Distributions

Course: Prof. Tony Dorlas 2011
Typset: Cathal Ormond

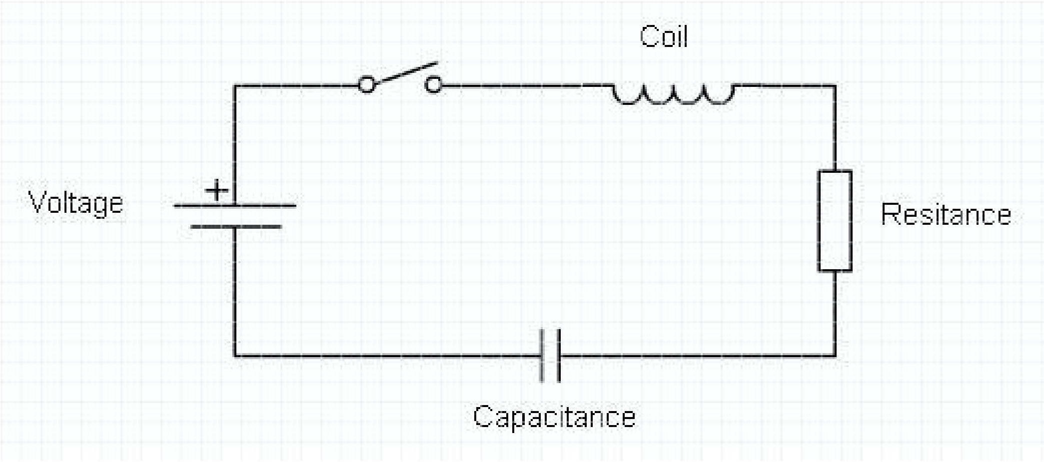
March 28, 2011

Contents

Chapter 1

Introduction

We will consider an example from physics where we require the use of distributions to solve problems. Consider the following circuit diagram:



We have a circuit with voltage V, resistance R, capacitance C and inductance in the coil L. When we flip the switch at time $t = 0$, we have a constant voltage flowing through the circuit. Thus, we have $V(t) = V_0\theta(t)$, where V_0 is a constant, and $\theta(t)$ is the Heaviside function given by

$$\theta(t) = \left\{ \begin{array}{ll} 1 & t \geq 0 \\ 0 & t < 0 \end{array} \right\} = \mathbf{1}_{\mathbb{R}^+}(t)$$

If we let $I(t)$ be the current flowing through the circuit at time t, then we also have

$$V(t) = V_0\theta(t) = LI'(t) + RI(t) + \frac{1}{C}\int_0^t I(\tau)\, d\tau$$

If we differentiate this, we will have an ODE in I, so we get:

$$V'(t) = V_0\theta'(t) = LI''(t) + RI'(t) + \frac{1}{C}I(t)$$

Let $\theta'(t) = \delta(t)$. This will be called the delta function. What we want is that $\displaystyle\int_{-\infty}^{t} \delta(\tau)\ d\tau = \theta(t)$. Our problem with the above is that we know that $\delta = 0$ a.e., and so $\int \delta(t)\ dt = 0$, so δ is not a function.

Suppose that δ did exist, and let f be any function. Then we have

$$\int_{-\infty}^{\infty} f(t)\delta(t)\ dt = \int_{\{0\}} f(t)\delta(t)\ dt + \int_{\mathbb{R}\backslash\{0\}} f(t)\delta(t)\ dt = f(0)$$

By substitution, this gives us

$$\int_{-\infty}^{\infty} f(t-x)\delta(x)\ dx = f(t)$$

Thus, if we have a solution to the above ODE, a solution to

$$LI''(t) + RI'(t) + \frac{1}{C}I(t) = f(t)$$

will be $(f * I)(t)$. We know that as a measure, δ is in fact well defined. Recall the following:

Theorem 1.1 (Riesz Representation Theorem)**.** *Let μ be a positive Radon measure on $\mathbb{R}^n$, which is a locally compact space. Then*

$$L(f) = \int f\ d\mu$$

defines a positive linear form on $\kappa(\mathbb{R}^n)$, where

$$\kappa(\mathbb{R}^n) = \{F : \mathbb{R}^n \to \mathbb{R} \mid f \text{ is continuous and } supp(f) \text{ is compact }\}$$

Conversely, if L is a positive linear form that satisfies the above, then there exists a unique Radon Measure such that

$$L(f) = \int f\ d\mu$$

We can generalise this even further.

Theorem 1.2 (Riesz Representation Theorem in $\mathbb{C}$)**.** *Let μ be a complex-valued Radon measure on $\mathbb{R}^n$, which is a locally compact space. Then*

$$L(f) = \int f\ d\mu$$

defines a positive linear form on $\kappa_{\mathbb{C}}(\mathbb{R}^n)$, where

$$\kappa_{\mathbb{C}}(\mathbb{R}^n) = \{F : \mathbb{R}^n \to \mathbb{C} \mid f \text{ is continuous and } supp(f) \text{ is compact }\}$$

Conversely, if L is a continuous linear form that satisfies the above, then there exists a unique complex-valued Radon Measure such that

$$L(f) = \int f\ d\mu$$

Our definition of δ is not quite sufficient: we may wish to differentiate δ. To do this, we take the integration by parts formula as a definition, so for a function $\varphi \in \kappa(\mathbb{R})$

$$\begin{aligned}\int \dot{\delta}(t)\varphi \; dt &= [\delta(t)\varphi(t)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(t)\dot{\varphi}(t) \; dt \\ &= -\int_{-\infty}^{\infty} \delta(t)\dot{\varphi}(t) \; dt \\ &= \dot{\varphi}(0)\end{aligned}$$

as φ has compact support. Our problem with this is that we only assumed φ to be continuous, not differentiable.

Definition 1.3 (Differential Space)**.** Given an open domain $\Omega \subseteq \mathbb{R}^n$, we make the following definitions

$$\mathcal{D}(\Omega) = \{\varphi : \Omega \to \mathbb{C} \mid \varphi \text{ is } C^\infty, \operatorname{supp}(\varphi) \text{ is compact } \} = C^\infty(\Omega) \cap \kappa_{\mathbb{C}}(\Omega)$$

$$\mathcal{D}_K = \{\varphi \in \mathcal{D}(\Omega) \mid \operatorname{supp}(\varphi) \subseteq K\}$$

A sequence $(\varphi_n)_{n=1}^{\infty}$ in $\mathcal{D}_K$ is said to converge to $\varphi \in \mathcal{D}_K$ if the following hold:

- $\|\varphi_n - \varphi\|_\infty$
- $\partial^p \varphi_n \to \partial^p \varphi$ uniformly for all multi-indices p

where $p \in \mathbb{N}_0^n = \{(p_1, p_2, \ldots, p_n) \mid p_i \in \mathbb{N}, i = 1, 2, \ldots, n\}$ and

$$\|\psi\|_\infty = \sup_{x \in \mathbb{R}^n} |\psi(x)|$$

$$\partial^p \varphi(x) = \frac{\partial^{p_1}}{\partial x_1^{p_1}} \frac{\partial^{p_2}}{\partial x_2^{p_2}} \cdots \frac{\partial^{p_n}}{\partial x_n^{p_n}} \varphi(x)$$

We'll use the notation $\varphi_n \xrightarrow[\mathcal{D}_K]{} \varphi$

Definition 1.4 (Distribution)**.** A <u>Distribution</u> on Ω is a continuous linear map $T : \mathcal{D}(\Omega) \to \mathbb{C}$, in the sense that if $\varphi_n \to \varphi$ in $\mathcal{D}_K$, then $T(\varphi_n) \to T(\varphi)$ for all $K \subseteq \Omega$. The set of all distributions on Ω is denoted $\mathcal{D}'(\Omega)$.

Our first problem is that $\mathbb{D}(\Omega)$ may be empty. We'll show that it is not.

Proposition 1.5. *If $f \in \mathcal{L}^1_{loc}(\mathbb{R})^n$ and φ is a complex valued Radon measure on $\Omega \subseteq \mathbb{R}^n$, then the following*

$$T_f(\varphi) = \int f(x)\varphi(x) \; dx \qquad\qquad T_\mu(\varphi) = \int \varphi \; d\mu$$

define distributions for all $\varphi \in \mathcal{D}(\Omega)$.

