathbb{R}$ is continuous, so $f$ is bounded by the Extreme Value Theorem and uniformly continuous by Theorem 9.12. To show that $f$ is integrable, let $\varepsilon > 0$ be given. Then there exists some $\delta > 0$ such that

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \frac{\varepsilon}{b-a} \tag{11.1}$$

for all $x, y \in [a,b]$. Consider a partition $P = \{x_0, x_1, \ldots, x_n\}$ consisting of equally spaced points. Since $f$ attains both a minimum and a maximum value on each $[x_k, x_{k+1}]$, one has

$$m_k = \inf\{f(x) : x_k \leq x \leq x_{k+1}\} = \min\{f(x) : x_k \leq x \leq x_{k+1}\} = f(y_k),$$
$$M_k = \sup\{f(x) : x_k \leq x \leq x_{k+1}\} = \max\{f(x) : x_k \leq x \leq x_{k+1}\} = f(z_k)$$

for some points $y_k, z_k \in [x_k, x_{k+1}]$. If we now assume that $n$ is sufficiently large, then

$$|z_k - y_k| \leq |x_{k+1} - x_k| = \frac{b-a}{n} < \delta$$

for each $0 \leq k \leq n-1$. According to equation (11.1), this also implies that

$$M_k - m_k = f(z_k) - f(y_k) = |f(z_k) - f(y_k)| < \frac{\varepsilon}{b-a}.$$

Thus, the lower and upper Darboux sums that correspond to the partition $P$ satisfy

$$U(f,P) - L(f,P) = \sum_{k=0}^{n-1} (M_k - m_k) \cdot (x_{k+1} - x_k) < \sum_{k=0}^{n-1} \frac{\varepsilon}{b-a} \cdot \frac{b-a}{n} = \varepsilon.$$

This verifies the Riemann integrability condition and it also completes the proof. ■

**Theorem 11.2 – Additivity of integrals**

Suppose that $a < c < b$. If a function $f$ is integrable on both $[a, c]$ and $[c, b]$, then the function $f$ is integrable on $[a, b]$ and one has the identity

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

**Proof.** Let $\varepsilon > 0$ be given. Then there exists a partition $P_1$ of $[a, c]$ such that

$$U(f, P_1) - L(f, P_1) < \varepsilon/2$$

and there similarly exists a partition $P_2$ of $[c, b]$ such that

$$U(f, P_2) - L(f, P_2) < \varepsilon/2.$$

The union $P = P_1 \cup P_2$ is then a partition of $[a, b]$ which is easily seen to satisfy

$$L(f, P) = L(f, P_1) + L(f, P_2),$$
$$U(f, P) = U(f, P_1) + U(f, P_2).$$

In particular, one has $U(f, P) - L(f, P) < \varepsilon$ and $f$ is integrable on $[a, b]$ as well.

Finally, we prove the identity for the integral of $f$. Given any partition $P_1$ of $[a, c]$ and any partition $P_2$ of $[c, b]$, the definition of the integral gives

$$L(f, P_1) \leq \int_a^c f(x)\,dx \leq U(f, P_1), \qquad L(f, P_2) \leq \int_c^b f(x)\,dx \leq U(f, P_2).$$

Once we now add these equations together, we may conclude that

$$L(f, P) \leq \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \leq U(f, P),$$

where $P = P_1 \cup P_2$ is a partition of $[a, b]$. On the other hand, we must also have

$$L(f, P) \leq \int_a^b f(x)\,dx \leq U(f, P)$$

by the definition of the integral. As we have already seen, the left hand side $L(f, P)$ and the right hand side $U(f, P)$ can be made arbitrarily close to one another. Thus, any two expressions that lie between them must be equal and the proof is complete. ■

**Theorem 11.3 – Sums of integrable functions**

Suppose that $f, g$ are integrable on $[a, b]$. Then $f + g$ is also integrable on $[a, b]$ and

