Category Theory

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Chapter 1

Types, Composition and Identities

1.1 Programs

A program (function) f applied to an argument x is denoted $f \cdot x$ or $f(x)$. We will develop some notation before we continue:

- $f \circ g \cdot x = f \cdot (g \cdot x)$
- $< f, g > \cdot x =< f \cdot x, g \cdot x >$
- $[f, g] \cdot < t, x >= \begin{cases} f \cdot x & \text{if } t = l \\ g \cdot x & \text{if } t = r \end{cases}$

We also define the following primitive functions:

- $\text{id} \ \cdot x = x$
- $\text{outl} \ \cdot (x, y) = x$
- $\text{outr} \ \cdot (x, y) = y$
- $\text{inl} \ \cdot x =< l, x >$
- $\text{inr} \ \cdot x =< r, x >$
- $\text{zero} \ \cdot x = 0$
- $\text{succ} \ \cdot x = x + 1$

The above notation is quite abstract, so we can think of them in familiar terms by using set notation:

- If $f : A \to B$, then $x \in A \mapsto f \cdot x \in B$.
- If $f : A \to B, g : B \to C$, then $g \circ f : A \to C$.
- If $f : T \to A, g : T \to B$, then $< f, g >: T \to A \times B$.
- If $f : A \to T, g : B \to T$, then $[f, g] : A + B \to T$.

We can also consider the above defined functions in terms of set theory:

- $\text{id} \ : A \to A$

- outl $: A \times B \to A$
- outr $: A \times B \to B$
- inl $: A \to A + B$
- inr $: B \to A + B$
- zero $: 1 \to \mathbb{N}$
- succ $: \mathbb{N} \to \mathbb{N}$

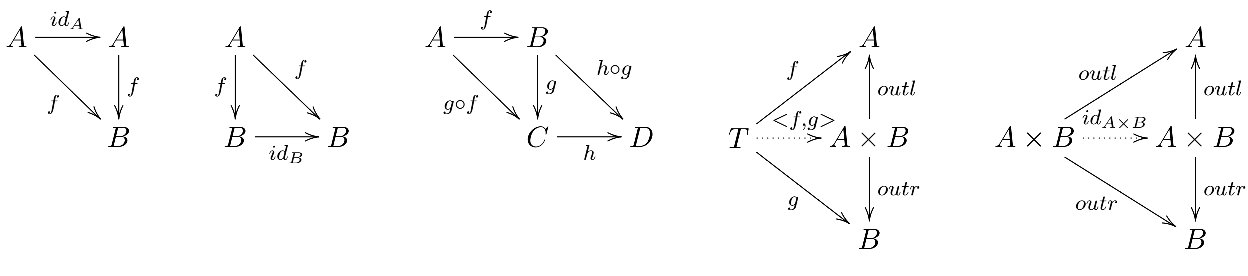
where 1 is the set containing one element, sometimes denoted $\{*\}$. It is common to denoted such functions by what are called commuting diagrams. For example, we denote the fact that if $f : A \to B, g : B \to C$ then $g \circ f : A \to C$ by the following commuting diagram:

1.2 Functional Laws

We have a set of laws that apply to all programs/functions:

- <u>Identity Law</u>: If $f : A \to B$, then $f \circ \text{id}_A = f : A \to B$
- <u>Identity Law</u>: If $f : A \to B$, then $\text{id}_B \circ f = f : A \to B$
- <u>Associativity Law</u>: If $f : A \to B, g : B \to C, h : C \to D$, then $h \circ (g \circ f) = (h \circ g) \circ f : A \to D$
- If $A : T \to A, g : T \to B$ then outl $\circ < f, g >= f : T \to A$
- If $A : T \to A, g : T \to B$ then outr $\circ < f, g >= g : T \to B$
- $<$ outl$_{A,B}$, outr$_{A,B}$ $>= \text{id}_{A \times B} : A \times B \to A \times B$

We can represent the above by commuting diagrams:



where we have combined the 4^{th} and 5^{th} conditions in the second last diagram.

We shall denote by $(A \Rightarrow B)$ the set of all functions from A to B. We wish to separate the following two concepts:

- functional programs and their laws
- the meaning of functions as defined by their application.

For example, given $f(x) = x^2$ we wish to differentiate between f and x^2. We will usually consider simply f and its properties. To do this, we use categories and functions, instead of sets and mappings.

