The cohomology of twisted coalgebras in species

Mariano Suárez-Álvarez and Pedro Tamaroff

Contents

§1. The monoidal category of species ................................. 2
§2. The exponential species, its representations and their cohomology 3
§3. Cohomology ............................................................. 5
§4. An alternative description for cohomology ............................ 11
§5. Products ................................................................. 19
§6. Künneth formulas ....................................................... 25
§7. Free species with restrictions and the boolean species .................. 27
§8. Examples ................................................................. 29
§9. Deformations ........................................................... 36
§1. The monoidal category of species

(1.1) We write $\text{Fin}$ the category whose objects are the finite sets and whose morphisms are the bijective functions. If $\mathcal{C}$ is a category, then the category of species in $\mathcal{C}$ is the functor category $\text{Fun}(\text{Fin}^{\text{op}}, \mathcal{C})$ of contravariant \(^1\) functors $\text{Fin}^{\text{op}} \to \mathcal{C}$, and we write it simply $\text{Sp}(\mathcal{C})$. If $\mathcal{C} = \text{Set}$, we write $\text{Sp}$ instead of $\text{Sp(\text{Set})}$ and call its objects species in sets or set-valued species. On the other hand, if $\mathcal{C} = \mathbb{k}\text{Mod}$, the category of $\mathbb{k}$-modules, we write $\mathbb{k}\text{Sp}$ instead of $\text{Sp}(\mathbb{k}\text{Mod})$ and call its objects linear species; being a category of functors from a small category into a Grothendieck category, this category is itself a Grothendieck category.

Let $\mathbb{k}(-) : \text{Set} \to \mathbb{k}\text{Mod}$ be the functor which maps a set $I$ to the module $\mathbb{k}I$ freely spanned by $I$ and each function $\tau : I \to J$ in $\text{Set}$ to the linear function $k\tau : \mathbb{k}I \to \mathbb{k}J$ which restricts to $\tau$. Composition with $\mathbb{k}(-)$ gives a functor $\mathbb{k}\text{Sp} \to \mathbb{k}\text{Sp}$. If $x \in \text{Sp}$, we write $\mathbb{k}x$ its image in $\mathbb{k}\text{Sp}$ under this functor and call it the linearization of $x$.

(1.2) If $x$ and $y$ are linear species, there is a linear species $x \otimes y$ such that whenever $I$ and $J$ are objects of $\text{Fin}$ and $\tau : I \to J$ is a morphism in that category we have

$$ (x \otimes y)[I] = \bigoplus_{(S,T)\vdash I} x[S] \otimes y[T], \quad (x \otimes y)[\tau] = \bigoplus_{(S,T)\vdash I} x[S] \otimes y[\tau(T)]. $$

Here $\tau_S$ denotes the restriction of $\tau$ to a bijection $\tau \to \tau(S)$; applying $x$ to it we get a linear map $x[\tau_S] : x[f(S)] \to x[S]$, and similarly for $\tau_T$, so that we have a linear map $x[\tau_S] \otimes y[\tau_T] : x[\tau(S)] \otimes x[\tau(T)] \to x[S] \otimes x[T]$.

The morphism $(x \otimes y)[\tau]$ described above is then the direct sum

$$ \bigoplus_{(S,T)\vdash I} x[\tau_S] \otimes y[\tau_T] : \bigoplus_{(S,T)\vdash I} x[\tau(S)] \otimes x[\tau(T)] \to \bigoplus_{(S,T)\vdash I} x[S] \otimes x[T], $$

whose domain is $(x \otimes y)[J]$, up to the permutation of the direct summands induced by the bijection $\tau$, and whose codomain is equal to $(x \otimes y)[I]$.

If $u : x \to x'$ and $v : y \to y'$ are morphisms of linear species, there is a morphism $u \odot v : x \otimes y \to x' \otimes y'$ of linear species such that for each $I \in \text{Fin}$ we have

$$ (u \odot v)[I] = \bigoplus_{(S,T)\vdash I} u[S] \otimes v[T]. $$

In this way we obtain a bilinear bifunctor $\otimes : \mathbb{k}\text{Sp} \times \mathbb{k}\text{Sp} \to \mathbb{k}\text{Sp}$, which is in fact the product in a symmetric monoidal structure $\langle \mathbb{k}\text{Sp}, \otimes, \mathbb{k}1, a, l, r, c \rangle$ on the category $\mathbb{k}\text{Sp}$,

\(^1\)Species are usually defined to be covariant functors on $\text{Fin}$. Since $\text{Fin}$ is isomorphic to its opposite category, our choice does not change anything significant, and simplifies some considerations that we make below.
with unit object the linearization of the set-valued species 1 such that $1[\emptyset] = \{\emptyset\}$, $1[I] = \emptyset$ for all non-empty $I \in \text{Fin}$, and $1[\tau] : 1[J] \to 1[I]$ the identity map for all bijections $\tau : I \to J$ of $\text{Fin}$, and associator $\alpha$, unitors $l$ and $r$ and symmetry $c$ induced in the obvious way from those of the usual monoidal category $\text{Mod}$. We call this the Cauchy product. The category $\text{Sp}$ is therefore an abelian symmetric monoidal category, as in [2, Section 1.1.4], which we will pretend to be strict. In particular, we can talk about (co)algebras in $\text{Sp}$, which we call twisted (co)algebras, their (bi)(co)modules and so on. Many twisted coalgebras arise from a sequence of combinatorial structures, such as matroids, for example, which admit restriction and contraction operations. We like to think of these as combinatorial twisted coalgebras.

Let $x$ be a species, $I$ a finite set and $\tau : J \to I$ a bijection. It will be notationally useful to simply write $\tau z$ for $x[\tau](z)$ whenever $z \in x[I]$.

§2. The exponential species, its representations and their cohomology

(2.1) There is species in sets $e$, the exponential set-valued species, which maps each $I \in \text{Fin}$ to the set $e[I] = \{I\}$ and each function $f : I \to J$ in $\text{Fin}$ to the unique function $e[f] : e[J] \to e[I]$; if $I \in \text{Fin}$, we will write $e_I$ for the element of the set $e[I]$.

The linearization $k e$, which we call simply the exponential species and will simply write $e$, is a commutative and cocommutative bimonoid in $\text{Sp}$ with respect to the multiplication, unit, comultiplication and counit given by the morphisms $\mu : e \otimes e \to e$, $\eta : k 1 \to e$, $\Delta : e \to e \otimes e$ and $\varepsilon : e \to k 1$ of $\text{Sp}$ such that for each $I \in \text{Fin}$

$$\mu[I](e_S \otimes e_T) = e_I \quad \text{if } (S, T) \prec I,$$

$$\eta[\emptyset](1) = e_\emptyset,$$

$$\Delta[I](e_I) = \sum_{(S,T) \vdash I} e_S \otimes e_T,$$

$$\varepsilon[I](e_I) = \begin{cases} 1, & \text{if } I = \emptyset; \\ 0, & \text{if not}. \end{cases}$$

(2.2) The category $\text{Mod}$ of left $e$-comodules admits a straightforward description as a category of functors:

Lemma. Let $\text{Fin}_\leq$ the category of finite sets and injective functions, and for each $I \in \text{Fin}$ and each subset $T \subseteq I$ denote $\iota_T : T \to I$ the inclusion function.

(i) If $(x, \lambda)$ is a left $e$-comodule, so that $x$ is a linear species and $\lambda : x \to e \otimes x$ is a morphism in $\text{Sp}$ giving a counitary and coassociative coaction of $e$ on $x$, there is a unique functor $x^\lambda : \text{Fin}_\leq^{\text{op}} \to \text{Mod}$ whose restriction to $\text{Fin}$ is $x$ and such that whenever $I \in \text{Fin}$ and $z \in x[I]$ we have

$$\lambda[I](z) = \sum_{(S,T) \vdash I} e_S \otimes x^\lambda[I](z) \in \bigoplus_{(S,T) \vdash I} e[S] \otimes x[T] = (e \otimes x)[I].$$
(ii) There is an equivalence of categories

\[ \text{eMod} \to \text{Fun}(\text{Fin}^{\text{op}}, \text{}_k\text{Mod}) \]

which maps a left e-comodule \((x, \lambda)\) to the functor \(x^\lambda\) and which is the identity on morphisms.

If \((p, \lambda)\) is a left e-comodule and \(I \in \text{Fin}\), \(z \in x[I]\) and \(T \subseteq I\), we will write \(z \Downarrow T\) instead of \(x^\lambda[I](z)\) and call it the right restriction of \(z\) to \(T\). It follows at once that the expression \(z \Downarrow T\) depends linearly on \(z\), that \(\lambda[I](z) = \sum_{(S,T) \vdash I} e_S \otimes (z \Downarrow T)\) (1) and that for each bijection \(f : J \to I\) in \(\text{Fin}\) and each subset \(U \subseteq J\) we have

\[ x[f](z) \Downarrow U = x[f|_U](z \Downarrow f(U))\] (2)

and counitarity and coassociativity of the coaction \(\lambda\) imply that

\[ z \Downarrow I = z \] (3)

and that for all \(U \subseteq T\) we have

\[ (z \Downarrow T) \Downarrow U = z \Downarrow U. \] (4)

Conversely, if \(x\) is a linear species and for each \(I \in \text{Fin}\) and each subset \(T \subseteq I\) we have a linear map \(z \in x[I] \to z \Downarrow T \in x[T]\) such that the conditions (2), (3) and (4) are satisfied, then there is a coaction \(\lambda : x \to \text{e} \otimes x\) of \(\text{e}\) on \(x\) given by the formula (1). It is in this way that a left \(\text{e}\)-comodule structure on a linear species and a choice of right restrictions on it amount to the same thing. From now on, we will write a left \(\text{e}\)-comodules \((x, \lambda)\) simply as \(x\) and work almost exclusively with the corresponding right restrictions.

(2.3) Lemma (2.2) suggests the following definition: we call a functor \(x : \text{Fin} \to \text{Set}\) a set-valued species with right restrictions. As in the linear case, such an functor is determined by the set-valued species \(x|\text{Fin}\) obtained by restricting \(x\) to the subcategory \(\text{Fin}\) together with a choice, for each \(I \in \text{Fin}\) and each subset \(S \subseteq I\), of a map \(? \Downarrow S : x[I] \to x[S]\) such that

\[ x[f](z) \Downarrow U = x[f|_U](z \Downarrow f(U)), \]

\[ z \Downarrow I = z, \]

\[ (z \Downarrow S) \Downarrow T = z \Downarrow T \]

whenever \(I \in \text{Fin}\), \(z \in x[I]\), \(T \subseteq S \subseteq I\) and \(f : J \to I\) is a bijection.

The linearization \(kx\) of a set-valued species with right restrictions gives rise in an obvious way to a functor \(\text{Fin} \to \text{}_k\text{Mod}\) and, in view of the lemma, to a left \(\text{e}\)-comodule.
We consider now the category $\mathbf{eMod}_\mathbf{e}$ of $\mathbf{e}$-bicomodules. Its objects can be described in terms similar to those used in (2.2): a $\mathbf{e}$-bicomodule can be viewed as a linear species $x$ endowed with a choice, for each $I \in \text{Fin}$ and each subset $S \subseteq I$, of two linear functions

$$z \in x[I] \mapsto z \downharpoonleft S \in x[S], \quad z \in x[I] \mapsto z \upharpoonright S \in x[S],$$

which we call left and the right restrictions, such that for every $I \in \text{Fin}$ and every $z \in x[I]$ we have

- $x[f](z) \downharpoonleft U = x[f|_{f(U)}](z \downharpoonleft f(U))$ and $x[f](z) \upharpoonright U = x[f|_{f(U)}](z \upharpoonright f(U))$ whenever $f : J \rightarrow I$ is a bijection in $\text{Fin}$ and $U \subseteq J$,
- $z \downharpoonleft T \upharpoonleft U = z \downharpoonleft U$ and $z \upharpoonright T \upharpoonright U = z \upharpoonright U$ whenever $U \subseteq T \subseteq I$,
- $z \downharpoonleft I = z \upharpoonright I$, and
- $z \downharpoonleft (S \cup T) \upharpoonleft T = z \upharpoonright (T \cup U) \downharpoonleft T$ whenever $(S,T,U)$ is a decomposition of $I$.

Indeed, using these restrictions maps we can construct a left coaction $\lambda : x \rightarrow \mathbf{e} \otimes x$ and a right coaction $\rho : x \rightarrow x \otimes \mathbf{e}$ putting, for each $I \in \text{Fin}$ and each $z \in x[I]$,

$$\lambda[I](z) = \sum_{(S,T) \vdash I} e_S \otimes z \upharpoonright T, \quad \rho[I](z) = \sum_{(S,T) \vdash I} z \downharpoonleft S \otimes e_T,$$

and these coactions turn $x$ into a $\mathbf{e}$-bicomodule. For example, when we view the comonoid $\mathbf{e}$ itself as a $\mathbf{e}$-bicomodule in the canonical way, the corresponding left and right restrictions are such that for each $I \in \text{Fin}$ and each subset $S \subseteq I$ we have $e_I \downharpoonleft S = e_S$ and $e_I \upharpoonright S = e_S$.

As with left comodules, there is a linearization process for bicomodules: a set-valued species with left and right restrictions is a set-valued species $x$ endowed with a choice, for each $I \in \text{Fin}$ and each subset $S \subseteq I$, of functions

$$z \in x[I] \mapsto z \downharpoonleft S \in x[S], \quad z \in x[I] \mapsto z \upharpoonright S \in x[S],$$

subject to the same four conditions as before. The linearization $kx$ can be turned into a $\mathbf{e}$-bicomodule in the obvious way. Most of the bicomodules that interest us arise like this; the canonical bicomodule structure $\mathbf{e}$ is of this form, for example.

§3. Cohomology

(3.1) We put on the category $\mathbf{eMod}_\mathbf{e}$ the exact structure consisting of those short exact sequences which split in $\mathbf{kSp}$. If $x$ is a $\mathbf{e}$-bicomodule, then the cohomology of $x$ is

$$H^\bullet(x) = \text{Ext}^\bullet(x, \mathbf{e})$$

with the $\text{Ext}$ taken in $\mathbf{eMod}_\mathbf{e}$ and the second argument $\mathbf{e}$ viewed as a $\mathbf{e}$-bicomodule in the usual way. This cohomology can be computed using a standard complex, which we now describe.
There is a cosimplicial $e$-bicomodule $B^i$ with
- $B^p = e^\otimes(p+2)$ for each $p \in \mathbb{N}_0$, viewed as an $e$-bicomodule with the left and right “regular” coactions

$$\Delta \otimes \text{id}_e^{(p+1)} : B^p \to e \otimes B^p, \quad \text{id}_e^{(p+1)} \otimes \Delta : B^p \to B^p \otimes e;$$

- coface morphisms $d_i^p : B^{p-1} \to B^p$ for each $p \in \mathbb{N}$ and each $i \in [0, p]$ given by

$$d_i^p = \text{id}_e^{(i+1)} \otimes \Delta \otimes \text{id}_e^{(p-i+1)};$$

- codegeneracy morphisms $s_i^p : B^{p+1} \to B^p$ for each $p \in \mathbb{N}_0$ and each $i \in [0, p]$ given by

$$s_i^p = \text{id}_e^{(i+1)} \otimes \varepsilon \otimes \text{id}_e^{(p-i+1)}.$$

For each $p \geq 0$ one can see easily that the object $B^p$ is injective in the exact category $\mathcal{E} \text{Mod}^e$, that the cochain complex $B^\bullet$ associated to $B^i$ is exact in positive degrees and that $H^*(B^\bullet) \cong e$. This means that $B^\bullet$ is an injective resolution of $e$ in the category of $e$-bicomodules, and we therefore have for each $e$-bicomodule $x$ a canonical isomorphism

$$H^*(x) \cong H(\text{hom}(x, B^\bullet)), \quad (5)$$

with $\text{hom}$ being that of the category $\mathcal{E} \text{Mod}^e$.