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

**Proof.** We need to relate the lower/upper Darboux sums for the functions $f, g$ with the corresponding ones for their sum $f+g$. In order to do this, we shall first show that

$$\inf\{f(x)+g(x) : x \in A\} \geq \inf\{f(x) : x \in A\} + \inf\{g(x) : x \in A\}, \tag{A1}$$
$$\sup\{f(x)+g(x) : x \in A\} \leq \sup\{f(x) : x \in A\} + \sup\{g(x) : x \in A\}. \tag{A2}$$

These identities are valid for any nonempty set $A \subseteq [a,b]$. In fact, each $x_0 \in A$ satisfies

$$f(x_0) \geq \inf\{f(x) : x \in A\}, \qquad g(x_0) \geq \inf\{g(x) : x \in A\}$$

and one may simply add these equations together to conclude that

$$f(x_0) + g(x_0) \geq \inf\{f(x) : x \in A\} + \inf\{g(x) : x \in A\}.$$

Since the right hand side is a lower bound for all possible sums $f(x_0)+g(x_0)$ with $x_0 \in A$, it is smaller than the greatest lower bound and (A1) follows. The proof of (A2) is similar.

We now use (A1)-(A2) along with the definition of Darboux sums to show that

$$L(f+g,P) \geq L(f,P) + L(g,P), \tag{B1}$$
$$U(f+g,P) \leq U(f,P) + U(g,P) \tag{B2}$$

for any partition $P = \{x_0, x_1, \ldots, x_n\}$. When it comes to lower Darboux sums, one has

$$\begin{aligned} L(f+g,P) &= \sum_{k=0}^{n-1} \inf\{f(x)+g(x) : x_k \leq x \leq x_{k+1}\} \cdot (x_{k+1}-x_k) \\ &\geq \sum_{k=0}^{n-1} \inf\{f(x) : x_k \leq x \leq x_{k+1}\} \cdot (x_{k+1}-x_k) \\ &\quad + \sum_{k=0}^{n-1} \inf\{g(x) : x_k \leq x \leq x_{k+1}\} \cdot (x_{k+1}-x_k) \end{aligned}$$

because of (A1). This already establishes (B1), while the proof of (B2) is similar.

Next, we show that the function $f+g$ is integrable. Let $\varepsilon > 0$ be given. Since $f, g$ are both integrable on $[a,b]$, there exist partitions $P, Q$ such that

$$U(f,P) - L(f,P) < \varepsilon/2,$$
$$U(g,Q) - L(g,Q) < \varepsilon/2.$$

Consider their refinement $R = P \cup Q$. According to Theorem 10.3, this satisfies

$$U(f,R) - L(f,R) \leq U(f,P) - L(f,P) < \varepsilon/2,$$
$$U(g,R) - L(g,R) \leq U(g,Q) - L(g,Q) < \varepsilon/2.$$

Once we now combine these inequalities with (B1) and (B2), we find that

$$U(f+g,R) - L(f+g,R) \leq [U(f,R) - L(f,R)] + [U(g,R) - L(g,R)] < \varepsilon.$$

This verifies the Riemann integrability condition, so $f+g$ is integrable as well.

Finally, we establish the formula for the integral of $f+g$. On one hand, we have

$$\begin{aligned} L(f,P)+L(g,P) \leq L(f+g,P) &\leq \int_a^b [f(x)+g(x)]\,dx \\ &\leq U(f+g,P) \leq U(f,P)+U(g,P) \end{aligned}$$

by (B1), (B2) and the definition of the integral. On the other hand, we also have

$$L(f,P)+L(g,P) \leq \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \leq U(f,P)+U(g,P)$$

by the definition of the integral. Now, the left hand side $L(f,P)+L(g,P)$ and the right hand side $U(f,P)+U(g,P)$ can be made arbitrarily close to one another. Thus, any two expressions that lie between them must be equal and the proof is complete. ■