Chapter 2

Categories

2.1 Definitions

We define a CATEGORY $\underline{\mathcal{C}}$ to contain the following data:

1. $\text{Obj}(\underline{\mathcal{C}})$, a class of objects.
2. $\text{Mor}(\underline{\mathcal{C}})$, a class called the morphisms of C.
3. $\text{dom}, \text{cod} : \text{Mor}(\underline{\mathcal{C}}) \to \text{Obj}(\underline{\mathcal{C}})$. For all $f \in \text{Mor}(\underline{\mathcal{C}})$, we call $\text{dom}(f)$ the deomain of f and $\text{cod}(f)$ the codomain of f.
4. $\text{id}_{\_} : \text{Obj}(\underline{\mathcal{C}}) \to \text{Mor}(\underline{\mathcal{C}})$. For all $A \in \text{Obj}(\underline{\mathcal{C}})$, $\text{id}_{\_}(A) = \text{id}_A$ is called the identity morphism for A.
5. $\circ : \text{Mor}(\underline{\mathcal{C}}) \times \text{Mor}(\underline{\mathcal{C}}) \rightharpoonup \text{Mor}(\underline{\mathcal{C}})$ a partial function called composition. For $f, g \in \text{Mor}(\underline{\mathcal{C}})$ we denote by $f \circ g$ the composite of g after f

subject to the following conditions:

- $\text{dom}(\text{id}_A) = A = \text{cod}(\text{id}_A)$
- $g \circ f \in \text{Mor}(\underline{\mathcal{C}}) \Leftrightarrow \text{cod}(f) = \text{dom}(g)$
- if $g \circ f$ is defined, then $\text{dom}(g \circ f) = \text{dom}(f)$ and $\text{cod}(g \circ f) = \text{cod}(g)$
- if $\text{dom}(f) = A$ and $\text{cod}(f) = B$, then $\text{id}_B \circ f = f = f \circ \text{id}_A$
- The associativity law holds on $\text{Mor}(\underline{\mathcal{C}})$

We denote by $\underline{\mathcal{C}}[A, B]$ or $\underline{\mathcal{C}}(A, B)$ the class of morphisms from A to B. If for all $A, B \in \text{Obj}(\underline{\mathcal{C}})$, $\underline{\mathcal{C}}[A, B]$ is a set, then these sets are called homomorphism (or simply hom) sets. We have the followind definitions:

- A category $\underline{\mathcal{C}}$ is called SMALL if $\text{Obj}(\underline{\mathcal{C}})$ is a set and for all $A, B \in \text{Obj}(\underline{\mathcal{C}})$, $\underline{\mathcal{C}}[A, B]$ is a (hom) set.
- A category $\underline{\mathcal{C}}$ is called LOCALLY SMALL if for all $A, B \in \text{Obj}(\underline{\mathcal{C}})$, $\underline{\mathcal{C}}[A, B]$ is a (hom) set.
- A category $\underline{\mathcal{C}}$ is called LARGE if $\text{Obj}(\underline{\mathcal{C}})$ is not a set.

2.2 Examples

The following are large categories.

2.2.1 From Sets

- **Set** a category of sets, whose objects are sets and whose morphisms are mappings between sets.
- **Pfn** a category of sets, whose objects are sets and whose morphisms are partial mappings between sets.
- **Rel** a category of sets and relations, whose objects are sets and whose morphisms are binary relations on the sets.
- **Set**$_f$ a category of sets, whose objects are finite sets and whose morphisms are mappings.
- **Set**$_*$ the category of pointed sets, whose objects are pairs of the form $(A, *_A)$ where A is a set and $*_A \in A$ and whose morphisms are mappings $f : A \to B$ such that $f(*_A) = *_B$, called base point preserving.
- **Set**$_\perp$ a category of sets, whose objects are sets which don't contain $\perp$ and whose morphisms are mappings $f : A \cup \{\perp\} \to B \cup \{\perp\}$ such that $f(\perp) =\perp$, called $\perp$-preserving .

2.2.2 Algrbraic Structures

- **Graph** the category of directed graphs, whose objects are directed graphs and whose morphisms are graph morphisms.
- **Mon** the category of monoids, whose objects are monoids and whose morphisms are monoid morphisms.
- **Grp** the category of groups, whose objects are groups and whose morphisms are group homomorphisms.
- **Ab** the category of Abelian Groups, whose objects are Abelian Groups and whose morphisms are group homomorphisms.
- **Rng** the category of rings, whose objects are rings and whose morphisms are ring homomorphisms.
- **CRng** the category of commutative rings, whose objects are commutative rings and whose morphisms are ring homomorphisms.
- **Vect**$_F$ the category of vector spaces, whose objects are vector spaces over the field F and whose morphisms are linear transformations.
- **Pre** the category of preorders, whose objects are preorders and whose morphisms are monotone (order preserving) mappings.
- **Pos** the category of posets, whose objects are posets and whose morphisms are monotone mappings.
- **M-Set** the category of M actions, whose objects are actions on a fixed monoid M and whose morphisms are M-action morphisms.