The cosimplicial module $\text{hom}(x, B^\bullet)$ whose cohomology appears on the left in (5) has an alternative description: it is isomorphic in a canonical way to the cosimplicial module $C^i(x)$ constructed as follows:

- The module of $p$-cochains is $C^p(x) = \text{hom}(x, e^\otimes p)$ for each $p \geq 0$, with the $\text{hom}$ taken now in the category $\mathfrak{s}\mathfrak{p}$. A $p$-cochain in $C^i(x)$ is therefore a morphism $\alpha : x \to e^\otimes p$ of linear species and it is determined by a choice of a linear function $[\alpha](S_1, \ldots, S_p) : x[I] \to k$ for each decomposition $(S_1, \ldots, S_p)$ of each finite set $I \in \text{Fin}$ in such a way that the equivariance condition that

$$[\alpha](\tau(S_1), \ldots, \tau(S_p))(z) = [\alpha](S_1, \ldots, S_p)(\tau z)$$

whenever $\tau : J \to I$ is a bijection in $\text{Fin}$ and $z \in x[I]$ holds; we then have

$$\alpha[I](z) = \sum_{(S_1, \ldots, S_p) \vdash I} [\alpha](S_1, \ldots, S_p)(z) e_{S_1} \otimes \cdots \otimes e_{S_p}$$

for each $I \in \text{Fin}$ and each $z \in x[I]$.

- The coface morphism $d_i^p : C^{p-1}(x) \to C^p(x)$, for each $p \in \mathbb{N}$ and each $i \in [0, p]$, is such that the coefficients of $d_i^p(\alpha)$, if $\alpha : x \to e^\otimes p$ is a $p$-cochain, are given by
This cohomology theory has, as usual, a simple description in low degrees. We whose cohomology modules are naturally isomorphic to is a short complex of modules whenever The normalized subcomplex \( N \) have a complex. We will show that each of its two cohomology modules are as described. Let \( \text{Proposition.} \) alternative description of cohomology that we will provide later.

The codegeneracy morphism \( s_i^p : C^{p+1}(x) \to C^p(x) \), for each \( p \in \mathbb{N}_0 \) and each \( i \in [0, p] \), is the function that maps a \((p + 1)\)-cochain \( \alpha : x \to e_{\leq p+1} \) to the \( p\)-cochain \( s_i^p(\alpha) : x \to e_{\leq p} \) whose coefficients are given by

\[
\| s_i^p(\alpha) \| (S_1, \ldots, S_p)(z) = \| \alpha \| (S_1, \ldots, S_i, a, S_{i+1}, \ldots, S_p)(z)
\]

for each decomposition \( (S_1, \ldots, S_p) \) of a finite set \( I \in \text{Fin} \) and each \( z \in x[I] \).

The normalized subcomplex \( N^\bullet(x) \) of the cochain complex \( C^\bullet(x) \) associated to the cosimplicial module \( C^\bullet(x) \) has as elements the cochains \( \alpha : x \to e_{\leq p} \) such that

\[
\| \alpha \| (S_1, \ldots, S_p) = 0
\]

whenever \( (S_1, \ldots, S_p) \) is a decomposition of a finite set \( I \in \text{Fin} \) with at least one empty block. Of course, the inclusion \( N^\bullet(x) \to C^\bullet(x) \) induces an isomorphism in cohomology and whenever convenient we may restrict our attention to normalized cochains.

(3.2) This cohomology theory has, as usual, a simple description in low degrees. We include the details, which are somewhat long, in order to hint at what is behind the alternative description of cohomology that we will provide later.

**Proposition.** Let \( x \) be a \( e \)-bicomodule and let \( \tau : \mathbb{I} \to \mathbb{I} \) be the transposition. There is a short complex of modules

\[
0 \longrightarrow x[0]^* \xrightarrow{\delta^0} x[1]^* \xrightarrow{\delta^1} x[2]^*
\]

with maps given by

\[
\delta^0(\alpha)(z) = a(z \cup \emptyset) - a(z \cap \emptyset) \quad \text{if } \alpha \in x[0]^* \text{ and } z \in x[1],
\]

\[
\delta^1(\alpha)(z) = a(z \cup \mathbb{I}) - a(z \cap \mathbb{I}) - a(x[\tau](z) \cup \mathbb{I}) + a(x[\tau](z) \cap \mathbb{I}) \quad \text{if } \alpha \in x[1]^* \text{ and } z \in x[2],
\]

whose cohomology modules are naturally isomorphic to \( H^0(x) \) and to \( H^1(x) \).

**Proof.** A computation which we omit shows that \( \delta^1 \circ \delta^0 = 0 \), and this means that we do have a complex. We will show that each of its two cohomology modules are as described.
As $k1[I] = 0$ for all nonempty sets $I$ in $\text{Fin}$, a 0-cochain $\alpha : x \to k1$ in the normalized complex $N^*(x)$ is completely determined by the linear map

$$\Phi^0(\alpha) : z \in x[[0]] \mapsto \|\alpha\|(z) \in k,$$

and this means that the function

$$\Phi^0 : \alpha \in N^0(x) \to \Phi^0(\alpha) \in x[[0]]^*$$

is injective. It is in fact a bijection, as one sees at once. To prove the claim of the proposition about $H^0(x)$ it is enough that we show that this map $\Phi^0$ restricts to an isomorphism from the kernel of $\delta^0$ to that of $d^0$. A 0-cochain $\alpha \in N^0(x)$ is 0-cocycle in $N^* (x)$ exactly when

$$\|d^0(\alpha)\|(I)(z) = \|\alpha\|(z^I_1) - \|\alpha\|(z^I_2) = 0$$

for all $I \in \text{Fin}$ and all $z \in x[I]$, and if that is the case it is clear that $d^0(\Phi^0(\alpha)) = 0$. We are therefore left with proving that every element in the kernel of $\delta^0$ is the image under $\Phi^0$ of a 0-cocycle in $N^* (x)$. Let then $a \in x[[0]]^*$ be such that $\delta^0(a) = 0$, so that $a(z \emptyset) = a(z \emptyset)$ for all $z \in x[I]$. We claim that

$$a(z \emptyset) = a(z \emptyset \{i\} \emptyset) \text{ if } I \in \text{Fin is nonempty, } i \in I \text{ and } z \in x[I]. \quad (6)$$

To see this we will prove more generally by descending induction on $r$ that

if $n \in \mathbb{N}$ and $I = \{i_1, \ldots, i_n\}$ has exactly $n$ elements, and we put $I_r = \{i_1, \ldots, i_r\}$ for each $r \in [n]$, then $a(z \| I_r \emptyset) = a(z \emptyset)$ for all $z \in x[I]$ and all $r \in [n]$; \quad (7)

the special case in which $r = 1$ gives (6). Let then $n$ and $I$ be as in (7). The equality asserted there is clear if $r = n$, for in that case $z \| I_r = z \| I = z$. On the other hand, If we suppose inductively that the equality holds for some $r \in [2, n]$, then we have that

$$a(z \| I_{r-1} \emptyset) = a(z \| I_r \| I_{r-1} \emptyset)$$

$$= a(z \| I_r \emptyset \{i_r\} \emptyset)$$

$$= a(x[lab,](z \| I_r \emptyset \{i_r\}) \emptyset)$$

and, since $x[lab,](z \| I_r \emptyset \{i_r\}) \in x[1]$ and in view on the hypothesis on $a$, this is

$$= a(x[lab,](z \| I_r \emptyset \{i_r\} \emptyset))$$

$$= a(z \| I_r \emptyset \{i_r\} \emptyset)$$

8
\[ a(z \parallel I_r \setminus \emptyset) = a(z \setminus \emptyset), \]

using the inductive hypothesis in the last step.

Having thus established the truth of our claim (6) above, let us show now that the 0-cochain \( \alpha : x \to 1 \) in \( N^0(x) \) such that \( \Phi^0(\alpha) = a \) is a 0-cocycle. Let \( I \in \text{Fin} \) and let \( z \in x[I] \). We have

\[ \|d^0(\alpha)\|(I)(z) = \|a\|(z \parallel \emptyset) - \|a\|(z \setminus \emptyset) = a(z \parallel \emptyset) - a(z \setminus \emptyset) \quad (8) \]

If \( I = \emptyset \), the rightmost member in this equality is zero, as \( z \parallel \emptyset = z = z \setminus \emptyset \) in that case. If instead \( I \) is not empty and \( i \) is any element of \( I \), then using (6) we see that

\[
\begin{align*}
a(z \parallel \emptyset) - a(z \setminus \emptyset) &= a(z \parallel \{i\} \parallel \emptyset) - a(z \parallel \{i\} \setminus \emptyset) \\
&= a(x[\text{lab}]_i(z \parallel \{i\}) \parallel \emptyset) - a(x[\text{lab}]_i(z \parallel \{i\}) \setminus \emptyset)
\end{align*}
\]

and, since \( x[\text{lab}]_i(z \parallel \{i\}) \) is an element of \( x[1] \), the hypothesis on \( a \) implies that this is

\[ = 0. \]

In any case, the rightmost member of the equality (8) is zero and we may therefore conclude that \( d^0(\alpha) = 0 \), as we wanted.

We deal now with \( H^1(x) \). Let us fix a 1-cocycle \( \alpha : x \to e \) in the normalized complex \( N^\bullet(x) \) and consider the function

\[ \Phi^1(\alpha) : z \in x[1] \mapsto \|a\|(\{1\})(z) \in k. \]

Since \( \alpha \) is normalized, we have that \( a[\emptyset] = 0 \); on the other hand, that \( \alpha \) is a 1-cocycle means that whenever \( I \in \text{Fin}, \; z \in x[I] \) and \( (S, T) \vdash I \) we have

\[ \|a\|(I)(z) = \|a\|(T)(z \parallel T) + \|a\|(S)(z \setminus S). \quad (9) \]

In particular, taking \( I = \{2\} \) and \( z \in x[2] \) we have for the decomposition \( \{1\}, \{2\} \) of \( I \) that

\[
\begin{align*}
\|a\|(\{2\})(z) &= \|a\|(\{2\})(z \parallel \{2\}) + \|a\|(\{1\})(z \setminus \{1\}) \\
&= \|a\|(\{1\})(x[\sigma](z) \parallel \{1\}) + \|a\|(\{1\})(z \setminus \{1\}) \\
&= \Phi^1(\alpha)(x[\sigma](z) \parallel \{1\}) + \Phi^1(\alpha)(z \setminus \{1\})
\end{align*}
\]

and, similarly, for the decomposition \( \{2\}, \{1\} \) that

\[
\begin{align*}
\|a\|(\{2\})(z) &= \Phi^1(\alpha)(z \parallel \{1\}) + \Phi^1(\alpha)(x[\sigma](z) \setminus \{1\}),
\end{align*}
\]
and these two equalities imply that
\[
\Phi^1(\alpha)(x[\sigma](z) \parallel \{1\}) + \Phi^1(\alpha)(z \parallel \{1\}) = \Phi^1(\alpha)(z \parallel \{1\}) + \Phi^1(\alpha)(x[\sigma](z) \parallel \{1\}),
\]
which means precisely that \(\Phi^1(\alpha)\) belongs to the kernel of \(\delta^1\). The map \(\Phi^1\) thus restricts to a map
\[
\Phi^1 : Z^1(N^\bullet(x)) \to \ker \delta^1
\]
on the module of 1-cycles in \(N^\bullet(x)\). We want to show that this restriction is a bijection.

Let \(\alpha : x \to e\) be an element of \(Z^1(N^\bullet(x))\). Let \(I \in \text{Fin}\) be a nonempty finite set, let \(n = |I|\) and suppose that \(\sigma : [n] \to I\) is a bijection. For each \(j \in [n]\) put \(I_j = \sigma([j,n])\) and \(S_j = \{\sigma(j)\}\), and for each \(z \in x[I]\) set \(u_j(\sigma,z) = x[\text{lab}_{\sigma(j)}](z \parallel I_j \setminus S_j) \in x[1]\). Using the relation (9) and the naturality of \(\alpha\), we see that for each \(z \in x[I]\) we have
\[
\|\alpha\|(I)(z) = \sum_{j \in [n]} \|\alpha\|(S_j)(z \parallel I_j \setminus S_j) = \sum_{j \in [n]} \Phi^1(\alpha)(u_j(\sigma,z)).
\]
It follows from this and normalization that the 1-cocycle \(\alpha\) is completely determined by the function \(\Phi^1(\alpha)\): the function \(\Phi^1\) is therefore injective.