Proof. Let μ be some complex valued measure on $\mathbb{R}^n$, and suppose that $\varphi_n \to \varphi$ in $\mathcal{D}_K$. Then we have

$$\begin{aligned} |T_\mu(\varphi_n) - T_\mu(\varphi)| &= \left| \int (\varphi_n - \varphi)\, d\mu \right| \\ &\leq \int_K |\varphi_n - \varphi|\, d\mu \\ &\leq \|\varphi_n - \varphi\| \cdot |\mu|(K) \to 0 \end{aligned}$$

as $|\mu|(K) < \infty$. For $f \in \mathcal{L}^1_{\text{loc}}$, we have

$$\begin{aligned} |T_f(\varphi_n) - T_f(\varphi)| &= \left| \int (\varphi_n(x) - \varphi(x)) f(x)\, dx \right| \\ &\leq \int_K |\varphi_n - \varphi| \cdot |f(x)|\, dx \\ &\leq \|\varphi_n - \varphi\| \int_K |f(x)|\, dx \to 0 \end{aligned}$$

since f has a finite integral. ■

Lemma 1.6. *Define* $\lambda : \mathbb{R} \to \mathbb{R}$ *by*

$$\lambda(t) = \left\{ \begin{array}{ll} e^{-\frac{1}{t}} & t > 0 \\ 0 & t \leq 0 \end{array} \right\}$$

Then, $\lambda \in C^\infty(\mathbb{R})$

Proof. We'll show that

$$\left(\frac{d}{dt}\right)^n [\lambda(t)] = \left\{ \begin{array}{ll} P_n(\frac{1}{t})e^{-\frac{1}{t}} & t > 0 \\ 0 & t \geq 0 \end{array} \right\}$$

where P_n is a polynomial. We have

$$\frac{d}{dt}[\lambda(t)] = \left\{ \begin{array}{ll} \frac{1}{t^2}e^{-\frac{1}{t}} & t > 0 \\ 0 & t \geq 0 \end{array} \right\}$$

so $P_1(x) = x^2$. By induction, we have

$$\begin{aligned} \left(\frac{d}{dt}\right)^{n+1} [\lambda(t)] &= \frac{d}{dt}[P_n(\tfrac{1}{t})e^{-\frac{1}{t}}] \\ &= -\frac{P_n'(\frac{1}{t})}{t^2} e^{-\frac{1}{t}} + \frac{P_n(\frac{1}{t})}{t^2} e^{-\frac{1}{t}} \\ &= P_{n+1} e^{-\frac{1}{t}} \end{aligned}$$

where $P_{n+1}(x) = -x^2 P_x'(x) + x^2 P_n(x)$. We also note that we have

$$\lim_{t \searrow 0} p_n \left(\frac{1}{t}\right) e^{-\frac{1}{t}} = 0$$

as required. ■

Corollary 1.7. *Let $\rho(x) = c\lambda(1 - \|x\|^2)$ for $x \in \mathbb{R}^n$. Then $\rho \in \mathcal{D}(\mathbb{R}^n)$ and supp$(\rho) = \overline{B}(0;1)$.*

In practise, we will take

$$c = \frac{1}{\int \lambda(1 - \|x\|^2)\, dx}$$

so that

$$\int_{\mathbb{R}^n} \rho(x)\, d^n x = 1$$

If Ω is arbitrary, we may have $\overline{B}(0;1) \nsubseteq \Omega$, which is problematic. However, we may shift ρ by a change of variables, and shrink its codomain by considering

$$\rho_\epsilon(x) = \frac{1}{\epsilon^n}\rho\left(\frac{x}{\epsilon}\right)$$

This gives that $\text{supp}(\rho_\epsilon) = \overline{B}(0;\epsilon)$ and $***$

Definition 1.8. For $T_1, T_2 \in \mathcal{D}'(\Omega)$, we define

$$(T_1 + T_2)(\varphi) = T_1(\varphi) + T_2(\varphi)$$

for all $\varphi \in \mathcal{D}(\Omega)$. Also, for all $\alpha \in C^\infty(\Omega)$, we define

$$(\alpha T)(\varphi) = T(\alpha\varphi)$$

Proposition 1.9. *$\mathcal{D}'(\Omega)$ is a linear space, and if $\alpha \in C^\infty(\Omega)$ and $T \in \mathcal{D}'(\Omega)$, then $\alpha T \in \mathcal{D}'(\Omega)$*

Proof. The linearity of the space follows easily from the definitions of addition and scalar multiplication. It remains to show that αT is a distribution. We need to show that if $\varphi_n \to \varphi$ in $\mathcal{D}_K$, then $\alpha\varphi_n \to \alpha\varphi$ in $\mathcal{D}_k$. First note that if $\varphi_n \to \varphi$ uniformly, then $\alpha\varphi_n \to \alpha\varphi$ uniformly because

$$\|\alpha\varphi_n - \alpha\varphi\|_\infty \leq \|\alpha|_K\|_\infty \cdot \|\varphi_n - \varphi\|_\infty$$

For the derivatives, we use the Leibniz formula which will be proved later, i.e.

$$\partial^m(\alpha\varphi) = \sum_{p+q=m} \frac{m!}{p!q!}(\partial^p\alpha)(\partial^q\varphi)$$

where p, q are multi-indices. ■

Definition 1.10 (Derivatives)**.** If $p \in \mathbb{N}_0^n$ is a multi-index and $T \in \mathcal{D}'(\Omega)$, then the p^th derivative of T is defined by

$$(\partial^p T)(\varphi) = (-1)^{|p|}T(\partial^p\varphi)$$

for $\varphi \in \mathcal{D}(\Omega)$, where $|p| = p_1 + p_2 + \cdots + p_n$.

Proposition 1.11. *If $f \in C^{|p|}(\Omega)$, then $\partial^p T_f = T_{\partial^p f}$*

Proof. By direct computation, we have

$$\begin{aligned}(\partial^p T)_f(\varphi) &= (-1)^{|p|} T_f(\partial^p \varphi)\\ &= (-1)^{|p|} \int (\partial^p \varphi)(x) f(x)\, dx\\ &= \int \varphi(x)(\partial^p f)(x)\, dx\\ &= T_{\partial^p f}(\varphi)\end{aligned}$$

where we integrate by parts a sufficient number of times. ■

Proposition 1.12 (Leibniz Rule). *If $T \in \mathcal{D}'(\Omega)$ and $\alpha \in C^\infty(\Omega)$, then for any multi-index m,*

$$\partial^m(\alpha T) = \sum_{p+q=m} \frac{m!}{p!q!} (\partial^p \alpha)(\partial^q T)$$

where the sum is over multi-indices p and q, and if $m = (m_1, m_2, \ldots, m_n)$, then

$$m! = \prod_{i=1}^{n} m_i!$$

Proof. We'll compute this by induction. If $|m| = 1$, then $\partial^m = \frac{\partial}{\partial x_i}$ for some i. Hence:

$$\begin{aligned}\left[\frac{\partial}{\partial x_i}(\alpha T)\right](\varphi) &= -(\alpha T)\left(\frac{\partial \varphi}{\partial x_i}\right)\\ &= -T\left(\alpha \frac{\partial \varphi}{\partial x_i}\right)\\ &= -T\left(\frac{\partial}{\partial x_i}(\alpha\varphi) - \frac{\partial \alpha}{\partial x_i}\varphi\right)\\ &= \left(\frac{\partial}{\partial x_i} T\right)(\alpha\varphi) + \left(\frac{\partial \alpha}{\partial x_i} T\right)(\varphi)\\ &= \left(\alpha \frac{\partial T}{\partial x_i}\right)(\varphi) + \left(\frac{\partial \alpha}{\partial x_i} T\right)(\varphi)\end{aligned}$$

as required. Now, we assume that the formula holds for all m such that $|m| = k$, and consider $|m| = k + 1$. Letting e_i be a vector of length $k + 1$ whose j^{th} coordinate is δ_{ij}, we have

$$\begin{aligned}\partial^m(\alpha T) &= \frac{\partial}{\partial x_i} \partial^{(m-e_i))}(\alpha T)\\ &= \frac{\partial}{\partial x_i} \sum_{p+q=m-e_i} \frac{(m-e_i)!}{p!q!} (\partial^p \alpha)(\partial^q T)\\ &= \sum_{p+q=m-e_i} \frac{(m-e_i)!}{p!q!} [(\partial^{(p+e_i)} \alpha)(\partial^q T) + (\partial^p \alpha)(\partial^{(q+e_i)} T)]\\ &= \sum_{p'+q'=m} \frac{(m-e_i)!}{p'!q'!} [p'_i(\partial^{p'} \alpha)(\partial^{q'} T) + q'_i(\partial^{p'} \alpha)(\partial^{q'} T)]\end{aligned}$$