2.2.3 Topological Spaces

- **Top** a category of topological spaces, whose objects are topological spaces and whose morphisms are continuous mappings.
- $\mathbf{Top}_h$ a category of topological spaces, whose objects are topological spaces and whose morphisms are homotopy classes of continuous mappings.
- $\mathbf{Top}_*$ a category of topological spaces, whose objects are topological spaces with base points and whose morphisms are base point preserving continuous mappings.

Isomorphism

A morphism $f \in \underline{\mathcal{C}}[A, B]$ is called an <u>Isomorphism</u> if there exists some $g \in \underline{\mathcal{C}}[A, B]$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. We call g the inverse for f sometimes denoted f^{-1}, and also say that A and B are isomorphic, denoted $A \cong B$.

2.2.4 Proposition

If $g_1, g_2 \in \underline{\mathcal{C}}[A, B]$ are inverses for $f \in \underline{\mathcal{C}}[A, B]$, then $g_1 = g_2$

Proof

$$g_1 = g_1 \circ \mathrm{id}_B = g_1 \circ (f \circ g_2) = (g_1 \circ f) \circ g_2 = \mathrm{id}_A \circ g_2 = g_2$$

2.2.5 Proposition

Identity morphisms are isomorphisms

Proof

This follows directly from the definitions of isomorphism and identity.

2.2.6 Proposition

The composition of two isomorphisms is an isomorphism.

Proof

Let f and g be isomorphisms with inverses f^{-1} and g^{-1} respectively. We'll show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

$$\begin{aligned}(g \circ f)^{-1} = f^{-1} \circ g^{-1} \quad &\Leftrightarrow (g \circ f) \circ f^{-1} \circ g^{-1} = \mathrm{id} \\ &\Leftrightarrow g \circ (f \circ f^{-1}) \circ g^{-1} = \mathrm{id} \\ &\Leftrightarrow g \circ id \circ g^{-1} = \mathrm{id} \\ &\Leftrightarrow g \circ g^{-1} = \mathrm{id}\end{aligned}$$

the last statement of which is clearly true. The other direction, i.e. showing that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$ follows similarly.

The following are small categories.

- Given a preorder $P = (P, \preceq)$, the objects are elements of P and the morphisms are given by $f : x \to y$ exists iff $x \preceq y$.

- Given a set S, the objects are elements of S and the morphisms are simply the identity morphisms.
- Given a monoid $M =< M, *, u >$, the objects are the single object $\bigstar$ and the morphisms are the mappings $x : \bigstar \to \bigstar$ for all $x \in M$
- Given a graph $G =< N, E, s, t >$, the objects are the nodes in N and the morphisms are paths between nodes.
- **0** the empty category, graph with no objects and no morphisms, generated from the empty graph.
- **1** the trivial category containing one object and the identity mapping, generated from a graph with one node and no edges.
- **2** the category containing two points and three mappings (two identity mappings) generated from the graph with two nodes and one edge.
- $\circlearrowleft$ the category with one object and one mapping, generated from a graph with one node and one edge.

A function Rwhich assigns to each pair A, B in a category $\underline{\mathcal{C}}$ a binary relation $R_{A,B}$ on the hom class $\underline{\mathcal{C}}[A, B]$ is called a congruence or relation on $\underline{\mathcal{C}}$ if:

- for all $A, B \in \text{Obj}(\underline{\mathcal{C}})$, $R_{A,B}$ is a reflexive, symmetric and transitive relation on $\underline{\mathcal{C}}[A, B]$.
- for all $A, B, A', B' \in \text{Obj}(\underline{\mathcal{C}})$ and for all $f, f' \in \underline{\mathcal{C}}[A, B], g \in \underline{\mathcal{C}}[A, A'], h \in \underline{\mathcal{C}}[B, B']$ we have $f R_{A,B} f' \Rightarrow (h \circ f \circ g) R_{A',B'} (h \circ f' \circ g)$

2.2.7 Proposition

Given any function R which assigns to each pair A, B in a category $\underline{\mathcal{C}}$ a binary relation $R_{A,B}$ on the hom class $\underline{\mathcal{C}}[A, B]$, then there exists a least congruence R' on $\underline{\mathcal{C}}$ with $R \subseteq R'$.