To show that it is surjective, let now \(a \in x[1]^*\) be an element of \(\ker \delta^1\). Let \(I\) be a nonempty finite set of cardinal \(n = |I|\) and let \(z \in x[I]\). We claim that
\[
\text{if } \sigma, \sigma' : [n] \to I \text{ are bijections then } \sum_{j=1}^n a(u_j(\sigma,z)) = \sum_{j=1}^n a(u_j(\sigma',z)). \tag{11}
\]
To prove this, it is enough to consider the case in which \(\sigma\) and \(\sigma'\) differ by a simple transposition, that is, in which there exists a \(k \in [n-1]\) with \(\sigma'(j) = \sigma(j)\) if \(j \in [n] \setminus \{k, k+1\}\), \(\sigma'(k) = \sigma(k+1)\) and \(\sigma'(k+1) = \sigma(k)\). As above, for each \(j \in [n]\) we consider the sets \(I_j = \sigma([j,n])\), \(S_j = \{\sigma(j)\}\), \(I'_j = \sigma'([j,n])\) and \(S'_j = \{\sigma'(j)\}\), so that \(u_j(\sigma,z) = x[\text{lab}_{\sigma(j)}](z \parallel I_j \setminus S_j)\) and \(u_j(\sigma',z) = x[\text{lab}_{\sigma'(j)}](z \parallel I'_j \setminus S'_j)\). As \(I_j = I'_j\) and \(S_j = S'_j\) for all \(j \in [n] \setminus \{k, k+1\}\), we clearly have that \(u_j(\sigma,z) = u_j(\sigma',z)\) for such values of \(j\). It follows from this that in order to show that the equality in (11) holds we have to prove simply that
\[
a(u_k(\sigma,z)) + a(u_{k+1}(\sigma,z)) = a(u_k(\sigma',z)) + a(u_{k+1}(\sigma',z)). \tag{12}
\]
If we put \(\bar{z} = z \parallel I_k \setminus \{\sigma(k), \sigma(k+1)\}\), then
\[
\begin{align*}
z \parallel I_k \setminus S_k &= \bar{z} \parallel \{\sigma(k)\}, & z \parallel I_{k+1} \setminus S_{k+1} &= \bar{z} \parallel \{\sigma(k+1)\}, \\
z \parallel I'_k \setminus S'_k &= \bar{z} \parallel \{\sigma(k+1)\}, & z \parallel I'_{k+1} \setminus S'_{k+1} &= \bar{z} \parallel \{\sigma(k)\}.
\end{align*}
\]
Set \( v = x[\text{lab}(k),\sigma(k+1)](\bar{z}) \in x[2] \). The functoriality of \( x \) implies that

\[
\begin{align*}
u_k(\sigma, z) &= v \sslash \{1\}, & u_{k+1}(\sigma, z) &= x[\sigma](v) \sslash \{1\}, \\u_k(\sigma', z) &= x[\sigma](v) \sslash \{1\}, & u_{k+1}(\sigma', z) &= v \sslash \{1\},
\end{align*}
\]

so that the equality (12) is equivalent to

\[
a(v \sslash \{1\}) + a(x[\tau](v) \sslash \{1\}) = a(x[\tau](v) \sslash \{1\}) + a(v \sslash \{1\}),
\]

which holds because \( \delta^1(a) = 0 \). This proves our claim (11).

It is now easy to see, using that claim, that there is a morphism of linear species \( \alpha : x \rightarrow e \) such that for each \( I \in \text{Fin} \) of cardinal \( n = |I| \), each \( z \in x[I] \) and each bijection \( \sigma : \{n\} \rightarrow I \) we have

\[
\| \alpha \|(I)(z) = \sum_{j=1}^n a(u_j(\sigma, z)),
\]

that it is a 1-cocycle and that \( \Phi^1(\alpha) \) is the function \( a \) with which we started. This concludes the proof that the restriction (10) is surjective.

To complete the proof of the proposition, it remains only that we show that that the image under that restriction of the module 0-coboundaries in \( Z^1(N^\bullet(x)) \) is precisely the image of \( \delta^0 \). We leave that easy task to the reader. \( \square \)

### §4. An alternative description for cohomology

(4.1) The statement of Proposition (3.2) suggests that there should be a certain alternative description of the cohomology \( H^\bullet(x) \) of a \( e \)-bicomodule \( x \), which we now set out to obtain. The plan is to filter \( x \) by certain special subbicomodules and then to proceed as one does with the filtration of a CW-complex by its skeleta. To do this, we will need a few preliminary results.

(4.2) We begin with a very simple observation:

**Proposition.** Let \( r \in \mathbb{N}_0 \). If \( x \) is a \( e \)-bicomodule such that for each \( I \in \text{Fin} \) with \( |I| > r \) we have \( x[I] = 0 \), then \( H^p(x) = 0 \) for all \( p > r \).

**Proof.** Let \( p > r \) and let \( \alpha : x \rightarrow e^\otimes p \) be a normalized \( p \)-cochain in the complex \( N^\bullet(x) \). If \( I \in \text{Fin} \), then

- either \( |I| \leq r \) and therefore every \( p \)-decomposition \((S_1, \ldots, S_p)\) of \( I \) has an empty part, so that \( \|\alpha\|(S_1, \ldots, S_p)(z) = 0 \) for all \( z \in x[I] \) because of normalization,
- or \( |I| > r \) and the map \( \alpha[I] : x[I] \rightarrow e^\otimes p[I] \) is zero simply because its domain is the zero module.

We thus see that the normalized complex \( N^\bullet(x) \) vanishes above degree \( r \) and that of course so does its cohomology. \( \square \)
Next we deal with \( e \)-bicomodules \( x \) which are concentrated in one cardinal, that is, for which there exists an \( r \in \mathbb{N}_0 \) such that \( x[I] = 0 \) whenever \( I \in \text{Fin} \) has cardinal different from \( r \). As the following computation shows, their cohomology is concentrated in one degree and they therefore behave similarly to wedges of spheres.

**Proposition.** Let \( x \) be a \( e \)-bicomodule, let \( r \in \mathbb{N}_0 \) be such that \( x \) is concentrated in cardinal \( r \), and consider the right action of the symmetric group \( S_r \) on \( x[r] \) such that \( z \triangleleft \sigma = x[\sigma](z) \) if \( z \in x[r] \) and \( \sigma \in S_r \). If \( x[r] \) is a projective right \( kS_r \)-module, then \( H^p(x) = 0 \) for all \( p \neq r \) and there is an isomorphism

\[
H^r(x) \xrightarrow{\cong} \text{hom}_{S_r}(x[r], sgn_r),
\]

with \( sgn_r \) denoting the sign representation of \( S_r \) on \( k \). Moreover, if \( \alpha : x \to e^{\otimes r} \) is an \( r \)-cocycle in the complex \( N^*(x) \), the image image under this isomorphism of its cohomology class is the function \( \tilde{\alpha} : x[r] \to sgn_r \) such that

\[
\tilde{\alpha}(z) = \sum_{\sigma \in S_r} sgn(\sigma) \cdot [\alpha](\{1\}, \ldots, \{r\})(x[\sigma](z))
\]

for each \( z \in x[r] \).

**Proof.** Let \( \Sigma_p(r) \) be the augmented semisimplicial set with

- \( \Sigma_p(r) \) the set of all \((p+2)\)-compositions of \([r]\) for each \( p \geq -1 \) and
- boundary maps \( d^p_i : \Sigma_p(r) \to \Sigma_{p-1}(r) \), for each \( p \geq 0 \) and each \( i \in [0, p] \), given by

\[
d^p_i(S_0, \ldots, S_{p+1}) = (S_0, \ldots, S_{i-1}, S_i \cup S_{i+1}, S_{i+2}, \ldots, S_{p+1})
\]

for each \((S_0, \ldots, S_{p+1}) \in \Sigma_p(r) \).

We write \( k\Sigma_*(r) \) for the chain complex corresponding to the semisimplicial set \( \Sigma_p(r) \) with coefficients in \( k \) and \( k\Sigma_*(r)^* \) the dual cochain complex, so that the cohomology of the latter is the reduced cohomology of \( \Sigma_p(r) \). For each \( p \geq -1 \), the symmetric group \( S_r \) acts on the left on the set \( \Sigma_p(r) \) so that

\[
\sigma \triangleright (S_0, \ldots, S_{p+1}) = (\sigma(S_0), \ldots, \sigma(S_{p+1}))
\]

for all \( \sigma \in S_r \) and all \((S_0, \ldots, S_{p+1}) \in \Sigma_p(r) \), and the boundary maps of \( \Sigma_p(r) \) are clearly \( S_r \)-equivariant. This action passes on to complexes, and in particular \( k\Sigma_*(r)^* \) is a cochain complex of right \( S_r \)-modules.

The semisimplicial complex \( \Sigma_p(r) \) is isomorphic to the Coxeter complex of the Coxeter group of type \( A_{r-1} \), described for example by P. Abramenko y K.S. Brown in [1, §1.5], and its geometric realization is therefore homeomorphic to a sphere of dimension \( r-2 \). In particular, we have \( H^p(k\Sigma_*(r)^*) = 0 \) for all \( p \in \mathbb{Z} \setminus \{r-2\} \) and \( H^{r-2}(k\Sigma_*(r)^*) \cong k \).
A canonical \((r - 2)\)-cocycle whose cohomology class freely generates this last cohomology group as a module is the map \(\zeta_r : \Sigma_{r-2}(r) \to \Bbbk\) which sends an \(r\)-composition \((S_1, \ldots, S_r) \in \Sigma_{r-2}(r)\) to \(1\) if \(S_i = \{i\}\) for all \(i \in [r]\), and to \(0\) if not. The action right action of \(S_r\) on \(H^{r-2}(k\Sigma_\bullet(r)^*)\) is such that \(\zeta_r \circ \sigma = \text{sgn}(\sigma)\zeta_r\) for all \(\sigma \in S_r\).

If \(p \geq 0\) and \(\alpha : x \to e\otimes^p\) is a \(p\)-cochain in the complex \(N^\bullet(x)\) of normalized cochains which computes the cohomology of the \(e\)-bicomodule \(x\), there is a linear map \(\Phi^p(\alpha) : x[r] \to k\Sigma_{p-2}(r)^*\) given by

\[
\Psi^p(\alpha)(z)(S_1, \ldots, S_p) = \|\alpha\|(S_1, \ldots, S_p)(z).
\]

for all \(z \in x[r]\) and all \((S_1, \ldots, S_p) \in \Sigma_{p-2}(r)\), and the equivariance condition on morphisms of species implies that this map \(\Psi^p(\alpha)\) is \(S_r\)-equivariant. We see in this way that we actually have a linear map \(\Psi^p : N^p(x) \to \text{hom}_{S_r}(x[r], k\Sigma_{p-2}(r)^*)\), and this map is in fact a bijection because the species \(x\) is concentrated in one cardinal. A direct calculation shows that the maps obtained in this way for various \(p\) collect into an isomorphism of cochain complexes of vector spaces

\[
\Psi^\bullet : N^\bullet(x) \to \text{hom}_{S_r}(x[r], k\Sigma_\bullet(r)^*[-2]),
\]

again because \(x\) is concentrated in one cardinal. If \(x[r]\) is a projective \(S_r\)-module, then the functor \(\text{hom}_{S_r}(x[r], -)\) commutes with taking cohomology and the isomorphism \(\Psi^\bullet\) therefore induces in turn an isomorphism

\[
H(\Psi^\bullet) : H^\bullet(x) \to \text{hom}_{S_r}(x[r], H(k\Sigma_\bullet(r)^*)[-2]).
\]

Our description of the cohomology of the cochain complex \(k\Sigma_\bullet(r)^*\) immediately implies then that \(H^p(x) = 0\) if \(p \neq r\) and that \(H^r(x) \cong \text{hom}_{S_r}(x[r], \text{sgn}_r)\), as the proposition claims.

\(\square\)

(4.4) The hypothesis involving projectivity that appears in the statement of Proposition (4.3) is necessary. We say that a \(e\)-bicomodule is \textit{relatively projective} if for all \(r \in \BbbN_0\) the \(kS_r\)-module \(x[r]\) is projective; this condition implies that the underlying object of \(k\text{Sp}\) is projective in that category. If the ground ring is a field of characteristic zero, every \(e\)-bicomodule is relatively projective. On the other hand, it is not \textit{necessary} to restrict \(k\) in this way: for example, the species of linear orders that we describe below in (8.3) is relatively projective independently of the ground ring being used — in fact, it is precisely to allow for this example that we do not simply restrict our consideration to fields of characteristic zero.

(4.5) If \(x\) is a \(e\)-bicomodule and \(r \in \BbbN_0\), there are \(e\)-bicomodules \(x_{\leq r}, x_r, \) and \(x_{\geq r}\) obtained by truncating \(x\) that on the objects of \(\text{Fin}\) are as follows:

\[
x_{\leq r}[I] = \begin{cases} x[I], & \text{if } |I| \leq r; \\ 0, & \text{if not;} \end{cases} \quad x_r[I] = \begin{cases} x[I], & \text{if } |I| = r; \\ 0, & \text{if not;} \end{cases}
\]
and
\[ x_{\geq r}[I] = \begin{cases} x[I], & \text{if } |I| \geq r, \\ 0, & \text{if not}. \end{cases} \]

The first one of these is a subbicomodule of \( x \) and there is in fact a short exact sequence
\[ 0 \longrightarrow x_{\leq r} \longrightarrow x \longrightarrow x_{\geq r+1} \longrightarrow 0 \]
(4.6) The subbicomodule \( x_{\leq r} \) will play the role of the \( r \)-dimensional skeleton of \( x \). Much as in the context of CW-spaces, where a skeleton has vanishing cohomology in degrees larger than its dimension, we here have the following result:

**Proposition.** Let \( x \) be a relatively projective e-bicomodule. If \( r \geq 1 \), then \( H^r(x_{\geq r}) = 0 \).

**Proof.** Let us fix \( r \geq 0 \). There is a chain of inclusions of e-bicomodules
\[ (x_{\geq r+1})_{\leq r+1} \subseteq (x_{\geq r+1})_{\leq r+2} \subseteq (x_{\geq r+1})_{\leq r+3} \subseteq \cdots \]
from which we obtain, applying the contravariant functor \( C^\bullet(\cdot) \), an inverse system of cochain complexes of modules
\[ C^\bullet((x_{\geq r+1})_{\leq r+1}) \leftarrow C^\bullet((x_{\geq r+1})_{\leq r+2}) \leftarrow C^\bullet((x_{\geq r+1})_{\leq r+3}) \leftarrow \cdots \] (13)
For each \( p \geq 0 \) the inverse system of modules obtained from this at degree \( p \),
\[ C^p((x_{\geq r+1})_{\leq r+1}) \leftarrow C^p((x_{\geq r+1})_{\leq r+2}) \leftarrow C^p((x_{\geq r+1})_{\leq r+3}) \leftarrow \cdots \]
has all its morphisms surjective, so that it satisfies the Mittag-Leffler condition: indeed, for each \( s \geq r+1 \) the arrow \( C^p((x_{\geq r+1})_{\leq r+1}) \to C^p((x_{\geq r+1})_{\leq s}) \) is the result of applying the functor \( \text{hom}_{kSp}(-, e^{\otimes p}) \) to the morphism \( (x_{\geq r+1})_{\leq r} \to (x_{\geq r+1})_{\leq s+1} \), which is a split monomorphism in \( kSp \).

From the collection of morphisms \( C^\bullet(x_{\geq r+1}) \to C^\bullet((x_{\geq r+1})_{\leq s}) \) with \( s \geq r+1 \) we obtain a morphism
\[ C^\bullet(x_{\geq r+1}) \to \lim_{s \geq r+1} C^\bullet((x_{\geq r+1})_{\leq s}) \]
and it is easy to check that it is in fact an isomorphism. On the other hand, taking cohomology in (13) we obtain an inverse system of graded modules
\[ H^\bullet((x_{\geq r+1})_{\leq r+1}) \leftarrow H^\bullet((x_{\geq r+1})_{\leq r+2}) \leftarrow H^\bullet((x_{\geq r+1})_{\leq r+3}) \leftarrow \cdots \] (14)
and from the collection of morphisms \( H^\bullet(x_{\geq r+1}) \to H^\bullet((x_{\geq r+1})_{\leq s}) \) with \( s \geq r+1 \) a morphism
\[ H^\bullet(x_{\geq r+1}) \to \lim_{s \geq r+1} H^\bullet((x_{\geq r+1})_{\leq s}). \]
We claim that the inverse system obtained from (14) by looking at degree \( r - 1 \) has all its morphisms surjective, so that it also satisfies the Mittag-Leffler condition. To see this, we need only observe that if \( s \geq r + 1 \) we have a short exact sequence

\[
0 \longrightarrow (x_{\geq r + 1})_{\leq s} \longrightarrow (x_{\geq r + 1})_{\leq s + 1} \longrightarrow (x_{\geq r + 1})_{s + 1} \longrightarrow 0
\]

in whose corresponding long exact sequence for cohomology there is a segment of the form

\[
H^{r-1}((x_{\geq r + 1})_{\leq s + 1}) \longrightarrow H^{r-1}((x_{\geq r + 1})_{\leq s}) \longrightarrow H^r((x_{\geq r + 1})_{s + 1}) = 0
\]

with the zero coming from Proposition (4.3).

