# Techniques of Analysis

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# Chapter 1

# Introduction

We'll assume that, where appropriate, we are working in a metric space $(X, d)$, a normed space $(X, ||.||)$ or a topological space $(X, \tau)$. If we require more than one space, we'll use the metric space $(Y, d')$ and $(Z, d'')$, the normed spaces $(Y, ||.||')$ and $(Z, ||.||'')$ and the topological spaces $(Y, \tau')$ and $(Z, \tau'')$. Subscripts may alternately be used to denote different metrics, norms or topologies, i.e. $||.||_1$, $d_2$, $\tau_3$ etc. Normed spaces are all assumed to be over $\mathbb{R}$

We note that we can consider each metric space as a normed space by letting $d(x, y) = ||x - y||$, and so the tests for metric spaces are the same as those for normed spaces.

If we wish to disprove, e.g. that a set is not closed, it suffices to show that any one the tests for closed sets does not hold.

# Chapter 2

# Sets

## 2.1 Open Sets

We can test whether a set $A$ is open as follows:
Metric Space

- (**Def.**) If for all $x \in A$, there exists some $r > 0$ such that $B(x, r) \subset A$. Equivalently, if it equals its interior.
- If $X \setminus A$ is a closed set.
- If we can decompose $A$ into some countable sequence of open sets $A_1, A_2, \ldots$, with $A = \bigcup_{n=1}^{\infty} A_n$. Also, if we can decompose $A$ into some finite sequence of open sets $A_1, A_2, \ldots, A_n$ with $A = \bigcap_{k=1}^{n} A_k$.
- If there exists some continuous mapping $T : X \to Y$, with $T(A)$ open. Typically, we'll let $Y = \mathbb{R}^n$.

Topological Spaces

- (**Def.**) If $A \in \tau$

Topological Spaces

- (**Def.**) If for all $x \in A$, there exists some $r > 0$ such that $\{y \in X \mid ||x - y|| < r\} \subset A$.

## 2.2 Closed Sets

We can test whether a set $A$ is open as follows:
Metric Space

- (**Def.**) If it contains all its limit points, i.e. if there is a sequence $(x_n)$ in $A \setminus \{x\}$ which converges to $x$, then $x \in A$.
- If $X \setminus A$ is an open set. Equivalently, if it equals its closure.

- If we can decompose $A$ into some countable sequence of closed sets $A_1, A_2, \ldots$, with $A = \bigcap_{n=1}^{\infty} A_n$. Also, if we can decompose $A$ into some finite sequence of closed sets $A_1, A_2, \ldots, A_n$ with $A = \bigcup_{k=1}^{n} A_k$.
- If there exists some continuous mapping $T : X \to Y$, with $T(A)$ closed . Typically, we'll let $Y = \mathbb{R}^n$.
- If $(A, d_A)$ is complete.

Topological Spaces

- (**Def.**) If $X \setminus A \in \tau$
- If $A$ is a compact subset of $\mathbb{R}^n$.

### 2.2.1 Note

One way to show that a set $A$ is not open/closed is to show that, if it is in a connected metric space, that it is closed/open. As $\mathbb{R}^n$ is connected, we can also consider the image of $A$ in $\mathbb{R}^n$ under any continuous map.

# Chapter 3

# Sequences

## 3.1 Convergence

A sequence $(x_n)_{n=1}^{\infty}$ converges to a point $x$:
Metric Spaces

- (**Def.**) If for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ whenever $n \geq N$.

Topological Spaces

- (**Def.**) If for all open neighbourhoods $U$ of $x$ there exists some $N \in \mathbb{N}$ such that $x_n \in U$ whenever $n \geq N$.

## 3.2 Cauchy Sequences

We can test if a sequence $(x_n)$ is cauchy as follows:
Metric Space

- (**Def.**) If for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ whenever $m, n > N$.
- If the sequence converges.

## 3.3 Completeness

We can test if a subspace $(A, d_A)$ of $X$ is complete as follows:
Metric Spaces

- (**Def.**) If every cauchy sequence in $A$ converges to a point in $A$.
- If $(X, d)$ is complete and $A$ is closed.