Quotient category

Given a category $\underline{\mathcal{C}}$ and a function R which assings to each pair A, B in a category $\underline{\mathcal{C}}$ a binary relation $R_{A,B}$ on the hom class $\underline{\mathcal{C}}[A, B]$, then there exits a <u>Quotient</u> category $\underline{\mathcal{C}}/R$ whose objects are objects of the category $\underline{\mathcal{C}}$ and whose objects are the hom classes $(\underline{\mathcal{C}}/R)[A, B] := \underline{\mathcal{C}}[A, B]/R'_{A,B}$, where $R'_{A,B}$ is the least congruence of $\underline{\mathcal{C}}$ containing R.

Dual Category

Given a category $\underline{\mathcal{C}}$, we define the <u>Dual</u> category, denoted $\underline{\mathcal{C}}^{\text{op}}$, by

- $\text{Obj}(\underline{\mathcal{C}}^{\text{op}}) = \text{Obj}(\underline{\mathcal{C}})$
- for all $A, B \in \text{Obj}(\underline{\mathcal{C}})$, $\underline{\mathcal{C}}^{\text{op}}[A, B] := \underline{\mathcal{C}}[B, A]$
- $\text{dom}^{\text{op}}(f) = \text{cod}(f)$ and $\text{cod}^{\text{op}}(f) = \text{dom}(f)$
- $\text{id}_A^{\text{op}} = \text{id}_A$
- $f \circ^{\text{op}} g := g \circ f$

Product Category

Given categories $\underline{\mathcal{C}}, \underline{\mathcal{D}}$, we define the PRODUCT category, denoted $\underline{\mathcal{C}} \times \underline{\mathcal{D}}$, by

- $\mathrm{Obj}(\underline{\mathcal{C}} \times \underline{\mathcal{D}}) = \mathrm{Obj}(\underline{\mathcal{C}}) \times \mathrm{Obj}(\underline{\mathcal{D}})$
- for all $A, A' \in \mathrm{Obj}(\underline{\mathcal{C}})$ and $B, B' \in \mathrm{Obj}(\underline{\mathcal{D}})$ we define $\underline{\mathcal{C}} \times \underline{\mathcal{D}}[< A, B >, < A', B' >] := \underline{\mathcal{C}}[A, A'] \times \underline{\mathcal{C}}[B, B']$
- $\mathrm{dom}_{\underline{\mathcal{C}} \times \underline{\mathcal{D}}}(< f, g >) :=< \mathrm{dom}_{\underline{\mathcal{C}}}(f), \mathrm{dom}_{\underline{\mathcal{D}}}(g) >$, $\mathrm{cod}_{\underline{\mathcal{C}} \times \underline{\mathcal{D}}}(< f, g >) :=< \mathrm{cod}_{\underline{\mathcal{C}}}(f), \mathrm{cod}_{\underline{\mathcal{C}}}(g) >$
- $\mathrm{id}_{< A, B >} :=< \mathrm{id}_A, \mathrm{id}_B >$
- $< g, g' > \circ < f, f' > :=< g \circ f, g' \circ f' >$

Slice Category

Given a category $\underline{\mathcal{C}}$ and an element $X \in \mathrm{Obj}(\underline{\mathcal{C}})$, then we define the SLICE category over X, denoted $\underline{\mathcal{C}}/X$, by

- Objects: pairs $< A, f >$ where $A \in \mathrm{Obj}(\underline{\mathcal{C}})$ and $f \in \underline{\mathcal{C}}[A, X]$
- Morphisms: mappings $h :< A, f > \to < A', f' >$ where $h : A \to A'$ is a morphism in $\underline{\mathcal{C}}$ and $f = f' \circ h$

Chapter 3

Functors

3.1 Definitions

A (covariant) FUNCTOR $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ between categories $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ consists of an object mapping $F_{Obj} : \mathrm{Obj}(\underline{\mathcal{C}}) \to \mathrm{Obj}(\underline{\mathcal{D}})$ and a morphism mapping $F : \mathrm{Mor}(\underline{\mathcal{C}}) \to \mathrm{Mor}(\underline{\mathcal{D}})$ such that

- for all $f \in \mathrm{Mor}(\underline{\mathcal{C}})$, we have $\mathrm{dom}(F(f)) = F_{Obj}(\mathrm{dom}(f))$ and $\mathrm{cod}(F(f)) = F_{Obj}(\mathrm{cod}(f))$,
- for all $A \in \mathrm{Obj}(\underline{\mathcal{C}})$, we have $F(\mathrm{id}_A) = \mathrm{id}_{F_{Obj}(A)}$
- for all $A, B, C \in \mathrm{Obj}(\underline{\mathcal{C}})$ and for all $f \in \underline{\mathcal{C}}[A, B], g \in \underline{\mathcal{C}}[B, C]$, we have $F(g \circ f) = F(g) \circ F(f)$.