It follows from all this and [5, Proposition 13.2.3] that the morphism

\[
H^r(x_{\geq r + 1}) \cong \lim_{s \geq r + 1} H^r((x_{\geq r + 1})_{\leq s})
\]

is an isomorphism. To prove the proposition it is therefore enough that we show that

\[
H^r((x_{\geq r + 1})_{\leq s}) = 0 \quad \text{for all} \quad s \geq r + 1.
\]

(15)

This is clear if \( s = r + 1 \), for in that case the comodule \((x_{\geq r + 1})_{\leq s}\) is concentrated in cardinal \( r + 1 \) and we already have Proposition (4.3). Suppose then that \( s > r + 1 \) and, inductively, that \( H^r((x_{\geq r + 1})_{\leq s-1}) = 0 \). The long exact sequence in cohomology corresponding to the short exact sequence

\[
0 \longrightarrow (x_{\geq r + 1})_{\leq s-1} \longrightarrow (x_{\geq r + 1})_{\leq s} \longrightarrow (x_{\geq r + 1})_s \longrightarrow 0
\]

contains a segment of the form

\[
H^r((x_{\geq r + 1})_s) \longrightarrow H^r((x_{\geq r + 1})_{\leq s}) \longrightarrow H^r((x_{\geq r + 1})_{\leq s-1})
\]

Since the bicomodule \((x_{\geq r + 1})_s\) is concentrated in cardinal \( s \) and \( s \neq r \), this together with the induction hypothesis imply immediately that \( H^r((x_{\geq r + 1})_{\leq s}) = 0 \). This proves (15) and, as observed above, the proposition. \( \square \)

(4.7) Let \( x \) be a \( e \)-bicomodule. For each \( r \geq 0 \) we have a short exact sequence

\[
0 \longrightarrow x_r \xrightarrow{\phi_r} x_{\geq r} \xrightarrow{\psi_r} x_{\geq r + 1} \longrightarrow 0
\]

(\( F_r \))

Let \( \partial^x_r : H^s(x_r) \rightarrow H^{s+1}(x_{\geq r + 1}) \) be the connecting homomorphism appearing in the corresponding long exact sequence for cohomology and for each \( r \geq 0 \) let

\[
\delta^r : H^r(x_r) \rightarrow H^{r+1}(x_{r + 1})
\]
be the composition

$$H^r(x_r) \xrightarrow{\partial^r} H^{r+1}(x_{\geq r+1}) \xrightarrow{H^{r+1}(\phi_{r+1})} H^{r+1}(x_{r+1}).$$

Since $\partial^r \circ H^{r+1}(\phi_{r+1}) = 0$, it is clear that we have $\delta^{r+1} \circ \delta^r = 0$ for all $r \geq 0$ and we therefore have a cochain complex of modules

$$H^0(x_0) \xrightarrow{\delta^0} H^1(x_1) \xrightarrow{\delta^1} H^2(x_2) \xrightarrow{\delta^2} H^3(x_3) \xrightarrow{\delta^3} \ldots$$

which we denote $\mathcal{X}^\bullet(x)$. It is easy to see that this is a contravariant functor of $x$. 

(4.8) We can now give the promised description of cohomology:

**Proposition.** Let $x$ be a relatively projective $e$-bicoregular module. The cohomology of the complex $\mathcal{X}^\bullet(x)$ is naturally isomorphic to $H^\bullet(x)$.

**Proof.** Let us fix $r \geq 0$ and consider the commutative diagram in Figure 1 on the following page, in which the diagonal sequences are segments of long exact sequences corresponding to short exact sequences of the form $(F_r)$, so that they are exact, and the terms that are zero have been computed in Propositions (4.3) and (4.6).

If $u$ is an element of $H^r(x_{\geq r-1})$, there exists a $v \in H^r(x_{\geq r})$ such that $H^{r+1}(\psi_r) = u$, because $H^{r+1}(\psi_r)$ is surjective, and we have $\delta^r(H^r(\phi_r)(v)) = 0$, so that we have a class $[H^r(\phi_r)(v)]$ in $H^r(\mathcal{X}^\bullet(x))$. This cohomology class depends only on $u$ and not on the choice of $v$: if $v'$ is another element of $H^r(x_{\geq r})$ such that $H^{r+1}(\psi_r) = u$, then $v - v' = \delta^r_w$ for some $w \in H^{r-1}(x_r)$ and then $H^r(\phi_r)(v) - H^r(\phi_r)(v') = \delta^{r-1}(w)$. We obtain in this way a function

$$\Phi : H^r(x_{\geq r-1}) \rightarrow H^r(\mathcal{X}^\bullet(x)) \tag{16}$$

A little chase in the diagram proves that this function is an isomorphism of modules and it clearly is natural in $x$. Finally, from the short exact sequence

$$0 \rightarrow x_{\leq r-2} \rightarrow x \rightarrow x_{\geq r-1} \rightarrow 0$$

we obtain a long exact sequence which contains the segment

$$0 = H^{r-1}(x_{\leq r-2}) \rightarrow H^r(x_{\geq r-1}) \rightarrow H^r(x) \rightarrow H^r(x_{\leq r-2}) = 0$$

so that $H^r(x_{\geq r-1}) \cong H^r(x)$ naturally. Composing this isomorphism with the one in (16) we conclude that the proposition holds. \(\square\)
0 = H^r(x_{\geq r+1})

\[ H^r(x_{\geq r}) \]

\[ \partial_{r-1}^r \]

\[ H^{-1}(x_{r-1}) \]

\[ \delta_{r-1} \]

\[ H^r(x_r) \]

\[ \delta^r \]

\[ H^r(x_r) \]

\[ \delta^r \]

\[ H^{r+1}(x_{r+1}) \]

\[ \psi_r \]

\[ \phi_r \]

\[ 0 = H^{r+1}(x_{\geq r+2}) \]

Figure 1. The diagram used in the proof of Proposition (4.8).

(4.9) If \( x \) is a relatively projective \( e \)-bicomodule, we know from Proposition (4.3) that there is a canonical isomorphism \( X^*(x) \cong \text{hom}_{S_r}(x[r], sgn_r) \), which we will from now on view as an identification. In these terms, the differential of the complex \( X^*(x) \) often has a simple description.

For each \( z \in x[r+1] \) and each \( j \in [r+1] \) let us write \( z'_j \) for the element

\[ \lambda_{r+1,j}(z \setminus [r+1] \setminus j) \in x[r] \]

where the permutation \( \lambda_{r+1,j} \in S_{r+1} \) is such that

\[ \lambda_{r+1,j}(i) = \begin{cases} 
  i, & \text{if } i < j; \\
  i + 1, & \text{if } j \leq i \leq r; \\
  j, & \text{if } i = r + 1.
\end{cases} \]

We will write \( z''_j \) for the element obtained in this way by using the restriction to the other side. It is useful to think about \( z'_j \) as the structure obtained from \( z \) by deleting label \( j \) using the restriction maps to the left, and similarly for \( z''_j \).
Proposition. Suppose that the ground ring $\mathbb{k}$ contains $\mathbb{Q}$. Let $x$ be a $e$-bicomodule, let $r \in \mathbb{N}_0$. If $\phi$ is an element of $\mathcal{A}^r(x)$, then for each $z \in x[r+1]$ we have

$$(\delta^r \phi)(z) = \sum_{j=1}^{r+1} (-1)^j \left( \phi(z'_j) - \phi(z''_j) \right).$$

Notice that when $r \leq 1$ this expression coincides with the one that appears in Proposition (3.2) —this makes sense, as $\mathcal{A}^r(x) = x[r]^*$ when $r \in \{0, 1\}$ and $\mathcal{A}^2(x)$ can be viewed as a subspace of $x[2]^*$.

Proof. A straightforward computation using the definition given in (4.7) shows that the differential $\delta^r : \mathcal{A}^r(x) \rightarrow \mathcal{A}^{r+1}(x)$ is as follows: if $\phi : x[r] \rightarrow sgn_r$ is an element of $\mathcal{A}^r(x)$, then its coboundary $\delta^r \phi : x[r+1] \rightarrow sgn_{r+1}$ is such that for all $z \in x[r+1]$ one has

$$(\delta^r \phi)(z) = \frac{1}{r!} \sum_{\sigma \in S_{r+1}} sgn(\sigma) \cdot \left( \phi(x[\sigma \times \lambda_{r+1,1}](z) / [r]) + (-1)^{r+1} \phi(x[\sigma](z) \% [r]) \right). \quad (17)$$

If $\sigma \in S_{r+1}$, the permutation $\bar{\sigma} = \lambda_{r+1,1}^{-1} \circ \sigma$ fixes $r+1$, so that it maps the subset $[r]$ to itself and therefore

$$\phi(x[\sigma](z) \% [r]) = \phi(x[\lambda_{r+1,1} \circ \bar{\sigma}](z) \% [r])$$
$$= \phi(x[\bar{\sigma}](x[\lambda_{r+1,1} \circ \bar{\sigma}](z) \% [r]))$$
$$= \phi\left( x[\bar{\sigma}](z) \right) \phi(x[\lambda_{r+1,1} \circ \bar{\sigma}](z) \% [r])$$
$$= (-1)^{r+1-\sigma(r+1)} \cdot sgn(\bar{\sigma}) \cdot \phi(x[\lambda_{r+1,1} \circ \bar{\sigma}](z) \% [r]),$$

because $sgn(\lambda_{r+1,1}) = (-1)^{r+1-j}$. Similarly, the permutation $\bar{\tau} = \lambda_{r+1,1}^{-1} \circ \sigma \circ \lambda_{r+1,1}$, which fixes $r+1$, allows us to rewrite $\sigma \circ \lambda_{r+1,1}$ as $\lambda_{r+1,1}^{-1} \circ \bar{\sigma}$, and proceeding as before, we see that

$$\phi(x[\sigma \circ \lambda_{r+1,1}](z) / [r]) = (-1)^{1-\sigma(1)} \cdot \phi(x[\lambda_{r+1,1} \circ \bar{\sigma}](z) / [r]).$$

Using these two equalities to simplify the right hand side of (17) we find easily the expression for $\delta^r \phi$ given in the statement of the proposition. \hfill \square

(4.10) We can rephrase what we have done in this section in terms of spectral sequences, and doing so will streamline our treatment of products in the next section. We fix a relatively projective $e$-bicomodule $x$. There is a decreasing filtration on the complex $C^\bullet(x)$

$$\cdots \subseteq F^2C^\bullet(x) \subseteq F^1C^\bullet(x) \subseteq F^0C^\bullet(x) = C^\bullet(x)$$

with each $F^pC^\bullet(x)$ equal to the subcomplex of $C^\bullet(x)$ of cochains which vanish on $x_{\leq p-1}$. Applying the functor $C^\bullet(-)$ to the short exact sequence of left $e$-comodules

$$0 \longrightarrow x_{\leq p-1} \longrightarrow x \longrightarrow x_{\geq p} \longrightarrow 0$$

18
which is split in \(\mathcal{S}_p\) we obtain a short exact sequence of cochain complexes of modules

\[
0 \longrightarrow C^\bullet(x_{\geq p}) \longrightarrow C^\bullet(x) \longrightarrow C^\bullet(x_{\leq p-1}) \longrightarrow 0
\]

and we can therefore identify \(F^pC^\bullet(x)\) with \(C^\bullet(x_{\geq p})\). The spectral sequence \(E(x)\) that arises from the complex \(C^\bullet(x)\) endowed with this filtration has

\[
E_{0}^{p,q}(x) = \frac{F^pC^{p+q}(x)}{F^{p+1}C^{p+q}(x)} = \frac{C^{p+q}(x_{\geq p})}{C^{p+q}(x_{\geq p+1})} \cong C^{p+q}(x_p), \tag{18}
\]

with the isomorphism coming from a short exact sequence like \((\mathcal{F}_r)\) and, in particular, we have \(E_{0}^{0,q}(x) = 0\) unless \(p \geq 0\) and \(p + q \geq 0\).

The differential \(d_{0}^{p,q} : E_{0}^{p,q}(x) \to E_{0}^{p,q+1}(x)\) coincides, up to the isomorphism (18), with the differential \(d_{p+q} : C^{p+q}(x_p) \to C^{p+q+1}(x_p)\) and Proposition (4.3) then tells us, as we are supposing \(x\) to be relatively projective, that the first page of the spectral sequence has

\[
E_{1}^{p,q}(x) \cong \begin{cases} H^p(x_p), & \text{if } p \geq 0 \text{ and } q = 0; \\ 0, & \text{in any other case.} \end{cases}
\]

It is clear now that the spectral sequence degenerates at the second page. Moreover, the differential \(d_{1}^{p,0} : E_{1}^{p,0}(x) \to E_{1}^{p+1,0}(x)\) corresponds under the above isomorphism to the differential \(\delta^p : H^p(x_p) \to H^{p+1}(x_{p+1})\) constructed in (4.7) —this follows from the way the spectral sequence of a filtered complex is constructed—and then our Proposition (4.8) tells us that

\[
E_{\infty}^{p,q}(x) \cong E_{2}^{p,0}(x) \cong \begin{cases} H^p(x^\bullet(x)), & \text{if } p \geq 0 \text{ and } q = 0; \\ 0, & \text{in any other case.} \end{cases}
\]

The filtration of \(C^\bullet(x)\) is Hausdorff, exhaustive and complete—in the sense that the canonical morphism \(C^\bullet(x) \to \lim_{\leftarrow p} C^\bullet(x)/F^pC^\bullet(x)\) is an isomorphism. It follows from the Complete Convergence Theorem of [14, Theorem 5.5.10] that the spectral sequence converges to \(H(C^\bullet(x))\), that is, to the cohomology \(H^\bullet(x)\) of the \(e\)-bicomodule \(x\). The shape of the second page then gives us the isomorphism \(H^\bullet(x) \cong H(x^\bullet(x))\) of Proposition (4.8).