# Chapter 4

# Continuity

## 4.1 Continuity

A mapping $T : X \to Y$ is continuous if:
Metric Spaces

- (**Def.**) it is continuous at all points in $X$, i.e. for all $x \in X$ and for all $\epsilon > 0$ there exists some $\delta > 0$ such that $d'(T(x), T(y)) < \epsilon$ whenever $d(x, y) < \delta$.
- for all sequences $(x_n)$ in $X$ converging to $x$, the sequence $(T(x_n))$ converges to $T(x)$ in $Y$.
- for all $U$ open in $Y$, $T^{-1}(U)$ is an open set in $X$.
- it is the composition of continuous mappings.
- there exists some sequence of continuous functions $(T_n)_{n\in\mathbb{N}}$ with $T_n : X \to Y$ which converge uniformly to $T$.

Topological Spaces

- (**Def.**) for all $U$ open in $Y$, $T^{-1}(U)$ is an open set in $X$.

Normed Spaces

- $T$ is linear and there exists some $M \in \mathbb{R}$ such that $||T(x)|| \leq M||x||$ for all $x \in X$

## 4.2 Uniform Continuity

We can determine if a mapping $T : X \to Y$ is uniformly continuous as follows:
Metric Spaces

- (**Def.**) If for all $\epsilon > 0$ there exists some $\delta$ such that for all $x \in X$, $d'(T(x), T(x_0)) < \epsilon$ whenever $d(x, x_0) < \delta$.
- If $T$ is continuous in the normal sense and $X$ is compact.

# Chapter 5

# Specifics for Metric Spaces

## 5.1 Metrics

- Euclidean on $\mathbb{R}^n$: $d(x,y) = \sqrt{\sum_{k=1}^{n}(x_k - y_k)^2}$
- Discrete: $d(x,y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$
- Taxi-cab on $\mathbb{R}^2$: $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$
- Irish Rail on $\mathbb{R}^n$: $d(x,y) = \begin{cases} 0 & x = y \\ d_1(x,0) + d_1(0,y) & x \neq y \end{cases}$ where $d_1$ is the Euclidean metric.
- On $\mathbb{C}$: $d(z,w) = |z - w|$.
- On $L^\infty(I)$: $d(f,g) = \sup_{x \in I} |f(x) - g(x)|$
- On $L^1(I)$: $d(f,g) = \int_I |f(x) - g(x)|\, dx$
- On Sequence Spaces: $d(x,y) = \sup_{k \in \mathbb{N}} |x_k - y_k|$ where $x = (x_k)_{k \in \mathbb{N}}$ and $y = (y_k)_{k \in \mathbb{N}}$ are real sequences converging to the same point.

## 5.2 Convergence of Functions

Given a sequence of functions $(f_n)_{n \in \mathbb{N}}$ where $f_n : X \to Y$, and some $f : X \to Y$, we make the following definitions:

- **Pointwise:** The sequence converges pointwise to $f$ if for all $x_0 \in X$ and for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that $d'(f_n(x_0), f(x_0)) < \epsilon$ whenever $n \geq N$.
- **Uniformly:** The sequence converges uniformly to $f$ if for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that $d'(f_n(x_0), f(x_0)) < \epsilon$ whenever $n \geq N$, for all $x_0 \in X$.

The difference between the two is that $N$ is different for each $x_0$ for pointwise convergence and $N$ is fixed for every $x_0$ for uniform convergence.

## 5.3 Isometry

A mapping $T : X \to Y$ is called an isometry if for all $x, x_0 \in X$, $d(x, x_0) = d'(T(x), T(x_0))$, i.e. the mapping preserves the distance.

### 5.3.1 Properties

Given an isometry $t : X \to Y$, we have the following:

- $T$ is injective
- $T$ is continuous
- If $T$ is surjective, then it is a homeomorphism.