Usually, the subsript $_{\mathrm{Obj}}$ is dropped when the meaning is clear. Equivalently, we have a (covariant) functor $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ consists of a function $F : \mathrm{Obj}(\underline{\mathcal{C}}) \to \mathrm{Obj}(\underline{\mathcal{D}})$ and a family of functions $F[A, B] : \underline{\mathcal{C}}[A, B] \to \underline{\mathcal{D}}[F(A), F(B)]$ induced by pairs $< A, B >$ of objects of $\underline{\mathcal{C}}$ such that

- for all $A \in \mathrm{Obj}(\underline{\mathcal{C}})$, we have $F[A, A](\mathrm{id}_A) = \mathrm{id}_{F(a)}$
- for all $A, B, C \in \mathrm{Obj}(\underline{\mathcal{C}})$ and for all $f \in \underline{\mathcal{C}}[A, B], g \in \underline{\mathcal{C}}[B, C]$ we have $F[A, C](g \circ_{A,B,C} f) = F[B, C](g) \circ_{F(A),F(B),F(C)} F[A, B](f)$

3.2 More Definitions

- Given functors $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ and $G : \underline{\mathcal{D}} \to \underline{\mathcal{E}}$, we define the COMPOSITE functor $G \circ F : \underline{\mathcal{C}} \to \underline{\mathcal{E}}$ by the following:

$$\begin{array}{ccc} A & & G(F(A)) \\ \downarrow f & \longmapsto & \downarrow G(F(f)) \\ B & & G(F(B)) \end{array}$$

- Given a category $\underline{\mathcal{C}}$, the <u>IDENTITY</u> functor on $\underline{\mathcal{C}}$ denoted $\mathrm{id}_{\underline{\mathcal{C}}} : \underline{\mathcal{C}} \to \underline{\mathcal{C}}$ is given by

$$\begin{array}{ccc} A & & A \\ \Big\downarrow f & \longmapsto & \Big\downarrow f \\ B & & B \end{array}$$

- We denote by **Cat** the category of small categories and functors between small categories.
- We denote by **CAT** the "category" of categories and functors between categories.
- A functor $F : \underline{\mathcal{C}}^{\mathrm{op}} \to D$ is called a <u>CONTRAVARIANT</u> functor.

3.3 Examples of Functors

- We have the inclusion functor which, for example, maps **Set** to **Pfn** or **Pfn** to **Rel**.
- $\_{}_*$: **Pfn**→**Set**$_*$ with $f \mapsto f_*$, where $f_*(x) = \begin{cases} f(x) & \text{if } x \in A, x \in \mathrm{dom}(f) \\ *_B & \text{if } x \in A, x \notin \mathrm{dom}(f) \\ *_B & x = *_A \end{cases}$ for mappings of the form $f : A \to B$.
- D :**Set**$_*$ →**Pfn** with $f \mapsto Df$, where $\mathrm{dom}(Df) := \{x \in A \mid x \neq *_A, f(x) \neq *_B\}$ and $(Df)(x) = f(x)$ for all $x \in \mathrm{dom}(Df)$, where $f :< A, *_A > \to < B, *_B >$
- U :**Mon**→**Set** which is a forgetful functor (it "forgets" the monoid structure and just gives a set).
- V :**Cat**→**Graph**, a forgetful functor given by:

$$\begin{array}{ccc} < \mathrm{Obj}(\underline{\mathcal{C}}), \mathrm{Mor}(\underline{\mathcal{C}}), \mathrm{dom}_{\underline{\mathcal{C}}}, \mathrm{cod}_{\underline{\mathcal{C}}}, \mathrm{id}_{\underline{\mathcal{C}}}, \circ > & & < \mathrm{Obj}(\underline{\mathcal{C}}), \mathrm{Mor}(\underline{\mathcal{C}}), \mathrm{dom}_{\underline{\mathcal{C}}}, \mathrm{cod}_{\underline{\mathcal{C}}} > \\ \Big\downarrow F & \longmapsto & \Big\downarrow V(F) \\ < \mathrm{Obj}(\underline{\mathcal{D}}), \mathrm{Mor}(\underline{\mathcal{D}}), \mathrm{dom}_{\underline{\mathcal{D}}}, \mathrm{cod}_{\underline{\mathcal{D}}}, \mathrm{id}_{\underline{\mathcal{D}}}, \circ > & & < \mathrm{Obj}(\underline{\mathcal{D}}), \mathrm{Mor}(\underline{\mathcal{D}}), \mathrm{dom}_{\underline{\mathcal{D}}}, \mathrm{cod}_{\underline{\mathcal{D}}} > \end{array}$$