### §5. Products

(5.1) Since the functors \(e \otimes (-)\) and \((-) \otimes e\) are exact, there is a monoidal structure on the category \(\text{eMod}_e\) of \(e\)-bicomodules with monoidal functor given by the cotensor product. Let us simply recall that if \(x\) and \(y\) are \(e\)-bicomodules and \(\rho^x\) and \(\lambda^y\) are the right coaction of \(e\) on \(x\) and the left coaction of \(e\) on \(y\), then the cotensor product \([7]\)
of \( x \) and \( y \) is the \( e \)-bicomodule \( x \boxdot y \) whose underlying linear species is the equalizer of the morphisms \( \rho^x \otimes \text{id}_y \) and \( \text{id}_x \otimes \lambda^y \) in the category of linear species, so that we have an exact sequence

\[
0 \longrightarrow x \boxdot y \longrightarrow x \otimes y \overset{\rho^x \otimes \text{id}_y - \text{id}_x \otimes \lambda^y}{\longrightarrow} x \otimes e \otimes y
\]

in \( \mathcal{S}p \), and whose \( e \)-bicomodule structure is the unique one which makes the inclusion \( \iota \) in that sequence a morphism of \( e \)-bicomodules when we view \( x \otimes y \) as an \( e \)-bicomodule using the left coaction of \( x \) and the right coaction of \( y \); that such a bicomodule structure exists at all is a consequence of the exactness of the functors \( e \otimes ( - ) \) and \( ( - ) \otimes e \). The unit object in this monoidal category is \( e \) and, in particular, unitors give us an isomorphism \( \bar{\mu} : e \boxdot e \to e \).

(5.2) From the cochain complex \( B^\bullet \) corresponding to the cosimplicial \( e \)-bicomodule \( B^d \) of (3.1) we can construct a first quadrant double complex \( B^p \boxdot B^q \) and its totalization, which we will denote in the same way. There is a morphism of complexes of \( e \)-bicomodules \( \mu : B^p \boxdot B^q \to B^{p+q} \) with components \( \mu^{p,q} : B^p \boxdot B^q \to B^{p+q} \) such that the diagram

\[
\begin{array}{ccc}
B^p \boxdot B^q & \overset{\mu^{p,q}}{\longrightarrow} & B^{p+q} \\
\downarrow & & \downarrow \\
(e \otimes e^{\otimes p} \boxdot e) \boxdot (e \otimes e^{\otimes q} \boxdot e) & & e \otimes e^{\otimes (p+q)} \boxdot e \\
\downarrow & & \downarrow \\
e \otimes e^{\otimes p} \boxdot (e \boxdot e) \boxdot e^{\otimes q} \boxdot e & \overset{id \otimes (e \to e)}{\longrightarrow} & e \otimes e^{\otimes (p+q)} \boxdot e
\end{array}
\]

commutes. Using this morphism \( \mu \) we can construct an external product on cohomology: if \( x \) and \( y \) are \( e \)-bicomodules, there is a morphism of complexes of modules

\[
\vee : \text{hom}_{e \boxdot e}(x, B^\bullet) \otimes \text{hom}_{e \boxdot e}(y, B^\bullet) \to \text{hom}_{e \boxdot e}(x \boxdot y, B^\bullet)
\]  
(19)

such that for each \( p \)- and \( q \)-cochains \( \alpha : x \to B^p \) and \( \beta : y \to B^q \) the image \( \alpha \vee \beta \) of \( \alpha \otimes \beta \) under the map \( \vee \) is the composition

\[
x \boxdot y \overset{\alpha \boxdot \beta}{\longrightarrow} B^p \boxdot B^q \overset{\mu^{p,q}}{\longrightarrow} B^{p+q},
\]

and this product (19) induces on cohomology a product

\[
\vee : H^\bullet(x) \otimes H^\bullet(y) \to H^\bullet(x \boxdot y).
\]  
(20)

This product is associative in the hopefully obvious sense.

Of course, conjugating with the natural isomorphism \( \text{hom}_{e \boxdot e}(-, B^\bullet) \cong C^\bullet(-) \), we produce from the map \( \vee \) of (19) a map

\[
\vee : C^\bullet(x) \otimes C^\bullet(y) \to C^\bullet(x \boxdot y)
\]  
(21)
which induces the same multiplication (20) on cohomology. A crucial observation to make at this point is that this map factors through a map

\[ \vee : C^\bullet(x) \otimes C^\bullet(y) \to C^\bullet(x \otimes y) \]

where \( x \otimes y \) is the \( e \)-bicomodule obtained using the Hopf algebra structure of \( e \), and the factorization is simply induced by the inclusion of \( x \square y \hookrightarrow x \otimes y \). This means, in particular, that the formulas we will obtain next apply to this map, too.

(5.3) As usual, from the external product we have just described we can construct internal products. If \((x, \Delta, \varepsilon)\) is a comonoid in the monoidal category \( \mathcal{E}\text{Mod}^e \), with comultiplication and counit \( \Delta : x \to x \square x \) and \( \varepsilon : x \to e \), then \( H^\bullet(x) \) becomes an associative and unitary algebra with multiplication

\[ \sim : H^\bullet(x) \otimes H^\bullet(x) \to H^\bullet(x) \]

given by the composition

\[ H^\bullet(x) \otimes H^\bullet(x) \xrightarrow{\vee} H^\bullet(x \square x) \xrightarrow{H^\bullet(\Delta)} H^\bullet(x). \]

The unit element in \( H^\bullet(x) \) is the class of the counit \( \varepsilon \); notice that this makes sense, as \( \varepsilon \) is an element of \( H^0(x) = \text{Ext}_{e-e}^0(x, e) = \text{hom}_{e-e}(x, e) \). In general, the algebra \( H^\bullet(x) \) will be non-commutative: for example, if \( x \) is concentrated in cardinal 0, then the datum of \( x \) really amounts to that of the coalgebra \( x_J^0 K \), a classical coalgebra in \( _k\text{Mod} \), and \( H^\bullet(x) \) is the algebra dual to it, which may well be non-commutative.

(5.4) One can easily obtain explicit formulas for both the external product and the internal one on the complex \( C^\bullet(\_\_\_) \) by simply following their definitions. We will content ourselves with writing them out for future convenience.

Let \( x \) and \( y \) be \( e \)-bicomodules, and let \( \alpha : x \to e^{\otimes p} \) and \( \beta : y \to e^{\otimes q} \) be a \( p \)-cochain and a \( q \)-cochain in the complexes \( C^\bullet(x) \) and \( C^\bullet(y) \), respectively. The \((p + q)\)-cochain \( \alpha \vee \beta : x \square y \to e^{\otimes(p+q)} \) in the complex \( C^\bullet(x \square y) \) is the restriction to \( x \square y \) of the morphism of linear species \( \gamma : x \otimes y \to e^{\otimes(p+q)} \) which has coefficients, for all \( I \in \text{Fin}, (S, T) \vdash I, z \in x[S] \) and \( t \in y[T] \), so that \( z \otimes t \in (x \otimes y)[I] \), and each decomposition \((U_1, \ldots, U_{p+q})\) of \( I \), given by

\[ \|\gamma\|(U_1, \ldots, U_p)(z \otimes t) = \begin{cases} \|\alpha\|(U_1, \ldots, U_p)(z) \cdot \|\beta\|(U_{p+1}, \ldots, U_{p+q})(t), & \text{if } U_{1,p} = S; \\ 0, & \text{if not}. \end{cases} \]

This takes care of external products.
à la Sweedler, with each summand $z_{(S)} \otimes z^{(T)}$ appearing here standing for an element— not necessarily an elementary tensor— of the submodule $x[S] \otimes x[T]$ of $(x \otimes x)[I]$. If $\alpha : x \to e^{\otimes p}$ and $\beta : x \to e^{\otimes q}$ are a $p$- and a $q$-cochain in the complex $C^*(x)$, then their product $\alpha \smile \beta \in C^{p+q}(x)$ has coefficients given by

$$[\alpha \smile \beta](U_1, \ldots, U_{p+q})(z) = [\alpha](U_1, \ldots, U_p)(z_{(U_{1:p})}) \cdot [\beta](U_{p+1}, \ldots, U_{p+q})(z_{(U_{p+1:p+q})})$$

for all $I \in \mathsf{Fin}$, all decompositions $(U_1, \ldots, U_{p+q})$ of $I$ and all $z \in x[I]$. 

(5.5) The map $\vee$ of (21) is compatible with the filtrations on the complexes $C^*(x)$, $C^*(y)$ and $C^*(x \square y)$ described in (4.10), in the sense that $F^p C^*(x) \vee F^{p'} C^*(y) \subseteq F^{p+p'} C^*(x \square y)$ for all $p, p', q, q' \in \mathbb{N}_0$. We are therefore in the situation of [10, Theorem 2.14] (except that there only an internal product is considered) and from $\vee$ we obtain on our spectral sequences products $\vee : E^{p,q}_r(x) \otimes E^{p',q'}_r(y) \to E^{p+p',q+q'}_r(x \square y)$ giving a multiplicative structure which converges to the product $\vee : H^*(x) \otimes H^*(y) \to H^*(x \square y)$ on their limits. Since the $E_{\infty}$-pages of the spectral sequences are concentrated in one row, there are no extension problems in determining products. The upshot of all this is that the map $\vee$ of (21) induces a multiplication

$$\vee : \Lambda^\bullet(x) \otimes \Lambda^\bullet(y) \to \Lambda^\bullet(x \square y),$$

which is what the multiplicative structure on the spectral sequences amounts to at the level of the $E_1$-pages, and that this multiplication passes on to cohomology to give a multiplication

$$\vee : H(\Lambda^\bullet(x)) \otimes H(\Lambda^\bullet(y)) \to H(\Lambda^\bullet(x \square y))$$

which coincides, up to the isomorphism $H^*(-) \cong H(\Lambda^\bullet(-))$ of Proposition (4.8), with the external product (20). As before, if $x$ is a coalgebra in $e\mathsf{Mod}^e$, there is a corresponding internal product

$$\smile : \Lambda^\bullet(x) \otimes \Lambda^\bullet(x) \to \Lambda^\bullet(x),$$

inducing that of $H^*(x)$. 

22
(5.6) Going through the chain of isomorphisms involved, one can compute the following description of the map (22):

**Proposition.** Suppose that the ground ring \( k \) is a field of characteristic zero. Let \( x \) and \( y \) be e-bicomodules, let \( p, q \geq 0 \) and let \( \alpha : [p] \to \text{sgn}_p \) and \( \beta : [q] \to \text{sgn}_q \) be a \( p \)-cochain in \( \mathcal{X}^*(x) \) and a \( q \)-cochain in \( \mathcal{X}^*(y) \), respectively. The product

\[
\alpha \lor \beta : ([p] \otimes [q]) \to \text{sgn}_{p+q}
\]

is the element of \( \mathcal{X}^{p+q}(x \square y) \) which is the restriction to \( ([p] \otimes [q]) \) of the linear map \( \gamma : ([p] \square [q]) \to \text{sgn}_{p+q} \) such that whenever \( (S, T) \in [p+q] \), \( z \in x[S] \) and \( t \in y[T] \), we have

\[
\gamma(z \otimes t) = \begin{cases} (-1)^{\text{sch}(S, T)} p! q! \cdot \alpha(x[s](z)) \cdot \beta(y[t](t)), & \text{if } |S| = p; \\ 0, & \text{if not.} \end{cases}
\]

Here for each \( U \subseteq \mathbb{N} \) of cardinal \( r \) we are letting \( i_U : [r] \to [p+q] \) be the unique strictly increasing function with image equal to \( U \), and for each composition \((S, T)\) of \([p+q]\) we are writing \( \text{sch}(S, T) \) the number of elements of the set \( \{(i, j) \in S \times T : i > j\} \).

**Proof.** From the element \( \alpha : [p] \to \text{sgn}_p \) of \( \mathcal{X}^p(x) \) we get the \( p \)-cochain \( \tilde{\alpha} : x \to e^{\otimes p} \) in the complex \( C^*(x) \) which is concentrated in cardinal \( p \) and such that for each \( I \in \text{Fin} \) with \( p \) elements, each composition \( (S_1, \ldots, S_p) \vdash I \) and each \( z \in x[I] \) has

\[
\|\tilde{\alpha}\|(S_1, \ldots, S_p)(z) = \frac{1}{p!} \alpha(x[\sigma](z)),
\]

with \( \sigma : [p] \to I \) the bijection such that \( S_i = \{\sigma(i)\} \) for each \( i \in [p] \). In a similar way, from \( \beta \in \mathcal{X}^q(y) \) we obtain a \( q \)-cochain \( \tilde{\beta} : y \to e^{\otimes q} \). The product \( \tilde{\alpha} \lor \tilde{\beta} : x \square y \to e^{\otimes (p+q)} \) can be computed using the formulas of (5.4): it is concentrated in cardinal \( p+q \), and when \( I \in \text{Fin} \) has that cardinal, \( (S, T) \vdash I, z \in x[S], t \in y[T] \) and \( (U_1, \ldots, U_{p+q}) \) is a \((p+q)\)-composition of \( I \) we have

\[
\|\tilde{\alpha} \lor \tilde{\beta}\|(U_1, \ldots, U_{p+1})(z \otimes t) = \begin{cases} \frac{1}{p! q!} \alpha(x[\sigma](z)) \cdot \beta(y[\tau](t)), & \text{if } U_{1:p} = S; \\ 0, & \text{if not}; \end{cases}
\]

(24)

where in the first case \( \sigma : [p] \to S \) and \( \tau : [q] \to T \) are the bijections such that \( U_i = \{\sigma(i)\} \) for each \( i \in [p] \) and \( U_{p+i} = \{\tau(i)\} \) for each \( i \in [q] \). According to the final claim of Proposition (4.3), to the cochain \( \tilde{\alpha} \lor \tilde{\beta} \) corresponds the function \( \gamma : ([p] \square [q]) \to \text{sgn}_{p+q} \) in \( \mathcal{X}^{p+q}(x \square y) \) such that for each composition \((S, T)\) of \([p+q]\) and each choice of \( z \in x[S] \) and \( t \in x[T] \), has

\[
\gamma(z \otimes t) = \sum_{\sigma \in S_{p+q}} \text{sgn}(\sigma) \cdot \|\tilde{\alpha} \lor \tilde{\beta}\|(\{\sigma(1)\}, \ldots, \{\sigma(p+q)\})(z \otimes t)
\]

23
and, in view of the formula (24) for the coefficients of $\tilde{\alpha} \vee \tilde{\beta}$, this is

$$\frac{1}{plq!} \sum_{\substack{\sigma \in S_{p+q} \\
\sigma([p]) = S}} \text{sgn}(\sigma) \cdot \alpha(x[\sigma]_{[p]}^{S}(z)) \cdot \beta(x[\sigma]_{[p+1,p+q]}^{T} \circ \rho_{p,q})(t)$$

(25)

with $\rho_{p,q}: i \in \lbrack q \rbrack \mapsto p + i \in \lbrack p + 1, p + q \rbrack$.