# Chapter 6

# Specifics for Normed Spaces

## 6.1 Normed Spaces

- $(\mathbb{R}^n, ||.||)$, where $||.||$ is the Euclidean norm, $||x|| = d(x, 0)$, where $d$ is the Euclidean metric.
- $((C), |.|)$, where $|.|$ is the usual modulus function.
- $(L^p(I), ||.||_p)$, where $||f||_p = \left( \int_I |f(x)|^p \, dx \right)^{\frac{1}{p}}$
- $(c_0, ||.||)$, where $c_0$ is the set of all real sequences converging to 0 and $||x|| = \sup_{k \in \mathbb{N}} |x_k|$
- $(l^p, ||.||)$, where $l^p$ is the set of all real sequences such that $\sum_{n=1}^{\infty} (x_n)^p < \infty$ and $||x|| = \left( \sum_{n=1}^{\infty} (x_n)^p \right)^{\frac{1}{p}}$

## 6.2 Operator Norms

Given a linear operator $T : X \to Y$, the norm of $T$ is given by any of the following:

- (**Def.**) $\sup_{||x|| \leq 1} ||T(x)||$
- $\inf \{M \in \mathbb{R} \ : \ ||T(x)|| \leq M||x|| \ \forall x \in X\}$
- $\sup_{x \neq 0} \dfrac{||T(x)||}{||x||}$
- $\sup_{||x||=1} ||T(x)||$

The operator norm, as well as being a norm, has the property that if $S, T : X \to X$, then $||ST|| \leq ||S|| \cdot ||T||$

## 6.3 Equivalent Norms

If we have two norms $||.||_1$ and $||.||_2$ on a vector space $X$, then the norms are equivalent if:

- (**Def.**) There exists some $a, b \in \mathbb{R}$ such that $a||x||_1 \leq ||x||_2 \leq b||x||_1$ for all $x \in X$
- The norms generate the same open sets, i.e. A set $A$ is open with respect to $||.||_1$ iff $A$ is open with respect to $||.||_2$.
- A sequence $(x_n)$ in $X$ converges to $x$ with respect to $||.||_1$ iff $(x_n)$ converges to $x$ with respect to $||.||_2$

If our vector space is finite-dimensional, then all norms are equivalent.

## 6.4 Banach Spaces

A Banach space is a normed space that is complete with respect to the metric $d(x, y) = ||x-y||$.

Every finite dimensional normed vector space is a Banach space.

# Chapter 7

# Specifics for Topological Spaces

## 7.1 Topologies

Given some $X \neq \varnothing$ and $\tau \subseteq \mathcal{P}(X)$, then $\tau$ is a topology if:

- $\varnothing \in \tau$ and $X \in \tau$
- The union of any collection of sets in $\tau$ is also in $\tau$.
- The intersection of any finite collection of sets in $\tau$ is in $\tau$.

## 7.2 Hausdorff

A topological space $(X, \tau)$ is called a Hausdorff space on if for all $a, b \in X$ there exist some $U, V \in \tau$ with $a \in U$, $b \in V$ and $U \cap V = \varnothing$.

## 7.3 Connected Spaces

We can determine if a set $A \subseteq X$ is connected as follows:

- (**Def.**) If there does not exists a pair of non-empty sets $U$ and $V$ which are disjoint, open with $U \cup V = A$.
- If the only sets that are both open and closed in $(A, d_A)$ are $\varnothing$ and $A$
- If every continuous mapping $f : A \to \{0, 1\}$ is constant.
- If $A$ is a closed interval on the real line.
- If $T : Y \to A$ is continuous and surjective and $Y$ is connected.
- If $A$ can be decomposed into connected sets $A_1, A_2, \ldots, A_n$ with $A = A_1 \times A_2 \times \cdots \times A_n$.
- If $A$ is the closure of some connected space.
- If $A$ is path connected, i.e. if for all $x, y \in A$ there exists a continuous mapping $\alpha : [0, 1] \to A$ with $\alpha(0) = x$ and $\alpha(1) = y$.

## 7.4 Compactness

We can determine if a set $A \subseteq X$ is compact as follows:

- (**Def.**) Given and collection of open sets $\{U_\alpha \mid \alpha \in I\}$ such that $A = \bigcup_{\alpha \in I} U_\alpha$, there exists some finite sub-collection such that $A = \bigcup_{k=1}^{n} U_k$.
- If $A$ is closed and bounded subset of $\mathbb{R}^n$.
- If $T : Y \to A$ is continuous and surjective and $Y$ is compact.
- If $A$ can be decomposed into compact sets $A_1, A_2, \ldots, A_n$ with $A = A_1 \times A_2 \times \cdots \times A_n$.