- $\_{}_*$:**Graph**→**Cat**, a free functor given by

$$\begin{array}{ccc} < N, E, s, t > & & < N, P(E), s, t, \mathrm{id}, \circ > \\ \Big\downarrow h & \longmapsto & \Big\downarrow f \\ < N', E', s', t' > & & < N', P(E'), s', t', \mathrm{id}', \circ > \end{array}$$

where we define

  - $P(E) := \{e_1, \ldots, e_n \mid t(e_i) = s(e_{i+1}), 1 \leq i \leq n\} \cup \{id_A \mid A \subseteq N\}$, where the identity element is interpreted as the empty word.

- composition to be the concatenation for the paths which join head to tail.
- $h^* : G^* \to H^*$ is a functor between categories defined by:

$$
\begin{array}{ccc}
a \in N & & h_n(a) \in N' \\
\Big\downarrow{\scriptstyle e_1 e_2 \cdots e_n} & \longmapsto & \Big\downarrow{\scriptstyle h^*(e_1 e_2 \cdots e_n) := h_e(e_1) \cdots h_e(e_n)} \\
b \in N & & h_n(b) \in N'
\end{array}
$$

- I :**AbMon**→**Mon** an inclusion functor, which is the identity mapping on the objects of **AbMon**, and where **AbMon**$[A, A']$ $\subseteq$**Mon**$[A, A']$
- U :**M-Set**→**Set**, a forgetful functor.

Given a monoid $M =< M, \diamond, u >$, we can also consider two functors in the reverse order to the last example:

- $_{-*}$**Set**→**M-Set**, a free functor given by the following commutative diagram

$$
\begin{array}{ccc}
A & & < M \times A, \delta^* : M \times (M \times A) \to M \times A > \\
\Big\downarrow{\scriptstyle f} & \longmapsto & \Big\downarrow{\scriptstyle f^* := id_M \times f} \\
B & & < M \times B, \delta^* : M \times (M \times B) \to M \times B >
\end{array}
$$

  where for all $m, m' \in M, a \in A, b \in B$ we define $\delta^*(m, (m', a)) := (m \diamond m', a)$ and $\delta^*(m, (m', b)) := (m \diamond m', b)$

- $_{-*}$**Set**→**M-Set**, a free functor given by the following commutative diagram

$$
\begin{array}{ccc}
A & & < M \Rightarrow A, \delta^* : M \times (M \Rightarrow A) \to M \Rightarrow A > \\
\Big\downarrow{\scriptstyle f} & \longmapsto & \Big\downarrow{\scriptstyle f^* := id_M \times f} \\
B & & < M \Rightarrow B, \delta^* : M \times (M \Rightarrow B) \to M \Rightarrow B >
\end{array}
$$

  where for all $m, m' \in M, f \in M \Rightarrow A, g \in M \Rightarrow B$ we define $[\delta^*(m, f)](m') := f(m \diamond m')$ and $[\delta^*(m, g)](m') := g(m \diamond m')$

Chapter 4

Universal Properties

4.1 Terminal Object

A TERMINAL OBJECT in a category $\underline{\mathcal{C}}$ is an object 1 such that for all $A \in \mathrm{Obj}(\underline{\mathcal{C}})$ there exists a unique morphism from A to 1, i.e. $\underline{\mathcal{C}}[A, 1]$ contains one object. We'll denote this unique isomorphism by $<>_A$.

4.1.1 Proposition

If 1 and $1'$ are terminal objects of a category $\underline{\mathcal{C}}$, then there exists a unique isomorphism from 1 to $1'$.

Proof

As 1 is terminal, there exists a unique $<>_1 \colon 1 \to 1'$. Similarly, as $1'$ is terminal, there exists a unique $<>_{1'} \colon 1' \to 1$. Note that the only element in $\underline{\mathcal{C}}[1, 1]$ is id_1 and the only element in $\underline{\mathcal{C}}[1', 1']$ is $\mathrm{id}_{1'}$. However, $<>_{1'} \circ <>_1 \colon 1 \to 1$ and $<>_1 \circ <>_{1'} \colon 1' \to 1'$, so $<>_1$ and $<>_{1'}$ must be isomorphisms.