**Theorem 11.4 – Multiples of integrable functions**

Suppose that $f$ is integrable on $[a,b]$ and $c \in \mathbb{R}$. Then $cf$ is also integrable on $[a,b]$ and

$$\int_a^b [cf(x)]\,dx = c\int_a^b f(x)\,dx.$$

**Proof.** When $c=0$, the result follows by Example 10.5 since $cf$ is constant in that case. Suppose now that $c>0$. Given any partition $P=\{x_0,x_1,\ldots,x_n\}$ of $[a,b]$, one has

$$\begin{aligned} L(cf,P) &= \sum_{k=0}^{n-1} \inf\{cf(x) : x_k \leq x \leq x_{k+1}\}\cdot(x_{k+1}-x_k) \\ &= c\sum_{k=0}^{n-1} \inf\{f(x) : x_k \leq x \leq x_{k+1}\}\cdot(x_{k+1}-x_k) = cL(f,P) \end{aligned}$$

and similarly $U(cf,P)=cU(f,P)$. To show that $cf$ is integrable, let $\varepsilon>0$ be given. Using our assumption that $f$ is integrable, one may find a partition $P$ such that

$$U(f,P)-L(f,P)<\varepsilon/c \quad\Longrightarrow\quad U(cf,P)-L(cf,P)<\varepsilon.$$

This implies that $cf$ is integrable as well. In addition, the integral of $cf$ is given by

$$\begin{aligned} \int_a^b [cf(x)]\,dx &= \sup\{L(cf,P) : P \text{ is a partition of } [a,b]\} \\ &= \sup\{cL(f,P) : P \text{ is a partition of } [a,b]\} \end{aligned}$$

because $c>0$. Using the definition of the integral, we conclude that

$$\int_a^b [cf(x)]\,dx = c\sup\{L(f,P) : P \text{ is a partition of } [a,b]\} = c\int_a^b f(x)\,dx.$$

The case $c < 0$ is somewhat different because inequalities are reversed upon multiplication by a negative number. If it happens that $c < 0$, then our previous approach gives

$$\begin{aligned} L(cf, P) &= \sum_{k=0}^{n-1} \inf \{cf(x) : x_k \leq x \leq x_{k+1}\} \cdot (x_{k+1} - x_k) \\ &= c \sum_{k=0}^{n-1} \sup \{f(x) : x_k \leq x \leq x_{k+1}\} \cdot (x_{k+1} - x_k) = cU(f, P) \end{aligned}$$

and similarly $U(cf, P) = cL(f, P)$. This still implies that $cf$ is integrable because

$$U(cf, P) - L(cf, P) = cL(f, P) - cU(f, P) = |c| \cdot [U(f, P) - L(f, P)].$$

To show that the formula for the integral of $cf$ remains valid, we note that

$$\begin{aligned} \int_a^b [cf(x)]\, dx &= \sup \{L(cf, P) : P \text{ is a partition of } [a, b]\} \\ &= \sup \{cU(f, P) : P \text{ is a partition of } [a, b]\} \end{aligned}$$

as before. Since $c < 0$ in this case, however, the last equation gives

$$\int_a^b [cf(x)]\, dx = c \inf \{U(f, P) : P \text{ is a partition of } [a, b]\} = c \int_a^b f(x)\, dx$$

because the integral is defined in terms of both lower and upper Darboux sums. ■

**Theorem 11.5 – Integrals and inequalities**

Suppose that $f, g$ are integrable on $[a, b]$ and that $f(x) \geq g(x)$ for all $x \in [a, b]$. Then

$$\int_a^b f(x)\, dx \geq \int_a^b g(x)\, dx.$$

**Proof.** Consider the difference of the two functions $h = f - g = f + (-g)$. Since $f, g$ are both integrable on $[a, b]$, the same is true for $h$ by the last two theorems and

$$\int_a^b h(x)\, dx = \int_a^b [f(x) - g(x)]\, dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx.$$

To prove the given inequality, we need to show that this expression is non-negative. Now, let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ and consider the lower Darboux sum

$$L(h, P) = \sum_{k=0}^{n-1} m_k (x_{k+1} - x_k), \qquad m_k = \inf \{h(x) : x_k \leq x \leq x_{k+1}\}.$$

Since $h(x) = f(x) - g(x) \geq 0$ for all $x \in [a, b]$ by assumption, one has $m_k \geq 0$ for each $k$. This implies that $L(h, P) \geq 0$ for any partition $P$ and thus $\int_a^b h(x)\, dx \geq 0$ as well. ■