Hence, we need to show that

$$\frac{(m_i - 1)!}{p_i!q_i!}(p'_i + 1'_i) = \frac{m_i!}{p'_i!q'_i!}$$

This follows from the fact that $p'_i + q'_i = m_i$. ■

Theorem 1.13. *Every distribution $T \in \mathcal{D}'(\mathbb{R})$ has a primitive $S \in \mathcal{D}'(\mathbb{R})$, i.e. $S' = T$, which is unique up to an additive constant.*

Proof. We note that

$$S' = T \qquad \Leftrightarrow S'(\psi) = -S(\psi') = T(\psi)$$

for all test functions ψ. Therefore we need $S(\chi) = -T(\psi)$ for any $\chi = \psi'$, i.e. for $\chi \in \mathcal{D}(\mathbb{R})$ with

$$\int_{-\infty}^{\infty} \chi(t)\, dt = 0$$

Indeed, if $\chi = \psi'$, then

$$\int_{-\infty}^{\infty} \chi\, dt = \psi(t)|_{-\infty}^{\infty} = 0$$

On the other hand, if $\int_{-\infty}^{\infty} \chi(t)\, dt = 0$, then we can define

$$\psi(t) = \int_{-\infty}^{t} \chi(t')\, dt'$$

Now, choose some $\varphi_0 \in \mathcal{D}(\mathbb{R})$ with

$$\int_{-\infty}^{\infty} \varphi_0(t)\, dt = 1$$

Then given any $\varphi \in \mathcal{D}(\mathbb{R})$, we have the unique decomposition $\varphi = a\varphi_0 + \chi$, where $a = \int_{-\infty}^{\infty} \varphi(t)\, dt$. This gives

$$S(\varphi) = aS(\varphi_0) + S(\chi) = aS(\varphi_0) - T(\psi)$$

if $\chi = \psi'$. We now want to define $S(\varphi)$ by this expression, and need to show that it is a distribution. It thus remains to prove that $\varphi_n \to 0$, then $S(\varphi_n) \to 0$. Clearly, if $\varphi_n \to 0$ in $\mathcal{D}_K$, then $a_n \to 0$ an $\chi_n \to 0$ in $\mathcal{D}_{K'}$, where $K' = K \cup \operatorname{supp}(\varphi_0)$. We have that

$$\psi_n(t) = \int_{-\infty}^{t} \chi_n(t')\, dt' \in \mathcal{D}_H$$

where H is the smallest interval on the real line containing K'. Thus, $\chi_n \to \chi = 0$ and hence $T(\psi_n) \to T(\psi)$, as T is a distribution. It follows that $S(\varphi_n) \to 0$, so S is also a distribution. Clearly $S' = T$ because

$$S'(\varphi) = -S(\varphi') = T(\tilde{\psi}) = T(\varphi)$$

where

$$\varphi = a\varphi_0 + \chi \qquad \Rightarrow \qquad \varphi' = a\varphi_0 + \chi' = a\varphi_0 + \psi$$

and $\tilde{\chi} = \tilde{\psi}'$. Finally, if $S_1' = S_2' = T$, then we have

$$S_1(\varphi) - S_2(\varphi) = a(S_1(\varphi_0) - S_2(\varphi_0)) = \int_{-\infty}^{\infty} C\varphi(t)\, dt$$

where $C = S_1(\varphi_0) - S_2(\varphi_0)$. Thus, $S_1 - S_2 = C$. ■

Chapter 2

Regularisation and Localisation

Theorem 2.1. *Let μ be a Radon measure on $\mathbb{R}^n$. Then the set $\kappa_{\mathbb{C}}(\mathbb{R}^n)$ is dense in $\mathcal{L}^p(\mathbb{R}^n, \mu)$ for $p \in [1, \infty)$, where*

$$\mathcal{L}^p(\mathbb{R}^n), \mu) = \left\{ f : \mathbb{R}^n \to \mathbb{C} \ : \ \int |f|^p \, d\mu < \infty \right\}$$

Proof. Let $f \in \mathcal{L}^p(\mathbb{R}^n), \mu)$, and let $(K_m)_{m=1}^{\infty}$ be a sequence of sets in $\mathbb{R}^n$ increasing to $\mathbb{R}^n$, so $K_m \nearrow \mathbb{R}^n$. We define

$$f_m(x) = \left\{ \begin{array}{ll} 0 & f(x) = 0 \text{ or } x \notin K_m \\ \min\{m, f(x)\} \dfrac{f(x)}{|f(x)|} & f(x) \neq 0, x \in K_m \end{array} \right\}$$

for all $x \in \mathbb{R}^n$. Then we have $f_m \to f$ and therefore $|f_m(x) - f(x)| \searrow 0$, so

$$|f(x)| - |f_m(x) + f(x)| \nearrow |f(x)|$$

By the Monotone Convergence Theorem,

$$\begin{aligned} & \int |f|^p - |f_m - f|^p \, d\mu \nearrow \int |f|^p \, d\mu \\ \Rightarrow \quad & \int |f_m - f|^p \, d\mu \to 0 \end{aligned}$$

i.e. $f_m \to f$ in L^p norm. We conclude that we may assume f to be bounded and to have compact support. However, if this is true, we can approximate f uniformly by a simple function

$$g = \sum_{k=1}^{N} c_k \mathbf{1}_{A_k}$$

We can in fact replace the above sets A_k by compact sets K_k. If g is uniformly close to f, i.e. $|f - g| < \epsilon$, then for K compact,

$$\int_K |f - g|^p \, d\mu < \epsilon^p \mu(K) < \eta$$

where $\mu(K)$ must be finite as K is compact. The K_j are disjoint, compact sets, so there exist disjoint open sets $O_j \supseteq K_j$ with $O_j \subseteq K_j \cup B(0, \epsilon)$. By Urysohn's Lemma, there exist continuous functions $\varphi_j \in \kappa_{\mathbb{C}}(\mathbb{R}^n)$ with

$$\mathbf{1}_{K_j} \leq \varphi_j \leq \mathbf{1}_{O_j}$$

Define

$$\varphi = \sum_{k=1}^{N} c_k \varphi_k$$

Then we have

$$\int |\varphi - g|^p \; d\mu = \sum_{j=1}^{N} \int_{O_j \setminus K_j} |c_j|^p \; d\mu = \sum_{j=1}^{N} |c_j|^p \mu(O_j \setminus K_j) \to 0$$

as $\mu(O_j \setminus K_j) \to 0$, as required. ■

Theorem 2.2. *Let $g \in \mathcal{L}^1_{loc}(\mathbb{R}^n)$ and define g_ϵ by*

$$g_\epsilon(x) = (p_\epsilon * g)(x) = \int p_\epsilon(x - y) g(y) \; d^n y$$

(g_ϵ is called the <u>Regularisation</u> of g). We then have:

1. *$g_\epsilon \in C^\infty$ and $supp(g_\epsilon) \subseteq supp(g) \cup \overline{B}(0, \epsilon)$*
2. *If $g : \mathbb{R}^n \to \mathbb{C}$ is uniformly continuous, then $g\epsilon \to g$ uniformly as $\epsilon \to 0$.*
3. *If $g \in C^k_c(\mathbb{R}^n, \mathbb{C})$ (the space of C^k functions from $\mathbb{R}^n$ with compact support), then*

   $$(\partial^p g)_\epsilon = \partial^p g_\epsilon \to \partial^p g$$

   uniformly, and $\|\partial^p g_\epsilon - \partial^p g\|_\infty \to 0$ as $\epsilon \to 0$ for all $p \in \mathbb{N}_0^n$ with $|p| \leq k$.
4. *If $f \in \mathcal{L}^p(\mathbb{R}^n)$ with $p \in [0, \infty)$, then $f_\epsilon \in \mathcal{L}^p(\mathbb{R}^n)$, $\|f_\epsilon\|_p \leq \|f\|_p$ and $f_\epsilon \to f$ in $\mathcal{L}^p$ norm.*

Proof.