4.1.2 Proposition: Reflection Law

$<>_1 = \mathrm{id}_1$

Proof

$<>_1 = \mathrm{id}_1$ iff $\mathrm{id}_1 : 1 \to 1$, which is true.

4.1.3 Proposition: Fusion Law

If $f \in \underline{\mathcal{C}}[A, B]$, then $<>_B \circ f = <>_A$

Proof

$<>_B \circ f = <>_A$ iff $<>_B \circ f : 1 \to 1$, which is true.

4.2 Duality

Let $S(\underline{\mathcal{C}})$ be a statement about the objects and morphisms of a catagory $\underline{\mathcal{C}}$. Then we can form, by reversing the direction of all the morphisms in $S(\underline{\mathcal{C}})$, another statement $S^{\mathrm{op}}(\underline{\mathcal{C}}) = S(\underline{\mathcal{C}}^{\mathrm{op}})$ about $\underline{\mathcal{C}}$.

4.2.1 Proposition

For all $\underline{\mathcal{C}} \in$Obj(**CAT**), $S(\underline{\mathcal{C}})$ is equivalent to for all $\in$Obj(**CAT**), $S^{\mathrm{op}}(\underline{\mathcal{C}})$.

Proof

For all $\underline{\mathcal{C}}$, we have $S(\underline{\mathcal{C}}) \Rightarrow S(\underline{\mathcal{C}}^{\mathrm{op}}) \Leftrightarrow S^{\mathrm{op}}(\underline{\mathcal{C}})$. Similarly, for all $\underline{\mathcal{C}}$, we have $S^{\mathrm{op}}(\underline{\mathcal{C}}) \Rightarrow S^{\mathrm{op}}(\underline{\mathcal{C}}^{\mathrm{op}}) \Leftrightarrow S((\underline{\mathcal{C}}^{\mathrm{op}})^{\mathrm{op}}) \Leftrightarrow S(\underline{\mathcal{C}})$.

The question arises: what is the dual statement to the terminal object?

4.3 Initial Object

An <u>Initial Object</u> in a category $\underline{\mathcal{C}}$ is an object 0 such that for all $A \in \mathrm{Obj}(\underline{\mathcal{C}})$ there exists a unique morphism from 0 to A, i.e. $\underline{\mathcal{C}}[0, A]$ contains one object. We'll denote this unique isomorphism by $[\,]_A$.

The first following three propositions now follow by duality.

4.3.1 Proposition

If 0 and $0'$ are terminal objects of a category $\underline{\mathcal{C}}$, then there exists a unique isomorphism from 0 to $0'$.

4.3.2 Proposition: Reflection Law

$[\,]_0 = \mathrm{id}_0$

4.3.3 Proposition: Fusion Law

If $f \in \underline{\mathcal{C}}[A, B]$, then $f \circ [\,]_A = [\,]_B$

4.3.4 Proposition

The empty category **0** is an initial object in **Cat**.

4.3.5 Proposition

The trivial category **1** is a terminal object in **Cat**.

4.3.6 Proposition

The empty set $\varnothing$ is an initial object in **Set**.

4.3.7 Proposition

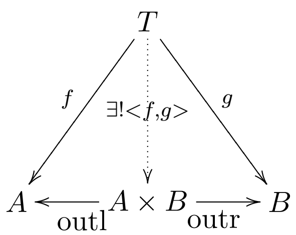
The singleton set $\{*\}$ is a terminal object in **Set**.

4.4 Binary Product

A BINARY PRODUCT of two objects A, B in a category $\underline{\mathcal{C}}$ is specified by

- an object $A \times B$ of $\text{Obj}(\underline{\mathcal{C}})$
- two projection morphisms outl: $A \times B \to A$ and outr: $A \times B \to B$

such that the following diagram commutes



We say that a category has binary products if a binary product exists for all pairs of objects.

4.4.1 Proposition

If $< P, \text{outl}, \text{outr} >$ and $< P', \text{outl}', \text{outr}' >$ are binary product for the objects A, B of a category $\underline{\mathcal{C}}$, then there exists a unique isomorphism $h : P \to P'$ such that $\text{outl} = \text{outl}' \circ h$ and $\text{outr} = \text{outr} \circ h$.