Let $\sigma_{S,T} \in S_{p+q}$ be the permutation such that the restrictions $\sigma_{S,T}|_{[p]} : [p] \rightarrow [p + q]$ and $\sigma_{S,T}|_{[p+1,p+q]} : [p+1, p+q] \rightarrow [p+q]$ are strictly increasing and have images $S$ and $T$, respectively. The number of inversions of $\sigma_{S,T}$ is precisely the number $\text{sch}(S, T)$ described in the statement of the proposition, so that its signature is $\text{sgn}(\sigma_{S,T}) = (-1)^{\text{sch}(S,T)}$. If $\sigma \in S_{p+q}$ is such that $\sigma([p]) = S$, then there exists permutations $l_{\sigma} \in S_{p}$ and $r_{\sigma} \in S_{q}$ such that $\sigma = \sigma_{S,T} \circ l_{\sigma} \sqcup r_{\sigma}$, and we have $\sigma|_{[p]} = \iota_{S} \circ l_{\sigma}$, $\sigma|_{[p+1,p+q]} \circ \rho_{p,q} = \iota_{T} \circ r_{\sigma}$ and $\text{sgn}(\sigma) = \text{sgn}(\sigma_{S,T}) \text{sgn}(l_{\sigma}) \text{sgn}(r_{\sigma})$. Using this to rewrite (25), we find that

$$\gamma(z \otimes t) = \frac{1}{plq!} \sum_{\substack{\sigma \in S_{p+q} \\
\sigma([p]) = S}} \text{sgn}(\sigma_{S,T}) \text{sgn}(l_{\sigma}) \text{sgn}(r_{\sigma}) \cdot \alpha(x[\iota_{S} \circ l_{\sigma}](z)) \cdot \beta(x[\iota_{T} \circ r_{\sigma}](t))$$

and the $S_{p}$- and $S_{q}$-equivariance of $\alpha$ and $\beta$ imply that this is

$$\gamma(z \otimes t) = \frac{1}{plq!} \sum_{\substack{\sigma \in S_{p+q} \\
\sigma([p]) = S}} \text{sgn}(\sigma_{S,T}) \cdot \alpha(x[\iota_{S}](z)) \cdot \beta(x[\iota_{T}](t)).$$

As the $plq!$ terms appearing in this sum are all equal, we see that the formula given in the statement holds.

(5.7) There is a corresponding formula for the map $\sim$ of (23):

**Corollary.** Suppose that the ground ring $\mathbb{k}$ is a field of characteristic zero. Let $x$ be a coalgebra in $\mathbb{k} \text{-} \text{Mod}^c$. Let $p$, $q \geq 0$ and let $\alpha : x[p] \rightarrow \text{sgn}_{p}$ and $\beta : x[q] \rightarrow \text{sgn}_{q}$ be a $p$- and a $q$-cochain on the complex $X^\bullet(x)$. The product $\alpha \sim \beta : x[p+q] \rightarrow \text{sgn}_{p+q}$ is such that whenever $z \in x[p+q]$ we have

$$(\alpha \sim \beta)(z) = \sum_{S,T \in [p+q]} (-1)^{\text{sch}(S,T)} \cdot \alpha(z(S)) \cdot \beta(z(T)).$$

**Proof.** This follows at once from the definition of $\sim$ and the result of Proposition (5.6). \hfill $\square$

\footnote{Here $l_{\sigma} \sqcup r_{\sigma}$ denotes the element of $S_{p+q}$ which maps $i \in \lbrack p + q \rbrack$ to $l_{\sigma}(i)$ if $i \leq p$, and to $p + r_{\sigma}(i - p)$ if $i > p$.}
We say that a coalgebra \( x \) in \( \mathsf{Mod}^e \) is **cocommutative** if the composition of its comultiplication \( x \to x \square x \) with the inclusion \( x \square x \to x \otimes x \) is cosymmetric. It should be noted that since the monoidal category \( \mathsf{Mod}^e \) is not symmetric we cannot really talk about cocommutative coalgebras there, and that we are therefore abusing language a little bit --- no confusion should arise, though.

A direct calculation using the expression given in Corollary (5.7) for the product proves at once the following important result:

**Corollary.** Suppose that the ground ring \( k \) is a field of characteristic zero. If \( x \) is a cocommutative coalgebra in \( \mathsf{Mod}^e \), then the cohomology algebra \( H^*(x) \) is graded-commutative.

The cohomology algebra is probably graded-commutative in general, for all ground rings and all cocommutative coalgebras in \( \mathsf{Mod}^e \), but we will not need this fact. On the other hand, the condition that the coalgebra be cocommutative is needed, as we observed already at the end of (5.3).

(5.9) Our main source of examples of coalgebras in \( \mathsf{Mod}^e \) is the following simple observation:

**Proposition.** Let \( x \) be a nonempty set-valued species with left and right restrictions, as in (2.4), and let \( kx \) be the \( e \)-bicomodule obtained by linearization from \( x \). There is a morphism of \( e \)-bicomodules \( \Delta : kx \to kx \otimes kx \) such that

\[
\Delta[I](z) = \sum_{(S,T) \vdash I} z \downarrow S \otimes z \uparrow T
\]

for each \( I \in \text{Fin} \) and each \( z \in x[I] \), it takes values in the subbicomodule \( kx \square kx \subseteq kx \otimes kx \) and the corresponding corestriction \( \Delta : kx \to kx \square kx \) is a coalgebraic comultiplication. This comultiplication is counital, with counit the morphism of bicomodules \( \varepsilon : kx \to e \) obtained by linearizing the unique morphism of set-values species \( x \to e \). On the other hand, this comultiplication is cocommutative if and only if the set-valued species with restrictions \( x \) is symmetric, in that \( z \downarrow S = z \uparrow S \) for all \( I \in \text{Fin} \), \( z \in x[I] \) and \( S \subseteq I \).

In what follows, we will usually consider every \( e \)-bicomodule whose underlying species is a linearization as a twisted coalgebra in \( \mathsf{Mod}^e \) in the way described in this proposition.

**Proof.** All the claims follow immediately from the definition of a set-valued species with left and right restrictions.

\[\square\]

**§6. Künneth formulas**

We now describe how to exploit the combinatorial complex to deduce a Künneth theorem for the Cauchy product. We already described an arrow \( \vee : \mathcal{A}^*(x) \otimes \mathcal{A}^*(y) \to \mathcal{A}^*(x \otimes y) \),
and we now claim it is an isomorphism.

**Proposition.** For each such \( p, q \in \mathbb{N} \) there is an isomorphism

\[
\phi^p : \text{hom}_{S_p \times S_q}(x[p] \otimes y[q], \sgn_p \otimes \sgn_q) \rightarrow \text{hom}_{S_{p+q}}((x \otimes y)^p(p+q), \sgn_{p+q})
\]

where \((x \otimes y)^p[p+q]\) is the space of summands \(x[S] \otimes y[T]\) with \(S\) of cardinality \(p\).

**Proof.** This is readily described as follows. For each decomposition \((S, T)\) of \(n\), let \(u = u_{S,T}\) be the unique bijection that assigns \(z\) to \((z + 1, \ldots, z + q)\). We claim this is \(S_{p+q}\)-equivariant. Note that the sign of \(u\) is \(\text{sch}(S, T)\). Indeed, if \(\tau\) is a permutation of \(n\) and \((S, T)\) is a decomposition of \(n\), we can write \(\tau = \xi \rho\) where \(\rho = \tau_1 \times \tau_2\) is a shuffle of \((S, T)\) and \(\xi\) is monotone over \(S\) and over \(T\). It is clear that if \((S', T')\) is the image of \((S, T)\) under \(\tau\) and if \(u' = u_{S', T'}\), then \(u = u' \xi\). Moreover, note that \(u(z \otimes w)\) is transported to \(u(z \otimes w)\) by \(u^{r-1} u^{-1}\), which belongs to \(S_p \times S_q\), and we now compute

\[
(-1)^{\tau} \phi^p(f)(\tau(z \otimes w)) = (-1)^{r+w'} f(u' \tau(z \otimes w)) = (-1)^{r+w'} f(u \tau(z \otimes w)) = (-1)^{r+w'+\rho} f(u \tau(z \otimes w)) = f^p(f)(z \otimes w)
\]

where the signs cancel by virtue of the identities \(\xi \rho = \tau\) and \(u' \xi = u\). This is what we wanted. \(\square\)

For each \(p, q \in \mathbb{N}\) there are canonical maps

\[
\text{hom}_{S_p}(x[p], \sgn_p) \otimes \text{hom}_{S_q}(y[q], \sgn_q) \rightarrow \text{hom}_{S_{p+q}}(x[p] \otimes y[q], \sgn_p \otimes \sgn_q)
\]

that are all isomorphisms if \(k\) is a field and \(x\) or \(y\) is finite in each arity, and they collect along with the maps \(\phi\) to recover the morphism

\[
\vee : \mathcal{X}^*(x) \otimes \mathcal{Y}^*(y) \rightarrow \mathcal{X}^*(x \otimes y).
\]

Explicitly, given maps \(f_p : x(p) \rightarrow k_p\) and \(g_q : y(q) \rightarrow k_q\), we have for each decomposition \((S, T)\) and \(z \otimes w \in x(S) \otimes y(T)\)

\[
(f_p \vee g_q)(z \otimes w) = (-1)^{\text{sch}(S,T)} f_p(u_S(z)) \otimes g_q(u_T(w)),
\]

where \(u = u_{S,T}\). We obtain now the main result of this section.
Theorem (Künneth formula). Suppose that $k$ is a field of characteristic zero and $x$ or $y$ is locally finite. The map $\vee : \mathcal{X}^\ast(x) \otimes \mathcal{X}^\ast(y) \to \mathcal{X}^\ast(x \otimes y)$ is an isomorphism of complexes. \qed

§7. Free species with restrictions and the boolean species

Define a bifunctor $\text{bool} : \text{Fin}_{\leq} \times \text{Fin} \to k_{\text{mod}}$ so that for each pair of finite sets $S$ and $I$, $\text{bool}(S, I) = k\{f : S \to I\}$ is the free $k$-module with basis the injections $S \to I$, and so that if $\iota : S' \to S$ is an injection and $\sigma : I \to I'$ is a bijection, we have a map

$$\text{bool}(\iota, \sigma) : \text{bool}(S, I) \to \text{bool}(S', I')$$

$$f \mapsto \sigma \circ f \circ \iota$$

It follows that for each fixed set $I$, we have a species with restrictions $\text{bool}^I$ so that $\text{bool}^I(S) = \text{bool}(S, I)$ and a species $\text{bool}_I$ so that $\text{bool}_I(S) = \text{bool}(I, S)$.

Theorem. Let $x$ be a species with restrictions. The assignment $I \mapsto \text{hom}_{\text{Fin}_{\leq}}(\text{bool}^I, x)$ defines a species that is naturally isomorphic to $x$, so that the forgetful functor $U : k\text{Sp}_{\leq} \to k\text{Sp}$ is, in this sense, represented by $\text{bool}$.

Proof. For each finite set $I$ let $1_I$ denote the identity of $I$, and consider a morphism $\alpha : \text{bool}^I \to x$ in $\text{Fin}_{\leq}$. This defines elements $z_I = \alpha_I(1_I)$, and the claim is the assignment

$$\eta_x : \text{hom}_{\text{Fin}_{\leq}}(\text{bool}^I, x) \to x$$

such that $\eta_{x,I}(\alpha) = \alpha_I(1_I)$ is an isomorphism in $\text{Fin}$. It is manifestly equivariant. Moreover, for a fixed finite set

$$\text{hom}_{\text{Fin}_{\leq}}(\text{bool}^I, x) \to x[I]$$

is bijective: if $\alpha : \text{bool}^I \to x$ is a natural transformation and $f : S \to I$ is an injection, then

$$\alpha_S(f) = \alpha_S(f \circ 1_I) = x^\lambda(f)(\alpha_I(1_I)).$$

which determines $\alpha$ uniquely. This shows that the inverse to $\eta$ is given by

$$\mu_x : x \to \text{hom}_{\text{Fin}_{\leq}}(\text{bool}^I, x)$$

such that $\mu_{x,I}(z)(f : S \to I) = x^\lambda(f)(z)$. This gives what we wanted. \qed

27
The use of corepresenting the forgetful functor as is explained by the following result, which now allows us to construct a left adjoint to this forgetful functor, and provides us with a description of free objects in the category of species with restrictions.

**Theorem** (MacLane-Moerdijk). Let $C$ be a small category, let $E$ be a cocomplete category and consider a functor $b : C \to E$, and the hom-functor $r : E \to \text{Fin}^{\text{C}^{\text{op}}}$ such that $r(e)(c) = \text{hom}_E(b(c), e)$. Then $r$ has a left adjoint $l : \text{Fin}^{\text{C}^{\text{op}}} \to E$ that preserves colimits such that for each presheaf $p$ of $\text{Fin}^{\text{C}^{\text{op}}}$,

$$l(p) = \lim_{\to} (\text{cl}(p) \to C \xrightarrow{b} E).$$

In our situation we would like to set $C = \text{Fin}$, which is only essentially small, replace $\text{Fin}$ by $\mathbb{k}\text{Mod}$, and set $E = \mathbb{k}\text{Sp}_{\leq}$. The desired left adjoint

$$F : \mathbb{k}\text{Sp} \to \mathbb{k}\text{Sp}_{\leq}$$

is then such that, for each species $x$ and each finite set $I$, we have

$$(Fx)(I) = \bigoplus_{n \geq 0} x_n \otimes_{S_n} \text{bool}^n(I).$$

Now observe that we may consider any species with restrictions as an $e$-bicomodule by declaring its right action to be that obtained from its restrictions and the left action to be trivial. With this convention, it is easy to see that for each finite set $I$ the species with restrictions $\text{bool}^I$ is such that its combinatorial complex is isomorphic to the dual of the usual Koszul complex of degree $\#I$ over $\mathbb{k}$, so that is is acyclic. This has the following consequence:

**Theorem.** Free species with restrictions, viewed as $e$-bicomodules in the way described above, are acyclic for the cohomology functor $H(\_) = \text{Ext}_{\text{ee}}(\_, e)$, provided they are also relatively projective. \( \Box \)

**Proof.** From the definitions it follows that we have an isomorphism of complexes

$$\mathcal{X}^\bullet(Fx) \to \prod_{p \geq 0} \text{hom}_{S_p}(x[p], \mathcal{X}^\bullet(\text{bool}^p)).$$

Now cohomology commutes with products and, since $x$ is weakly projective, it also commutes with the various $\text{hom}$s, so the result follows from the fact $\text{bool}^p$ is acyclic for any $p$. \( \Box \)

It would be desirable to have an analogous result describing the left adjoint to the forgetful functor from bicomodules to species, which addresses whether the free functor is $H$-acyclic or not.
§8. Examples

Cosymmetric bicomodules

(8.1) Since the monoidal category $\mathcal{Sp}$ is symmetric and $\mathbf{e}$ is a cocommutative comonoid there, it makes sense to talk about **cosymmetric $\mathbf{e}$-bicomodules**, those whose right coaction is the composition of their left coaction and the braiding. Such a bicomodule is entirely determined the underlying left $\mathbf{e}$-comodule — in fact, every left $\mathbf{e}$-comodule can be turned into a cosymmetric $\mathbf{e}$-bicomodule in the obvious way.