- Let $g \in \mathcal{L}_{\text{loc}}$. We wish to show that g_ϵ is differentiable, i.e.

  $$\frac{\partial}{\partial x_i} g_\epsilon(x) = \int \left(\frac{\partial}{\partial x_i} p_\epsilon(x - y) \right) g(y) \; d^n y$$

  To prove this, we use the dominated convergence theorem. We have

  $$\frac{\partial}{\partial x_i} g_\epsilon(x) = \lim_{h \to 0} \frac{g_\epsilon(x + he_i) - g_\epsilon(x)}{h} = \lim_{n \to \infty} \frac{g_\epsilon(x + h_n e_i) - g_\epsilon(x)}{h_n}$$

  where h_n is some sequence which converges to 0. Recalling the definition of g_ϵ, we note that

  $$\lim_{n \to \infty} \frac{g_\epsilon(x + h_n e_i) - g_\epsilon(x)}{h_n} = \int \left(\lim_{n \to \infty} \frac{p_\epsilon(x - y + h_n e_i) - p_\epsilon(x - y)}{h_n} \right) g(y) \; d^n y$$

But we know that
$$\frac{p_\epsilon(x-y+h_n e_i)-p_\epsilon(x-y)}{h_n} \to \frac{\partial}{\partial x_i} p_\epsilon(x-y)$$
and moreover, by the mean value theorem,
$$\begin{aligned}\left|\frac{p_\epsilon(x-y+h_n e_i)-p_\epsilon(x-y)}{h_n} g(y)\right| &= \left|\frac{\partial}{\partial x_i} p_\epsilon(x-y+\zeta e_i) g(y)\right| \\ &\leq C\mathbf{1}_K |g(y)| \\ &\in \mathcal{L}^1\end{aligned}$$

for some ζ, where $x-y \in \operatorname{supp}(p_\epsilon)$ and $K = \operatorname{supp}(p_\epsilon)+x$. We can repeat this procedure to show that all derivatives of g_ϵ exist. Hence, $g_\epsilon \in C^\infty$. Note that $p(x-y) = 0$ unless $x-y \in \overline{B}(0,\epsilon)$. So, $g_\epsilon(x) = 0$ unless $x-y \in \overline{B}(0,\epsilon)$ for some $y \in \operatorname{supp}(g)$.

- Suppose that g is uniformly continuous, i.e. for all $\eta > 0$ there exists some $\delta > 0$ such that
$$|x-x'| < \delta \qquad \Rightarrow \qquad |g(x)-g(x')| < \eta$$
Then we have
$$\begin{aligned} g_\epsilon(x) - g(x) &= \int p_\epsilon(x-y)g(y)\, dy - g(x) \\ &= \int g(x-z)p_\epsilon(z)\, dz - g(x) \\ &= \int (g(x-z)-g(x))p_\epsilon(z)\, dz \end{aligned}$$
where we make the change of variables $z = x-y$. Now, $p_\epsilon(z) = 0$ unless $z \in \overline{B}(0,\epsilon)$, so $|z| < \epsilon$, i.e. $|x-(x-z)| < \epsilon$. Hence, provided that $\delta < \epsilon$, we have
$$|g_\epsilon(x)-g(x)| \leq \int |g(x-z)-g(x)| p_\epsilon(z)\, dz < \eta \int p_\epsilon(z)\, dz = \eta$$

- This follows from the fact that $(\partial^p g_\epsilon) = \partial^p g_\epsilon$, combined with the previous part. We have:
$$\begin{aligned} \partial^p g_\epsilon &= \partial_x^p \int p_\epsilon(x-y)g(y)\, dy \\ &= \int \partial_x^p [p_\epsilon(x-y)g(y)]\, dy \\ &= \int \partial_x^p [p_\epsilon(z)g(x-z)]\, dz \\ &= \int p_\epsilon(z)\partial_x^p [g(x-z)]\, dz \\ &= (\partial^p g)_\epsilon(x) \end{aligned}$$

- If $f \in \mathcal{L}^p(\mathbb{R})$, then for all $\eta > 0$ there exists $g \in \kappa_{\mathbb{C}}(\mathbb{R}^n)$, then there exists some $g \in \kappa_{\mathbb{C}}(\mathbb{R}^n)$ with $\|f-g\|_p < \frac{\eta}{3}$, and for ϵ small enough we have $\|g-g_\epsilon\|_\infty < \epsilon'$, which gives
$$\int |g(x)-g_\epsilon(x)|\, dx < \frac{\eta}{3}$$

as g and g_ϵ have compact support. Now, we know that

$$|f_\epsilon(x)| = \left|\int p_\epsilon(x-y)f(y)\ dy\right| \leq \int p_\epsilon(x-y)|f(y)|\ dy$$

We also note that

$$\begin{aligned} f_\epsilon(x) &= \int p_\epsilon(x-y)f(y)\ dy \\ &= \frac{1}{\epsilon^n}\int p\left(\frac{x-y}{\epsilon^n}\right) f(y)\ d^n y \\ &= \int p(z)f(x-\epsilon z)\ d^n z \end{aligned}$$

by the change of variables $z = \frac{x-y}{\epsilon}$. Now, we combine the above two inequalities to see that

$$|f_\epsilon(x)| \leq \int p(z)|f(x-\epsilon z)|\ dz \leq \left[\int |f(x-\epsilon z)|^p p(z)\ dz\right]^{\frac{1}{p}} \left[\int p(z)\ dz\right]^{\frac{1}{q}}$$

where the final integral equals 1. Hence:

$$\begin{aligned} \|f_\epsilon\|_p^p &= \int |f_\epsilon(x)|^p\ dx \\ &\leq \iint |f(x-\epsilon z)|^p \rho(z)\ dz\ dx \\ &= \iint |f(y)|^p \rho_\epsilon(x-y)\ dy\ dx \\ &= \iint |f(y)|^p \rho_\epsilon(u)\ dy\ du \\ &= \int |f(y)^p\ dy \\ &= \|f\|_p^p \end{aligned}$$

Finally, we have

$$\begin{aligned} \|f-f_\epsilon\|_p &\leq \|f-g\|_p + \|g-g_\epsilon\|_p + \|g_\epsilon - f_\epsilon\|_p \\ &\leq \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3} \\ &< \eta \end{aligned}$$

as required.

■

Corollary 2.3. *If* $\mathcal{L}^1_{loc}(\mathbb{R}^n)$ *and* $T_f = 0$*, then* $f = 0$ *a.e. Similarly, if* μ *is a complex-valued measure on* $\mathbb{R}^n$ *and* $T_\mu = 0$ *the* $\mu = 0$

Proof. Suppose that $T_f = 0$ but $f \neq 0$ a.e., i.e. there exists $A \subseteq \mathbb{R}^n$ such that $\mu_L(A) > 0$ and $f(x) \neq=$ for all $x \in A$ (mu_L is the Lebesgue measure). Then we can divide A into $A = A_1 \cup A_2$ where $\Re f(x) \neq 0$ for all $x \in A_1$ and $\Im(x) \neq 0$ for all $x \in A_2$. Clearly, either $\mu_L(A_1) > 0$ or $\mu_L(A_2) > 0$. Suppose that $\mu_L(A_1) > 0$. Then $\Re f(x) < 0$ or $\Re f(x) > 0$. We decompose A_1 into

$A_1 = A_{1+} \cup A_{1-}$, for these two cases. Again, either $\mu_L(A_{1+}) > 0$ or $\mu_L(A_{1-}) > 0$. If $\mu_L(A_{1+}) > 0$, then we have

$$\int_{A_{1+}} (\Re f)(x)\, dx > 0$$

As $g = \mathbf{1}_{A_{1+}} \in \mathcal{L}^1_{\text{loc}}$, we have

$$\int g\, d\mu_L = \lim_{\epsilon \to 0} \int g_\epsilon\, d\mu_L = \lim_{\epsilon \to 0} T_{\mu_L}(g_\epsilon) = 0$$

which is a contradiction. For a general measure μ, we take $g \in \kappa_{\mathbb{C}}(\mathbb{R}^n)$, and hence

$$\int g\, d\mu = \lim_{\epsilon \to 0} \int g_\epsilon\, d\mu = \lim_{\epsilon \to 0} T_\mu(g_\epsilon) = 0$$

and hence $\mu = 0$ by Reisz' theorem. ■

Remember that if $f \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}^n)$, we define the support to be the complement of the largest open set O such that $f|_O = 0$. To prove that we can also define the support for a distribution, we first prove a theorem about the existence of a "partition of unity".

Lemma 2.4 (Urysohn's C^∞ Lemma)**.** *Given a compact set $K \subseteq \mathbb{R}^n$ and an open set $O \supseteq K$, there exists a non-negative function $h \in \mathcal{D}(\mathbb{R}^n)$ such that*

$$\mathbf{1}_K \leq h \leq \mathbf{1}_O$$

Proof. Let $\epsilon = d(K, O^c)$ and set

$$H = K + \overline{B}\left(0, \frac{\epsilon}{3}\right) \qquad U = K + B\left(0, \frac{2\epsilon}{3}\right)$$

For convenience, we will let $\eta = \frac{\epsilon}{3}$. There exists, by Urysohn's lemma, a continuous function $f \in \kappa(\mathbb{R}^n)$

$$\mathbf{1}_H \leq f \leq \mathbf{1}_U$$

Write $h = f_\eta$, then $h \in C^\infty$ and $\text{supp}(h) \subseteq \text{supp}(f) + \overline{B}(0, \eta)$. As U is compact, we must have $h \in \mathcal{D}(O)$. If $x \in K$, then we have

$$\begin{aligned} h(x) &= \int f(x-y)\rho_\eta(y)\, dy \\ &= \int_{\overline{B}(0,\eta)} f(x-y)\rho_\eta(y)\, dy \\ &= \int \rho_\eta(y)\, dy \\ &= 1 \end{aligned}$$

This follows because $\rho(y) = 0$ unless $y \in \overline{B}(0, \eta)$, and hence $x - y \in H$, so $f(x-y) = 1$. Clearly, $0 \leq h \leq 1$, since $0 \leq f \leq 1$. Thus, if $x \notin O$, we have $x - y \in U^c$, and so $f(x-y) = 1$. ■

Definition 2.5 (Paracompactness)**.** An open cover $\{U_i\}_{i \in I}$ of a subset A of a Hausdorff space X is called Locally Finite if for every $x \in A$, there exists some open set O which contains x and which has a non-empty intersection with only finitely many elements of the open cover. A is called Paracompact if every open cover has a locally finite subcover.

Lemma 2.6. *Let $\Omega \subseteq \mathbb{R}^n$ be open. There exists a family $\{\Omega_n\}_{n\in\mathbb{N}}$ of open sets $\Omega_n \subseteq \Omega$ such that $\Omega = \bigcup\limits_{n\in\mathbb{N}} \Omega_n$, $\overline{\Omega}_n$ is compact for all $n \in \mathbb{N}$, and $\overline{\Omega}_n \subseteq \Omega_{n+1}$ for all $n \in \mathbb{N}$.*

Proof. Consider the dyadic intervals of the form

$$\prod_{i=1}^{n} [r_i 2^{-p}, (r_i+1)2^{-p}]$$

for $r_i \in \mathbb{Z}$ and $p \in \mathbb{N}$. We can write $\Omega = \bigcup\limits_{k=1}^{\infty} I_k$ where I_k are diadic intervals. Define

$$\Omega_1 = (I_1)^{\circ}$$

This gives $\overline{\Omega}_1 = I_1$, which is compact. There exists an open set U_1 such that

$$\overline{\Omega}_1 \subseteq U_1 \subseteq \overline{U}_1 \subseteq \Omega$$

and $\overline{U}_1$ is compact. Define

$$\Omega_2 = \left(\bigcup_{k=1}^{n_2} I_k \right)^{\circ}$$

where n_2 is the smallest natural number such that $\overline{U}_1 \subseteq \bigcup\limits_{k=1}^{n_2} I_k$. Then, $\overline{U}_2 = \bigcup\limits_{k=1}^{n_2} I_k$ is compact. Continuing in this way, we get the required family of subsets. ■

Definition 2.7 (Refinement)**.** A cover $\{V_j\}_{j\in J}$ of a topological space X is called a <u>Refinement</u> of another cover $\{U_i\}_{i\in I}$ if for each $j \in J$ there exists some $i \in I$ such that $V_j \subseteq U_i$.

Proposition 2.8. *Every open subset $\Omega \subseteq \mathbb{R}^n$ is paracompact.*

Proof. Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be a family of subsets of Ω as in the above lemma, and let $\{U_i\}_{i\in I}$ be an open cover of Ω. Since $\overline{\Omega}_m$ is compact and we have $\overline{\Omega}_m \subseteq \bigcup\limits_{i\in I} U_i$, there exists a finite subset $I_m \subseteq I$ such that

$$\overline{\Omega}_m \subseteq \bigcup_{i\in I_m} U_i$$

For $m = 1$, let $k_1 = |I_1|$ *** ■

Note that a locally finite cover of an open set $\Omega \subseteq \mathbb{R}^n$ is always countable, as the intersection with $\overline{\Omega}_n$ is finite for all $n \in \mathbb{N}$.

Lemma 2.9. *All metric spaces are normal, i.e. given closed, disjoint subsets $A, B \subseteq X$, there exist disjoint open sets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$.*

Proof. For all $x \in A$, there exists $\delta_x > 0$ such that $B(x, \delta_x) \cap B = \varnothing$. Similarly, for all $y \in B$, there exists $\epsilon_y > 0$ such that $B(y, \epsilon_y) \cap A = \varnothing$. Define

$$U = \bigcup_{x \in A} B\left(x, \frac{\delta_x}{3}\right) \qquad V = \bigcup_{y \in B} B\left(y, \frac{\epsilon_y}{3}\right)$$

It is clear that these sets are open and that $A \subseteq U$, $B \subseteq V$. We will show that these are disjoint. Suppose that $z \in U \cap V$. Then for some $x \in A$ and $y \in B$ we have

$$d(x, z) < \frac{\delta_x}{3} \qquad d(y, z) < \frac{\epsilon_y}{3}$$

This implies that

$$d(x, y) < \frac{\delta_x}{3} + \frac{\epsilon_y}{3} < \max\{\delta_x, \epsilon_y\}$$

However, this would imply that $y \in B(x, \delta_x)$ or that $x \in B(y, \epsilon_y)$, which is a contradiction. ■

Lemma 2.10 (Shrinking Lemma). *Let $\{V_n\}_{n \in \mathbb{N}}$ be a locally finite open cover of an open set Ω such that $\overline{V}_n \subseteq \Omega$ for all $n \in \mathbb{N}$. Then, there exists an open over $\{W_n\}_{n \in \mathbb{N}}$ such that $\overline{W}_n \subseteq V_n$ for all $n \in \mathbb{N}$.*

Proof. We will construct the required open inductively. Let $W_0 = \varnothing$, and suppose that W_j has been constructed for all $k \leq j$ such that

$$\Omega \subseteq \left(\bigcup_{j<k} W_j\right) \cup \left(\bigcup_{j \geq k} V_j\right)$$

Define

$$G_k = V_k \setminus \left[\left(\bigcup_{j<k} W_j\right) \cup \left(\bigcup_{j>k} V_j\right)\right]$$