We can completely compute the cohomology of cosymmetric bicomodules:

**Proposition.** Let $\mathbf{x}$ be a relatively projective cosymmetric $\mathbf{e}$-bicomodule. For all $p \in \mathbb{N}_0$ we have

$$H^p(\mathbf{x}) \cong \text{hom}_{S_p}(\mathbf{x}[p], \text{sgn}_p).$$

If $\mathbf{x}$ is a coalgebra in $\mathbf{e}\text{-Mod}^\mathbf{e}$ and the ground ring $\mathbb{k}$ contains $\mathbb{Q}$, then the multiplication $\cdot$ on $H^\bullet(\mathbf{x})$ is given by the formula of Corollary (5.7).

**Proof.** As $\mathbf{x}$ is relatively projective, we can compute its cohomology via the complex $\mathcal{X}^\bullet(\mathbf{x})$. On the other hand, since $\mathbf{x}$ is cosymmetric, the description of the differential on this complex given in Proposition (4.9) makes it clear that it is identically zero. Both claims of the proposition follow immediately from this.

(8.2) Of course, the description given in this Proposition (8.1) is as good as the control we have over the hom-spaces appearing there, and sometimes we do not have any. A very useful observation in this regard is the following:

**Lemma.** Suppose that $2$ is neither zero nor a divisor of zero in the commutative ring $\mathbb{k}$. Let $\mathbf{x}$ be a set-valued species with left and right restrictions, let $\mathbb{k}\mathbf{x}$ be the $\mathbf{e}$-bicomodule obtained from $\mathbf{x}$ by linearization, as in (2.3), and let $p \in \mathbb{N}_0$. A $p$-cochain $\phi: \mathbb{k}\mathbf{x}[p] \to \text{sgn}_p$ in the complex $\mathcal{X}^\bullet(\mathbb{k}\mathbf{x})$ vanishes on any element $z$ of $\mathbf{x}[p]$ that has an odd automorphism.

Here by an automorphism of an element $z \in \mathbf{x}[p]$ we mean a permutation $\sigma \in S_p$ such that $\mathbf{x}[\sigma](z) = z$.

**Proof.** If $\phi \in \mathcal{X}^p(\mathbb{k}\mathbf{x})$ and $z \in \mathbf{x}[p]$ has an odd automorphism $\sigma \in S_p$, then

$$\phi(z) = \phi(\mathbf{x}[\sigma](z)) = \text{sgn}(\sigma)\phi(z) = -\phi(z)$$

and the result follows from the hypothesis on $\mathbb{k}$.

Linear orders

(8.3) Let $\mathbf{L}$ be the set-valued species with right restrictions such that
• for each \( I \in \text{Fin} \) the set \( L[I] \) is that of all total orders on \( I \), which we view as subsets of \( I \times I \) in the usual way, and
• for each injection \( f : J \to I \) in \( \text{Fin} \), the function \( L[f] : L[I] \to L[J] \) maps a total order \( z \) on \( I \) to the unique total order \( z' \) on \( J \) such that for each \( a, b \in J \) we have \((a, b) \in z' \) iff \((f(a), f(b)) \in z\).

If \( I \in \text{Fin} \), \( z \in L[I] \) and \( S \subseteq I \) we write \( z|_S = z \cap S \times S \), which is an element of \( L[S] \).

The linearization \( kL \) is a left \( e \)-comodule and, as described above, a cosymmetric \( e \)-comodule. Since for each \( p \geq 0 \) the group \( S_p \) acts simply-transitively on the set \( L[p] \), the \( kS_p \)-module \( kL[p] \) is free of rank 1 no matter what the ring \( k \) is and \( kL \) is thus a relatively projective \( e \)-bicomodule. It is a twisted coalgebra with comultiplication \( \Delta : kL \to kL \otimes kL \) and counit \( \varepsilon : kL \to e \) given by

\[
\Delta[I](z) = \sum_{(S,T) \vdash I} z|_S \otimes z|_T, \quad \varepsilon[I](z) = e_I
\]

for all \( I \in \text{Fin} \) and all \( z \in L[I] \).

**Proposition.** It is possible to choose for each \( p \geq 0 \) a non-zero element \( \alpha_p \) in in the \( k \)-module \( H^p(kL) \) which freely generates it in such a way that in the cohomology algebra \( H^\bullet(kL) \) we have

\[
\alpha_{2p} \sim \alpha_{2q} = \left(\frac{p + q}{p}\right)\alpha_{2(p+q)}, \quad \alpha_1 \sim \alpha_{2p} = \alpha_{2p+1}, \quad \alpha_1 \sim \alpha_{2p+1} = 0
\]

for all \( p, q \geq 0 \). The element \( \alpha_1 \) can be chosen to be the cohomology class of the 1-cocycle \( \kappa : kL \to e \) such that \( [\kappa](I)(z) = |I| \) for all \( I \in \text{Fin} \) and all \( z \in L[I] \), and \( \alpha_2 \) that of the 2-cocycle \( \text{sch} : kL \to e \otimes e \otimes e \) with \( [\text{sch}](S,T)(z) = |(T \times S) \cap z| \) for all \( I \in \text{Fin} \), \( z \in L[I] \) and \( (S,T) \vdash I \).

We emphasize the fact that result is valid for all ground rings. The even part \( H^{ev}(kL) \) is the divided power polynomial \( k \)-algebra generated by \( \alpha_2 \), as in [12, Tag 07H4], and the whole cohomology can be described as the result of adding a variable of degree 1 to the even part. If \( k \) is a field of characteristic zero, then the cohomology algebra \( H^\bullet(kL) \) is isomorphic to a graded-commutative polynomial algebra \( k[X,Y] \) with \( X \) of degree 1 and \( Y \) of degree 2. It should be remarked that there is an evident connection between the 2-cocycle \( \text{sch} \) mentioned in the statement of this proposition and the statistic \( \text{sch} \) of 2-decompositions of \( [p] \) used in Proposition (5.6).

**Proof.** For each \( p \in \mathbb{N}_0 \) let \( z_p \) be the usual total order on the set \( [p] \). The action of \( S_p \) on \( L[p] \) is simply transitive, and it follows at once from this that for all \( p \in \mathbb{N}_0 \) the map

\[
\phi \in \text{hom}_{S_p}(kL[p], \text{sgn}_p) \mapsto \phi(z_p) \in k
\]
is an isomorphism. As \( kL \) is a cosymmetric e-bicomodule, Proposition (8.1) tells us that \( H^p(kL) \cong \text{hom}_{S_p}(kL[p], \text{sgn}_p) \) and, viewing this isomorphism as an identification, we therefore have that \( H^p(kL) \) is the free \( k \)-module generated by the \( S_p \)-equivariant map \( \alpha_p : kL[p] \to \text{sgn}_p \) such that \( \alpha_p(z_p) = 1 \).

Using the formula of Corollary (5.7) we see immediately that

\[
(\alpha_p \sim \alpha_q)(z_{p+q}) = \sum_{(S,T) \vdash [p+q] \atop \abs{S}=p} (-1)^{\text{sch}(S,T)}.
\]

Writing \( (p,q) \) the integer appearing on the right hand side in this equality, we have that \( (p,0) = (0,p) = 1 \) for all \( p \geq 0 \) and that \( (p,q) = (p-1,q) + (-1)^p(p,q-1) \) for all \( p,q \geq 0 \), as can be seen by grouping the terms in the sum that defines \( (p,q) \) according to whether \( 1 \) is in the block \( S \) of the decomposition \( (S,T) \) to which it corresponds or not. From this it follows easily that

\[
(2p,2q) = \binom{p+q}{p}, \quad (1,2p) = 1, \quad (1,2p+1) = 0
\]

for all \( p, q \geq 0 \), so that

\[
\alpha_{2p} - \alpha_{2q} = \binom{p+q}{p}\alpha_{2(p+q)}, \quad \alpha_1 - \alpha_{2p} = \alpha_{2p+1}, \quad \alpha_1 - \alpha_{2p+1} = 0.
\]

As a consequence of these formulas, we see at once that the cohomology algebra \( H^\bullet(kL) \), which is graded-commutative because \( kL \) is cocommutative, is freely generated by \( \alpha_1 \) and \( \alpha_2 \).

A simple verification shows that the cochains \( \kappa \) and \( \text{sch} \) described in the statement of the proposition are cocycles. To check for example that \( \text{sch} \) represents the class \( \alpha_2 \), in view of the construction carried out in the proof of Proposition (4.8), it suffices to make the following observation: to \( \text{sch} \) in \( C^2(kL) \) corresponds the restriction \( \text{sch} : kL_2 \to e^2 \), which is a 2-cocycle in the complex \( N^2(kL_2) \), and, according to Proposition (4.3), to the latter corresponds the map \( \text{sch}' : kL[2] \to \text{sgn}_2 \) such that

\[
\text{sch}'(z) = \text{sch}({1}, {2})(z) - \text{sch}({1}, {2})(L[\tau](z))
\]

for all \( z \in L[2] \) —here \( \tau \in S_2 \) is the transposition— from which it is obvious that \( \text{sch}(z_2) = 1 \) and, as a consequence of this, that \( \text{sch}' = \alpha_2 \).

\[\square\]

**Simplicial complexes**

(8.4) If \( K \) is an abstract simplicial complex on a set \( X \), then there is a set-valued species with right restrictions \( X_K \) such that
• for each $I \in \text{Fin}$ the set $x_K[I]$ is that of all functions $z : I \to X$ whose image is a simplex of $K$, and
• for each injection $f : J \to I$ in $\text{Fin}_<$ we have $x_K[f] : z \in x_K[I] \mapsto z \circ f \in x_K[J]$.

We view the elements of $x_K[I]$ as “singular simplices” in the complex $K$.

There are two natural ways in which we can turn the left $\mathcal{E}$-comodule $k \otimes x_K$ obtained by linearization of the set-valued species $x_K$ into a $\mathcal{E}$-bicomodule, and there are correspondingly two natural cohomologies to consider, and we consider both.

(8.5). **Proposition.** Let $k$ be a field of characteristic zero. Let $K$ be an abstract simplicial complex on a nonempty finite set $X$ and endow the linearization $k \otimes x_K$ of the set-valued species with right restrictions $x_K$ with its canonical structure of left $\mathcal{E}$-comodule.

(i) If we turn the left $\mathcal{E}$-comodule $k \otimes x_K$ into a cosymmetric $\mathcal{E}$-bicomodule, and into a coalgebra in $\mathcal{E}_{\text{Mod}}$ as in Proposition (5.9), then there is an isomorphism of graded algebras

$$H^*(k \otimes x_K) \cong k \{K\}$$

to the exterior face ring $k \{K\}$ of the simplicial complex $K$.

(ii) If instead we turn the left $\mathcal{E}$-comodule $k \otimes x_K$ into a $\mathcal{E}$-bicomodule with trivial right coaction, then there is an isomorphism of graded vector spaces

$$H^*(k \otimes x_K) \cong \overline{H}^*(\Sigma K)$$

to the reduced cohomology of the suspension $\Sigma K$ of $K$ with values in $k$.

Here the exterior face ring $k \{K\}$, as defined for example in [8, §5.2.1], is the quotient of the exterior algebra $\Lambda(kX)$ on the $k$-module $kX$ freely generated by the set $X$ of vertices by the ideal generated by the squarefree monomials $x_1 \cdots x_r$ in the elements of $X$ such that $\{x_1, \ldots, x_r\}$ is not a simplex in $K$. This is, of course, a graded-commutative analogue of the Stanley-Reisner ring of the abstract simplex complex $K$.

**Proof.** In view of the hypothesis on the ground ring $k$, the $\mathcal{E}$-bicomodule $k \otimes x_K$ is relatively projective in the two cases considered, and we can compute everything using the complex $\Lambda^*(k \otimes x_K)$.

In fact, as the bicomodule is cosymmetric, we know from Proposition (8.1) that there is an isomorphism $H^*(k \otimes x_K) \cong \Lambda^*(k \otimes x_K)$ of graded vector spaces. We view it as an identification, and then we are left with showing that $\Lambda^*(k \otimes x_K)$, endowed with the product of Corollary (5.7), is isomorphic to $k \{K\}$ as an algebra.

Let $p \geq 0$, let $F_p$ be the set of $(p - 1)$-simplices in $K$, so that in particular $F_0 = \{\emptyset\}$. We fix an arbitrary total order on the set $X$ of vertices, and for each $s \in F_p$ we denote $z_s : [p] \to X$ the element of $x_K[p]$ whose image is $s$ and which is a strictly increasing
function, and $m_s$ the monomial $z_s(1)z_s(2)\cdots z_s(p)$ of $\mathbb{k}\{K\}$; the set \(\{m_s : s \in F_p\}\) is a basis of $\mathbb{k}\{K\}_p$, the homogeneous component of degree $p$ of $\mathbb{k}\{K\}$.

The function $\Phi^p : \phi \in \mathcal{X}^p(\mathbb{k}x_K) \mapsto \sum_{s \in F_p} \phi(z_s)m_s \in \mathbb{k}\{K\}_p$ is an isomorphism of vector spaces. This follows at once from the following two observations:

- If we let $x_K^+\|p]$ and $x_K^-\|p]$ be the subsets of $x_K\|p]$ of the elements which are injective and of those which are not injective, respectively, then there is a direct sum decomposition $\mathbb{k}x_K\|p] = x_K^+\|p] \oplus x_K^-\|p]$ and every $\phi \in \mathcal{X}^p(\mathbb{k}x_K)$ vanishes on $x_K^-\|p]$. Indeed, if $\phi \in \mathcal{X}^p(\mathbb{k}x_K)$ and $z \in x_K^-\|p]$, there exists a transposition $\tau \in S_p$ such that $x[\tau](z) = z$ and therefore Lemma (8.2) tells us that $\phi(z) = 0$.
- The $S_p$-submodule $\mathbb{k}x_K^+\|p]$ is freely generated by the set $\{z_s : s \in F_p\}$.

We obtain in this way an isomorphism of graded vector spaces $\Phi : \mathcal{X}^\ast(\mathbb{k}x_K) \to \mathbb{k}\{K\}$, and it is multiplicative: indeed, if $s \in F_p$ and $t \in F_q$, then using the formula for the product on $\mathcal{X}^\ast(\mathbb{k}x_K)$ given in Corollary (5.7) we easily see that

$$\Phi(\Phi^{-1}(m_s) \cdot \Phi^{-1}(m_t)) = \begin{cases} 0, & \text{if } s \cup t \text{ is not an element of } F_{p+q}; \\ \pm m_{s\cup t}, & \text{if it is.} \end{cases}$$

Here the sign $\pm$ is that of the permutation that rearranges the letters of the word $z_s(1)\cdots z_s(p)z_t(1)\cdots z_t(q)$ in order to form $z_{s\cup t}(1)\cdots z_{s\cup t}(p+q)$. $\square$

It is worth to observe that we can consider a sub-bicomodule $\mathbb{k}x_K'$ of the symmetric bicomodule $\mathbb{k}x_K$ given by the functions who have image simplices of $K$ and are also injective. Then the inclusion $\mathbb{k}x_K' \hookrightarrow \mathbb{k}x_K$ is a quasi-isomorphism in any characteristic, and one can recover the cohomology algebra of $\mathbb{k}x_K$ in this more general situation.

(8.6) If we put on $\mathbb{k}x_K$ the trivial right $e$-comodule structure, cohomology changes significantly.

**Proposition.** Let $\mathbb{k}$ be a field of characteristic zero. Let $K$ be an abstract simplicial complex on a nonempty finite set $X$ and endow the linearization $\mathbb{k}x_K$ of the set-valued species with right restrictions $x_K$ with its canonical structure of left $e$-comodule. If we view $\mathbb{k}x_K$ as a $e$-bicomodule with its trivial right $e$-comodule structure, then there is an isomorphism of graded vector spaces

$$H^\ast(\mathbb{k}x_K) \cong \tilde{H}^\ast(SK, \mathbb{k})$$

from the cohomology of $\mathbb{k}x_K$ to the reduced cohomology of the suspension $SK$ of $K$ with values in $\mathbb{k}$.

**Proof.** In view of the hypothesis on $\mathbb{k}$, we can use Proposition (4.8) to compute $H^\ast(\mathbb{k}x_K)$. Let us fix a total order on the set $X$ and for each $p \geq -1$ let $K_p$ be the set of $p$-simplices in $K$, so that in particular $K_{-1} = \emptyset$; if $s \in K_p$, then we write $s = (x_0, \ldots, x_p)$ to
mean that \( s = \{x_0, \ldots, x_p\} \) and that the order of \( X \) restricted to \( s \) is \( x_0 < \cdots < x_p \). We write \( kK_p \) the linearization of the set \( K_p \) and \( kK_p^* \) its dual space, and we let \( kK^*_p \) be the augmented cochain complex of vector spaces

\[
kK^*_{p-1} \xrightarrow{d^{-1}} kK^*_0 \xrightarrow{d^0} kK^*_1 \xrightarrow{d^1} kK^*_2 \xrightarrow{d^2} \cdots
\]

with differentials given by

\[
d^p \phi(x_0, \ldots, x_{p+1}) = \sum_{i=0}^{p+1} (-1)^i \phi(x_0, \ldots, \hat{x}_i, \ldots, x_{p+1})
\]

for all \( p \geq -1 \) and all \( (x_0, \ldots, x_{p+1}) \in K_{p+1} \). The cohomology of this complex is the reduced cohomology of \( K \) with values in \( k \).

Let \( r \geq 0 \). If \( s \in K_{r-1} \), we write \( z_s \) the unique element of \( x_K[[r]] \) whose image is \( s \) and which is a strictly increasing function. The function \( F^r : \mathcal{F}^r(kx_K) \to kK^*_{r-1} \) such that \( F^r(\phi)(s) = \phi(z_s) \) for all \( s \in K_{r-1} \) and all \( \phi \in \mathcal{F}^r(kx_K) \) is an isomorphism: this follows at once from the following two observations:

- If we let \( x^+_K[[r]] \) and \( x^-_K[[r]] \) be the subsets of \( x_K[[r]] \) of the elements which are injective and of those which are not injective, respectively, then there is a direct sum decomposition \( kx_K[[r]] = kx^+_K[[r]] \oplus kx^-_K[[r]] \) and every \( \phi \in \mathcal{F}^r(kx_K) \) vanishes on \( kx^-_K[[r]] \). Indeed, if \( \phi \in \mathcal{F}^r(kx_K) \) and \( z \in x^-_K[[r]] \), there exists a transposition \( \tau \in S_r \) such that \( x[\tau](z) = z \) and therefore Lemma (8.2) tells us that \( \phi(z) = 0 \).
- The \( S_r \)-submodule \( kx^+_K[[r]] \) is freely generated by the set \( \{z_s : s \in K_{r-1}\} \).

The collection \((F^r)_{r \geq 0}\) of these isomorphisms gives an isomorphism of cochain complexes \( F^* : \mathcal{F}^*(kx_K) \to kK^*[-1] \) and it follows from this, of course, that there is an isomorphism \( H^*(kx_K) \cong H^*(K,k)[-1] \). Composing this with the standard isomorphism \( H^*(SK,k) \cong H^*(K,k)[-1] \) we see that the proposition holds.

\( \square \)

The blow-up of a set-valued species

(8.7) If \( I \) is a set, we denote \( \Pi(I) \) the set of all partitions of \( I \), that is, of all disjoint families \( \pi \) of nonempty subsets of \( I \) such that \( \bigcup_{b \in \pi} b = I \). If \( f : J \to I \) is an injective function, there is a function \( f^* : \Pi(I) \to \Pi(J) \) such that \( f^*(\pi) = \{f^{-1}(b) : b \in \pi\} \setminus \{\emptyset\} \) and, for each \( \pi \in \Pi(I) \), a function \( f_\pi : f^*(\pi) \to \pi \) such that \( f_\pi(b) \subseteq b \) for all \( b \in f^*(\pi) \).

If now \( x \) is a set-valued species with right restrictions, we can construct a new set-valued species with right restrictions \( x^+ \): for each \( I \in \text{Fin} \) we put

\[
x^+[I] = \bigsqcup_{\pi \in \Pi(I)} x[\pi]
\]

and for an injective function \( f : J \to I \) let

\[
x^+[f] : x^+[I] \to x^+[J]
\]
be the unique function whose restriction to each \( x[\pi] \) is \( x[f_\pi] : x[\pi] \to x[f^*(\pi)] \). In other words, if \( x \) maps each \( I \in \text{Fin} \) to the set of structures of some kind labeled with elements of \( I \), then \( x^+ \) maps each such \( I \) to the set of structures of the same kind labeled by subsets of \( I \) which form a partition of \( I \). For example, the set-valued species \( L^+ \) maps each set \( I \) to the set of all compositions of \( I \), which are nothing but total orders on \( I \).

It is useful to observe that if \( e^+ \) is the species of sets with \( e^+[\emptyset] = \emptyset \), then \( x^+ \) is obtained as the composition \( x \circ e^+ \).

**Proposition.** Let \( x \) be a set-valued species with right restrictions. For each \( I \in \text{Fin} \) let us write \( \pi_I \) the partition \( \pi_I = \{ \{ i \} : i \in I \} \in \Pi(I) \) and let us consider the bijection \( \iota_I : \pi_I \to I \) such that \( \iota_I(\{ i \}) = i \) for all \( i \in I \). There is a morphism of set-values species with restrictions \( \omega : x \to x^+ \) that for each \( I \in \text{Fin} \) has

\[
\omega[I] : z \in x[I] \mapsto x[\iota_I](z) \in x[\pi_I] \subseteq x^+[I]
\]

and the induced morphism \( H^*(kx^+) \to H^*(kx) \) from the cohomology of the linearization \( x^+ \) to that of \( x \) is an isomorphism.

**Proof.** In fact, the morphism of complexes \( \lambda^*(kx^+) \to \lambda^*(kx) \) induced by \( \omega \) is already an isomorphism: a \( p \)-cochain \( \phi : kx^+[[p]] \to \text{sgn}_p \) in the complex \( \lambda^p(kx^+) \) vanishes on the submodule of \( kx^+[[p]] \) spanned by the elements of \( x^+[[p]] \) which are not in \( x[\pi_I] \), as they have odd automorphisms. \( \square \)

(8.8) Let us consider two examples. First, taking \( x = e \), we obtain that \( x^+ \) is the species of partitions \( \Pi \), whose restriction maps are obtained by intersecting the blocks of a partition with a subset and discarding empty intersections. As a second example, taking \( x = L \) the species of linear orders, we obtain that \( L^+ \) is the species of compositions \( \Sigma \).

Thus \( H^*(\Pi) \) is a free commutative algebra generated by one element in degree 1, while \( H^*(\Sigma) \) is a free commutative algebra generated by an element in degree 1 and an element in degree 2.

**Other partial computations:** graphs

(8.9) If \( x \) is a set-valued species with restrictions taking finite sets as values and if \( k \) is a field of characteristic zero, the computation of the cohomology of the linearization \( kx \) in any finite range of degrees is a finite problem. For example, a computer calculation shows that if \( x \) is the set-valued species such that \( x[I] \) is, for every \( I \in \text{Fin} \), the set of simple graphs with \( I \) as vertex set, where the left and right coactions are given by graph induction and contraction, then \( H^p(\mathbb{Q}x) = 0 \) for all \( p \leq 7 \). This was done by the authors by starting with Brendan McKay’s lists of isomorphism types of simple graphs [11], filtering out types with odd automorphisms, and explicitly computing the matrices
of differentials using Mathematica. This suggest that the cohomology of this species should be zero.

§9. Deformations

Let \((A, \mathfrak{m})\) be a local commutative \(k\)-algebra with \(A/\mathfrak{m} \cong k\), and let us view this isomorphism as an equality. There are functors \(r: \mathcal{A} \mathcal{S} p \to k \mathcal{S} p\) and \(s: k \mathcal{S} p \to A \mathcal{S} p\) such that if \(x\) is in \(A \mathcal{S} p\), then we have \(r(x)[I] = k \otimes_A x[I]\) for each \(I \in \text{Fin}\), and if \(x\) is in \(k \mathcal{S} p\), then we have \(s(x)[I] = A \otimes x[I]\) for all \(I \in \text{Fin}\), and which on morphisms act in the natural way. We will write \(k \otimes_A x\) instead of \(r(x)\), and \(A \otimes x\) instead of \(s(x)\).

These two functors are bistrong monoidal in an obvious way, in the sense of \([2, \S 3.1]\), and in particular they preserve (bi/co)algebras and their representations. In particular, and since clearly \(k \otimes_A A e \cong e\), the functor \(r\) induces a functor \(A e \text{-} \text{Mod} \to e \text{-} \text{Mod}\).

If \(x\) is a \(e\)-bicomodule, a deformation of \(x\) along \(A\) is a pair \((\tilde{x}, \xi)\) in which \(\tilde{x}\) a \(A e\)-bicomodule whose underlying \(A\)-linear species is \(A \otimes x\) and \(\xi: A/\mathfrak{m} \otimes \tilde{x} \to x\) is an isomorphism in \(e \text{-} \text{Mod}\). If \(A = k[[h]]/\langle h^2 \rangle\) or \(A = k[[h]]\), we speak of infinitesimal or formal deformations.

(9.1) Fix a comonoid \(x\) in \(\mathcal{S} p_k\), with comultiplication \(\Delta: x \to x \otimes x\) and counit \(\varepsilon: x \to 1\), and set \(K = k[[t]]\), the algebra of formal powerseries in \(k\). We write \(c_t\) for the coalgebra in \(\mathcal{S} p_k\) obtained by extending scalars pointwise in \(x\). By a weak deformation of \(x\) along \(K\) we mean coalgebra structure \(\Delta_t: x_t \to x_t \otimes x_t\) of the form

\[
\Delta_t(S,T)(z) = \sum_{i \geq 0} \Delta_i(S,T)(z) c^0_i \otimes c^1_i t^i
\]

where \(\Delta_i: x \to e^2\) and \(\Delta_0 = 1\).

These deformation are weak in the sense we are only modifying the coefficients of our comultiplication by means of a powerseries coefficient and not modifying the higher order terms of the comultiplication. Had we done this, we would have obtained maps \(\Delta_i: x \to x^2\) that are cochains in \(\Omega^2(x,x)\) —we are only allowing for cochains in \(\Omega^2(x,e)\). The condition that \(\Delta_t\) is unital is equivalent to the collection of equalities

\[
\sum_{i+j=n} \left( \Delta_i(RS,T)(z) \Delta_j(R,S)(z^1 f g) - \Delta_i(R,ST)(z) \Delta_j(S,T)(z^0 g h) \right) = 0 \quad (D_n)
\]

for \(n \in \mathbb{N}_0\) and \((R,S,T)\) an arbitrary decomposition of a finite set. The condition that \(\Delta_i\) is counital is equivalent to each \(\Delta_i\) being normalized for \(i \geq 1\). In particular \((D_1)\) says that \(\Delta_1\) belongs to \(Z^2(x,e)\), as expected. Thus, every weak deformation of \(c_t\) of \(x\) gives a corresponding 2-cocycle.
Given cochains $\alpha, \beta \in C^2(x,e)$, we write $\alpha \star_{1,0} \beta$ and $\alpha \star_{0,1} \beta$ for the 3-cochains such that

$$(\alpha \star_{1,0} \beta)(R,S,T)(z) = \alpha(R,ST)(z)\beta(S,T)(z_{gh}),$$  
$$(\alpha \star_{0,1} \beta)(R,S,T)(z) = \alpha(RS,T)(z)\beta(R,T)(z_{1}^{f}).$$

In turn, we define a Gerstenhaber bracket using the following formulae:

$$\alpha \star \beta = \alpha \star_{1,0} \beta - \alpha \star_{0,1} \beta, \quad \{\alpha,\beta\} = \alpha \star \beta - \beta \star \alpha.$$  

With this at hand, we can write equation $(D_n)$ as $\sum_{i+j=n} \Delta_i \star \Delta_j = 0$. We say a 2-cocycle $\Delta_1$ is integrable if it arises in this way from a deformation of $x$. We note that from $(D_2)$ that if $\Delta_1$ is integrable, then its first obstruction $\sigma_1 = \Delta_1 \star \Delta_1$ must me a coboundary in $C^3(x,e)$. A lengthy calculation shows that

**Theorem.** The first obstruction $\sigma_1$ is a 3-cocycle.

(9.2) One can use every 2-cocycle to deform a coalgebra in the weak sense in characteristic zero, which is, in fact, the sense in which Aguiar and Mahajan intend to deform coalgebras in [2].

**Theorem.** Suppose $k$ is of characteristic zero and let $\Delta_1$ be a normalized 2-cocycle $x \rightarrow e^2$. Then there exists a weak deformation $x_t$ of $x$ corresponding to $\Delta_1$.

**Proof.** We set, for each $i \in \mathbb{N}_0$ and each decomposition $(S,T)$ of a finite set,

$$\Delta_i(S,T)(z) = \frac{1}{i!} \Delta_1(S,T)(c)^i$$

and observe that equation $(D_n)$ can be obtained considering

$$\Delta_1(RS,T)(c) + \Delta_1(R,S)(c^R_{ST}) = \Delta_1(R,ST)(c) + \Delta_1(S,T)(c^0_{ST})$$

and raising the left and right terms to the $n$th power. \qed

Remark that, if we write $\Delta_i(S,T)(c) = \exp(\Delta_1(S,T)(c)t)$, then the analogy with the deformations considered in [2] is made evident by the change of variables $q = e^t$. Moreover, what we did above shows one can integrate a cocycle up to degree $p-1$ when $k$ is of characteristic $p$, and in such case the $p$th obstruction to the integrability of $\Delta_1$ is

$$\sigma_p = -\frac{1}{p} \sum_{i=1}^{p-1} \binom{p}{i} \{\Delta^i, \Delta^{p-i}\}.$$
References