Then, G_k is closed. Indeed if $x \in \overline{G}_k \setminus G_k$, then $x \in \overline{V}_k \setminus V_k$, and therefore

$$x \in \left(\bigcup_{j<k} W_j\right) \cup \left(\bigcup_{j>k} V_j\right)$$

which implies that

$$x \notin \overline{G}_k = \overline{V}_k \setminus \left[\left(\bigcup_{j<k} W_j\right) \cup \left(\bigcup_{j>k} V_j\right)\right]$$

which is a contradiction. Thus, $G_k = \setminus G_k \subseteq V_k$, and hence $\overline{G}_k \cup V_k^c = \varnothing$. By the previous lemma, there exists disjoint open sets W_k, U_k such that $\overline{G}_k \subseteq W_k$ and $V_k^c \subseteq U_k$. This gives that $W_k \cap V_k^c = \varnothing$, and so $\overline{W}_k \cap V_k^c = \varnothing$, whence $\overline{W}_k \subseteq V_k$, as required in our statement. We have

$$G_k \subseteq W_k \subseteq \overline{W}_k \subseteq V_k$$

However, recalling how G_k was defined, we have

$$\begin{aligned}
V_k &\subseteq W_k \cup \left[\left(\bigcup_{j<k} W_j \right) \cup \left(\bigcup_{j>k} V_j \right) \right] \\
\Rightarrow \quad \Omega &\subseteq V_k \cup \left[\left(\bigcup_{j<k} W_j \right) \cup \left(\bigcup_{j>k} V_j \right) \right] \\
\Rightarrow \quad \Omega &\subseteq W_k \cup \left[\left(\bigcup_{j<k} W_j \right) \cup \left(\bigcup_{j>k} V_j \right) \right] \\
&\left[\left(\bigcup_{j\leq k} W_j \right) \cup \left(\bigcup_{j>k} V_j \right) \right]
\end{aligned}$$

■

Definition 2.11 (Partition of Unity)**.** Given an open cover $\{U_i\}_{i\in I}$ be an open cover of an open set $\Omega \subseteq \mathbb{R}^n$, a PARTITION OF UNITY subordinate to the open cover is a collection $\{\alpha_j\}_{j\in J}$ of functions $\alpha_j : \Omega \to [0,1]$ such that:

- $\alpha_j(x) \geq 0$ for all $j \in J$ and $\sum_{j\in J} \alpha_j(x) = 1$ for all $x \in \mathbb{R}^n$
- for every compact set K, there exists a finite subset $J_0 \subseteq J$ such that $\alpha_j(x) = 0$ for every $x \in K$, $j \notin J_0$
- $\alpha_j \in \mathcal{D}(\mathbb{R}^n)$ and for every $j \in J$, there exists some $i \in I$ such that $\operatorname{supp}(\alpha_i) \subseteq U_i$

Theorem 2.12. *Let $\{\Omega_i\}_{i\in I}$ be an open cover of an open set $\Omega \subseteq \mathbb{R}^n$. Then there exists a partition of unity subordinate to $\{\Omega_i\}_{i\in I}$.*

Proof. Combining above a proposition and a lemma above, we know that there exists a locally finite sub-cover $\{U''_j\}_{j\in J''}$ such that $\overline{\Omega''_j}$ is compact. We can take

$$J'' = J \times \mathbb{N} \qquad \Omega''_{(j,n)} = \Omega_j \cup ()$$

*** ■

Chapter 3

Support of a Distribution

Proposition 3.1. *If* $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{R}^n$ *are open subsets and* $\varphi \in \mathcal{D}(\Omega_1)$*, then the test function* $\tilde{\varphi} : \Omega_2 \to \mathbb{C}$ *defined by*

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x) & x \in \Omega_1 \\ 0 & x \in \Omega_2 \setminus \Omega_1 \end{cases}$$

is in $\mathcal{D}(\Omega_2)$*. Conversely, if* $\psi \in \mathcal{D}(\Omega_2)$ *and supp*$(\psi) \subseteq \Omega_1$*, then* $\psi|_{\Omega_1} \in \mathcal{D}(\Omega_1)$*.*

Proof.

■

Definition 3.2. If $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{R}^n$, and $T \in \mathcal{D}'(\Omega_2)$, we abuse notation and write $T|_{\Omega_1}$ for the restriction of T to Ω_1. We say that $T_1, T_2 \in \mathcal{D}'(\Omega_2)$ agree on Ω_1 if $(T_1 - T_2)|_{\Omega_1}$.

Lemma 3.3. *If* $\Omega_1, \Omega_2 \subseteq \mathbb{R}^n$ *are open and* $\varphi \in \mathcal{D}(\Omega_1 \cup \Omega_2)$ *then there exists* $\varphi_1 \in \mathcal{D}(\Omega_1)$ *and* $\varphi_2 \in \mathcal{D}(\Omega_2)$ *such that* $\varphi = \varphi_1 + \varphi_2$*.*

Proof. *** ■

Lemma 3.4. *If* $\Omega_1, \Omega_2 \subseteq \Omega \subseteq \mathbb{R}^n$ *with* Ω_1, Ω_2 *open and* $T \in \mathcal{D}'(\mathbb{R}^n)$ *with* $T|_{\Omega_1} = 0$ *and* $T|_{\Omega_2} = 0$*, then* $T|_{\Omega_1 \cup \Omega_2} = 0$

Proof. *** ■

Proposition 3.5. *If* $\Omega = \bigcup_{i \in I} \Omega_i \subseteq \mathbb{R}^n$ *with* Ω_i *open for all* $i \in I$ *and* $T \in \mathcal{D}'(\mathbb{R}^n)$ *with* $T|_{\Omega_i} = 0$ *for all* $i \in I$*, then* $T|_{\Omega} = 0$

Proof. *** ■

We can conclude that there exists a largest open set O such that $T|_O = 0$. The complement of O is called the <u>SUPPORT</u> of T.

Example 3.6. Let $T = \delta_0$. Then, $T(\varphi) = \varphi(0)$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$, so supp$(T) = \{0\}$ since $T(\varphi) = 0$ iff $\varphi(0) = 0$.

Proposition 3.7. *If* μ *is som complex-valued Radon measure on* $\Omega \subseteq \mathbb{R}^n$*, then supp*$(T\mu) =$ *supp*(μ)

Proof. *** ■

Proposition 3.8. *Suppose that $\{\Omega_i\}_{i\in I}$ is an open cover of $\Omega \subseteq \mathbb{R}^n$, and that $T \in \mathcal{D}'(\Omega)$. If we set $T_i = T|_{\Omega_i}$, then*

$$T_i|_{\Omega_i\cap\Omega_j} = T_j|_{\Omega_i\cap\Omega_j}$$

Proof. *** ■

Theorem 3.9. *Let $\mathbb{R}^n \supseteq \Omega \subset \bigcup_{i\in I}\Omega_i$ with $\Omega_i \subseteq \mathbb{R}^n$, and suppose that for all $i \in I$ there exists some $T_i \in \mathcal{D}'(\Omega_i)$ such that*

$$T_i|_{\Omega_i\cap\Omega_j} = T_j|_{\Omega_i\cap\Omega_j}$$

Then there exists a unique $T \in \mathcal{D}'(\Omega)$ such that $T|_{\Omega_i} = T_i$

Proof. Let $\{\alpha_j\}_{j\in J}$ be a partition of unity subordinate to $\{\Omega_i\}_{i\in I}$. If $\operatorname{supp}(\varphi) = K \subseteq \Omega$ there exists a finite subset $J_0 \subseteq J$ such that $\alpha_j(x) = 0$ for all $x \in K$ and for all $j \in J \setminus J_0$. Then for all $j \in J_0$ there exists some $i(j) \in I$ such that $\operatorname{supp}(\alpha_j) \subseteq \Omega_{i(j)}$. Given φ we define

$$T = \sum_{j\in J_0} T_{i(j)}(\alpha_j\varphi)$$

We need to show that φ is well defined. Indeed, if $\varphi(\alpha_j) \subseteq \Omega_{i'}$, then $\operatorname{supp}(\alpha_j) \subseteq \Omega_i \cap \Omega_{i'}$, so

$$T_i(\alpha_j\varphi) = T_{i'}(\alpha_j\varphi)$$

We next need to show that T is a distribution and that $T|_{\Omega_i} = T_i$. T is clearly linear, so to prove that it is continuous, we need to show that id $\varphi_m \to \varphi$ in $\mathcal{D}_K$ for all $K \subseteq \Omega$, then $T(\varphi_m) \to T(\varphi)$. However, this will be true as

$$T(\varphi_m) = \sum_{j\in J_0} T_{i(j)}(\alpha_j\varphi_m) \to \sum_{j\in J_0} T_{i(j)}(\alpha_j\varphi) T(\varphi)$$

since $\alpha_j\varphi_m \to \alpha_j\varphi$ in $\mathcal{D}_{K\cap\operatorname{supp}(\alpha_j)}$ and hence $T_{i(j)}(\alpha\varphi_m) \to T_{i(j)}(\alpha\varphi)$. Thus T is a distribution. If $\varphi \in \mathcal{D}(\Omega_i)$ for $K = \operatorname{supp}(\varphi)$, we obtain J_0 as before, and obtain

$$T(\varphi) = \sum_{i\in J_0} T_{i(j)}(\alpha_j\varphi)$$

where $\operatorname{supp}(\alpha_j\varphi) \subseteq \operatorname{supp}(\alpha_j) \cap \operatorname{supp}(\varphi) \subseteq \Omega_{i(j)} \cap \Omega_i$. This gives $T_{i(j)}(\alpha_j\varphi) = T_i(\alpha_i\varphi)$, and thus on Ω

$$T(\varphi) = T_i\left(\sum_{j\in J}\alpha_i\varphi\right) = T_i(\varphi)$$

■

Chapter 4

Finite Order

Theorem 4.1. *Suppose that* $T : \mathcal{D}(\Omega) \to \mathbb{C}$ *is a linear map where* $\Omega \subseteq \mathbb{R}^n$ *is open open. Then* T *is a distribution iff for every compect set* $K \subseteq \Omega$ *there exists some* $m \in \mathbb{N}_0$ *and* $M \in (0, \infty)$ *such that*

$$|T(\varphi)| \leq M\|\varphi\|_m$$

for $\varphi \in \mathcal{D}_K$*, where*

$$\begin{aligned} \|\varphi\|_m &= \sup_{p \in \mathbb{N}_0^k} \{|\partial^p \varphi(x)| \ : \ x \in K, |p| \leq m\} \\ &= \max\{\|\partial^p \varphi\|_\infty \ : \ p \in \mathbb{N}_0^k, |p| \leq m\} \end{aligned}$$

Proof. Fix $K \subseteq \Omega$ compact. We claim that $(*)$ for all $\epsilon > 0$ there exists some $\delta > 0$ and $m \in \mathbb{N}_0$ such that for all $\varphi \in \mathcal{D}_K$ with $\|\varphi\|_m < \delta$, we have $|T(\varphi)| < \epsilon$. Indeed, suppose the contrary, that there exists some $\epsilon > 0$ such that for all $\delta > 0$ and $m \in \mathbb{N}_0$, there exists some $\varphi_{\delta,m} \in \mathcal{D}_K$ such that $\|\varphi_{\delta,m}\| < \delta$, but $|T(\varphi_{\delta,m})| \geq \epsilon$. We define $\varphi_k = \varphi_{\frac{1}{k},k}$ and consider the sequence $(\varphi_k)_{k=1}^\infty$, which we'll show converges to 0 in $\mathcal{D}_K$. Indeed, given $p \in \mathbb{N}_0^n$, we have

$$\|\partial^p \varphi_k\|_\infty \leq \|\varphi_k\|_{|p|} \leq \|\varphi_k\|_k \leq \delta = \frac{1}{k}$$

which converges to 0 whenever $k \geq |p|$, which contradicts $|T(\varphi_k)| > \epsilon$, and so $(*)$ holds. Now, fix $\epsilon > 0$ and let $M = \frac{\epsilon}{\delta}$, and take some $\varphi \in \mathcal{D}_K$. If $\|\varphi\|_m = 0$, then $\|\lambda\varphi_m\| = 0$ for all $\lambda > 0$, so

$$\begin{aligned} & |T(\lambda\varphi)| < \epsilon \\ \Rightarrow \quad & |T(\varphi)| < \frac{\epsilon}{\lambda} \\ \Rightarrow \quad & 0 = |T(\varphi)| \leq M\|\varphi\|_m = 0 \end{aligned}$$

Now, we suppose that $\|\varphi\|_m \neq 0$. Set $\lambda = \frac{\epsilon}{\|\varphi\|_m}$. Then $\|\lambda\varphi\|_m = \delta$, and so

$$\begin{aligned} & |T(\lambda\varphi)| \leq \epsilon \\ \Rightarrow \quad & |T(\varphi)| \leq \frac{\epsilon}{\lambda} = \frac{\epsilon}{\delta}\|\varphi\|_m = M\|\varphi\|_m \end{aligned}$$

as required. ■

Example 4.2. In our above theorem, m depends on the set K. For example, define:

$$T = \sum_{n=0}^{\infty} \delta_n^{(n)}$$

This is well defined, because if $\varphi \in \mathcal{D}_K$, then $J = \mathbb{N}_0 \cap K$ is finite, so

$$T(\varphi) = \sum_{n \in J} = (-1)^n \varphi^{(n)}(n)$$

However, if $\alpha \in \mathcal{D}(-1,1)$ and $\alpha(0) = 1$ we can define $\varphi_n = (x-n)^m \alpha(x-n)$. Thus if $x \notin (n-1, n+1)$, then $\varphi_n(x) = 0$, and so

$$\begin{aligned} T(\varphi_n) &= (-1)^n \frac{d^n}{dx^n}\varphi\bigg|_n = (-1)^n n! \\ &\Rightarrow |T(\varphi_n)| = n! \not\leq M\|\varphi\|_m \end{aligned}$$

since $\|\varphi\|_m \leq cn^m$, so we can always take n large enough.

Definition 4.3 (Order)**.** A distribution $T \in \mathcal{D}'(\Omega)$ is said to be of <u>Order</u> $\leq m$ if for all $K \subseteq \Omega$ compact, we have

$$|T(\varphi)| \leq M\|\varphi\|_m$$

for all $\varphi \in \mathcal{D}_K$, where $M \in (0, \infty)$. T is said to be of order m if it is of order $\leq m$ but not of order $\leq m-1$. T is said to be of finite order of there exists some $m \in \mathbb{N}_0$ such that T is of order $\leq m$.

Given $\Omega \subseteq \mathbb{R}^n$, we'll use the notation for all compact sets K:

$$\begin{aligned} \mathcal{D}^m(\Omega) &= \{\varphi \in C^m(\Omega) \ : \ \text{supp}(\varphi) \text{ is compact } \} \\ \mathcal{D}^m_K(\Omega) &= \{\varphi \in C^m(\Omega) \ : \ \text{supp}(\varphi) \subseteq K\} \end{aligned}$$

Whenever the meaning is clear, we will drop the Ω.

Proposition 4.4. *$\mathcal{D}^m_K$ is a normed space under $\|\cdot\|_m$*

Proof. If $\varphi_1, \varphi_2 \in \mathcal{D}^m_K$ and $\lambda \in \mathbb{C}$, then we have:

- $$\begin{aligned} \|\varphi_1 + \varphi_2\|_m &= \max_{|p| \leq m}\{\|\partial^p(\varphi_1 + \varphi_2)\|_\infty\} \\ &\leq \max_{|p| \leq m}\{\|\partial^p(\varphi_1)\|_\infty + \|\partial^p(\varphi_2)\|_\infty\} \\ &\leq \max_{|p_1| \leq m}\{\|\partial^{p_1}(\varphi_1)\|_\infty\} + \max_{|p_2| \leq m}\{\|\partial^{p_2}(\varphi_2)\|_\infty\} \\ &= \|\varphi_1\|_m + \|\varphi_2\|_m \end{aligned}$$

- $$\begin{aligned} \|\lambda\varphi_1\|_m &= \max_{|p| \leq m}\{\|\partial^p(\lambda\varphi_1)\|_\infty\} \\ &= \max_{|p| \leq m}\{|\lambda|\|\partial^p(\varphi_1)\|_\infty\} \\ &= |\lambda| \max_{|p| \leq m}\{\|\partial^p(\varphi_1)\|_\infty\} \\ &= |\lambda|\|\varphi_1\|_m \end{aligned}$$

- $$\|\varphi_1\|_m = 0 \quad \Leftrightarrow \quad \|\varphi_1\|_\infty = 0 \quad \Leftrightarrow \quad \varphi = 0$$

■

We can extend this proposition and see that $\mathcal{D}_K^m$ is actually a Banach space.

Proposition 4.5. *A linear map $L : \mathcal{D}^m(\Omega) \to \mathbb{C}$ is continuous iff there exists some $M > 0$ such that*

$$|L(\varphi)| \leq M\|\varphi\|_m$$

for all $\varphi \in \mathcal{D}_K^m$ with $K \subseteq \Omega$.

Proof. Trivial. ■

Proposition 4.6. *A linear map $L : \mathcal{D}^m(\Omega) \to \mathbb{C}$ is continuous if $L|_{\mathcal{D}_K^m}$ is continuous for all $K \subseteq \Omega$ compact.*

Proof. Also trivial. ■

Theorem 4.7. *If $L : \mathcal{D}_K^m \to \mathbb{C}$ is a continuous, linear map, then $L|_{D(\Omega)}$ is a distribution. Conversely, if T is a distribution in $\mathcal{D}'(\Omega)$ of finite order $\leq m$, then there exists a unique $\tilde{T} \in \mathcal{D}'^m_K$ such that $\tilde{T}|\mathcal{D}(\Omega) = T$*

Proof. If $L \in \mathcal{D}'^m(\Omega)$l et $T = L|_{\mathcal{D}(\Omega)}$ and suppose that $\varphi_k \to \varphi$ in $\mathcal{D}_K$. The we have $\|\varphi_k - \varphi\|_m to 0$, and hence

$$|T(\varphi) - T(\varphi_k)| = |L(\varphi - \varphi_k)| \leq M\|\varphi - \varphi_k\|_m \to 0$$

as $k \to \infty$. Hence, T is a distribution of order $\leq m$.

Conversely, let T by a distribution of order $\leq m$. Then for all $\varphi \in \mathcal{D}^m(\Omega)$, let $K = \text{supp}(\varphi)$. There exists a compact set K such that

$$K \subseteq H \subseteq H^\circ \subseteq \Omega$$

By theorem 2.2, there exists some $(\varphi_k)_{k=1}^\infty$ such that $\|\varphi_k - \varphi\|_m \to 0$ and $\varphi_k \to \varphi$ in $\mathcal{D}_H$. We now define:

$$\tilde{T}(\varphi) = \lim_{k\to\infty} T(\varphi_k)$$

This limit must exist because

$$|T(\varphi) - T(\varphi_k)| = |T(\varphi - \varphi_k)| \leq M\|\varphi - \varphi_k\|_m \to 0$$

$\tilde{T}(\varphi)$ is independent of the sequence chosen, because if $\|\psi_k i\varphi\|_m \to 0$ and $\psi_k \to \varphi$ in $\mathcal{D}_H$, then we have

$$|T(\varphi_k) - T(\psi_k)| \leq M\|\varphi_k - \psi_k\|_m \leq M(\|\varphi_k - \varphi\|_m + \|\varphi - \psi_k\|_m) \to 0$$

It remains to show uniqueness. If $\tilde{T}_1|_{\mathcal{D}(\Omega)} = \tilde{T}_2|_{\mathcal{D}(\Omega)}$, then for $\varphi \in \mathcal{D}^m(\Omega)$ take a sequence $\varphi_k \in \mathcal{D}(\Omega)$ such that $\varphi_k \to \varphi$ in $\mathcal{D}_H$. Then we have

$$\begin{aligned} |\tilde{T}_1(\varphi) - \tilde{T}_2(\varphi)| &\leq |\tilde{T}_1(\varphi - \varphi_k)| + |\tilde{T}_1(\varphi_k) - \tilde{T}_2(\varphi_k)| + |\tilde{T}_2(\varphi - \varphi_k)| \\ &\leq M_1\|\varphi - \varphi_k\|_m + M_2\|\varphi - \varphi_k\|_m \to 0 \end{aligned}$$

since $\tilde{T}_1(\varphi_k) = T(\varphi_k) = \tilde{T}_2(\varphi_k)$ for $\varphi_k \in \mathcal{D}$. ■

Proposition 4.8. *If $T \in \mathcal{D}'^m(\Omega)$ and $\alpha \in C^m(\Omega)$, then αT is a well defined distribution in $\mathcal{D}'^m(\Omega)$, and satisfies Leibniz' rule: if r is a multi-index such that $|r| \leq m$, then*

$$\partial^r(\alpha T) = \sum_{p+q=r} = \frac{r!}{p!q!}(\partial^p \alpha)(\partial^q T)$$

Proof. We define $(\alpha T)(\varphi) = T(\alpha\varphi)$. We'll show that $\alpha\varphi \in \mathcal{D}^m(\Omega)$ if $\varphi \in \mathcal{D}^m(\Omega)$. It then follows that $\alpha T \in \mathcal{D}^m(\Omega)$, as

$$\partial^r(\alpha\varphi) = \sum_{p+q=r} \frac{r!}{p!q!}(\partial^p \alpha)(\partial^q \varphi)$$

and if $|r| \leq m$, then $|p|, |q| \leq m$ and so $\partial^p \alpha$ and $\partial^q \varphi$ are well defined. Moreover,

$$|\partial^r(\alpha\varphi)| \leq \sum_{p+q=r} \frac{r!}{p!q!} \|\alpha\|_{m,K} \|\varphi\|$$

where $K = \text{supp}(\varphi)$ and

$$\|\alpha\|_{m,K} = \sup_{x \in K} \sup_{|p| \leq m} |\partial^p \alpha(x)|$$

Therefore, if $|T(\varphi)| \leq M\|\varphi\|_m$ for some $\varphi \in \mathcal{D}^m_K$,

$$|(\alpha T)(\varphi)| \leq M \left(\sum_{\substack{p+q=r \\ |r| \leq m}} \|\alpha\|_{m,K} \right) \|\varphi\|_m$$

where the sum in brackets is finite for fixed K. We conclude that $\alpha T \in \mathcal{D}'^m(\Omega)$, because it is continuous. The proof of Leibniz Rule is identical to the proof for C^∞. ■

Proposition 4.9. *If $f \in \mathcal{L}^1_{loc}(\Omega)$, then $T_f \in \mathcal{D}'^0(\Omega)$*

Proof. We need to show that for $\varphi \in C_K(\Omega)$

$$|T_f(\varphi)| \leq M\|\varphi\|_0 = M\|\varphi\|_\infty = \sup_{x \in \Omega} |\varphi(x)|$$

Indeed:

$$\begin{aligned} |T_f(\varphi)| &= \left| \int_\Omega f(x)\varphi(x)\, d^n x \right| \\ &\leq \int_\Omega |f(x)| \cdot |\varphi(x)|\, d^n x \\ &\leq \int_K |f(x)| \cdot |\varphi(x)|\, d^n x \\ &\leq \|\varphi\|_\infty \int_K |f(x)|\, d^n x \\ &= \|f\|_{1,K} \|\varphi\|_\infty \end{aligned}$$

as required. ■

Proposition 4.10. *A distribution $T \in \mathcal{D}')\Omega)$ is of order* 0 *iff there exists a compex-valued Radon measure μ on Ω such that $T = T_\mu$.*

Proof. If $T = T_\mu$, then for $\varphi \in \mathcal{D}(\Omega)$

$$|T_\mu(\varphi)| = \left| \int_\Omega \varphi(x)\ \mu(dx) \right| \leq \int_K |\varphi(x)|\ |\mu|(dx) \leq |\mu|(K)\|\varphi\|_\infty$$

The converse follows from the Riesz-Markov-Saks-Kakutani Representation Theorem. ■

Proposition 4.11. *If $T \in \mathcal{D}'(\Omega)$ is a distribution of order $\leq m$, then $\dfrac{\partial T}{\partial x_i}$ for $i = 1, 2, \ldots, n$ is a distribution of order $\leq m + 1$.*

Proof. Firstly note that $\varphi \in \mathcal{D}_K^{m+1}(\varphi)$ implies that $\dfrac{\partial \varphi}{\partial x_i} \in \mathcal{D}_K^m$. Hence

$$\begin{aligned}
\left| \frac{\partial T}{\partial x_i}(\varphi) \right| &= \left| -T\left(\frac{\partial \varphi}{\partial x_i} \right) \right| \\
&\leq M \left\| \frac{\partial \varphi}{\partial x_i} \right\|_m \\
&= M \sup_{x \in \Omega} \sup_{|p| \leq m} \left| \partial^p \frac{\partial}{\partial x_i} \varphi(x) \right| \\
&\leq M \|\varphi\|_{m+1}
\end{aligned}$$

■

Chapter 5

Distributions of Compact Support