Proof

As $< P', \text{outl}', \text{outr}' >$ is a binary product, we know that there exists some unique $< \text{outl}', \text{outr}' >: P' \to P$ such that

$$\text{outl}' = \text{outl} \circ < \text{outl}', \text{outr}' >$$

and

$$\text{outr}' = \text{outr} \circ < \text{outl}', \text{outr}' >$$

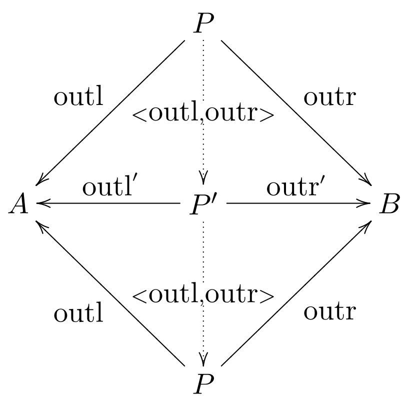
Similarly, as $< P, \text{outl}, \text{outr} >$ is a binary product, we know that there exists some unique $< \text{outl}, \text{outr} >: P \to P'$ such that

$$\text{outl} = \text{outl}' \circ < \text{outl}, \text{outr} >$$

and

$$\text{outr} = \text{outr}' \circ < \text{outl}, \text{outr} >$$

We will show that the following diagram commutes:



As per above, from $\text{outl}' = \text{outl} \circ < \text{outl}', \text{outr}' >$ and $\text{outl} = \text{outl}' \circ < \text{outl}, \text{outr} >$, we conclude that

$$\text{outl} \circ < \text{outl}', \text{outr}' > \circ < \text{outl}, \text{outr} >= \text{outl}$$

and so

$$< \text{outl}', \text{outr}' > \circ < \text{outl}, \text{outr} >= \text{id}_P$$

Using similar logic, but in the other direction, we can show that $< \text{outl}, \text{outr} > \circ < \text{outl}', \text{outr}' >= \text{id}_{P'}$ also. Thus, $< \text{outl}, \text{outr} >$ and $< \text{outl}', \text{outr}' >$ must be isomorphisms.

4.4.2 Proposition: Reflection Law

$< \text{outl}, \text{outr} >= \text{id}_{A \times B}$

4.4.3 Proposition: Fusion Law

$< f, g > \circ h =< f \circ h, g \circ h >$

4.5 Examples of Binary Products

4.5.1 Set

Let A and B be sets.

- $A \times B := \{< a, b > \mid a \in A, b \in B\}$
- $\text{outl} < a, b >= a$
- $\text{outr} < a, b >= b$

4.5.2 Mon

Let $< M, *, u >$ and $< M', *', u' >$ be monoids.

- $M \times M' :=< M \times M', \diamond, < u, u' >>$, where $< m, m' > \diamond < n, n' >=< m * n, m' *' n' >$.
- $\text{outl} < m, n >= m$
- $\text{outr} < m, n >= n$

4.5.3 Cat

Let $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ be monoids.

- $\underline{\mathcal{C}} \times \underline{\mathcal{D}}$ is the product category.
- $\text{outl}(< f, g >:< A, B >\rightarrow< C, D >) = (f : A \rightarrow C)$
- $\text{outl}(< f, g >:< A, B >\rightarrow< C, D >) = (g : B \rightarrow D)$

4.5.4 Pos

If $\underline{\mathcal{C}}, P, \preceq>$ is a category defined by a poset, then a binary product exists iff the poset has a greatest lower bound for all pairs of elements $p, q \in P$

- $p \times q := \sqcap\{p, q\} := p \sqcap q$
- $\text{outl} : p \sqcap q \rightarrow p$ if $p \sqcap q = p$
- $\text{outl} : p \sqcap q \rightarrow q$ if $p \sqcap q = q$

4.5.5 Proposition

If $\underline{\mathcal{C}}$ is a category with a specified binary product, then $\_ \times \_\underline{\mathcal{C}} \times \underline{\mathcal{C}} \rightarrow \underline{\mathcal{C}}$ given by

$$\begin{array}{ccc} (A, B) & & A \times B \\ \Big\downarrow{\scriptstyle (f,g)} & \longmapsto & \Big\downarrow{\scriptstyle f\times g=< f\circ \text{outl}\ , g\circ \text{outr}\ >} \\ (A', B') & & A' \times B' \end{array}$$

is a bifunctor.

Proof

We need to show the following

- for all $f, g \in \text{Mor}(\underline{\mathcal{C}})$, $\text{dom}(f \times g) = \text{dom}(f) \times \text{dom}(g)$ and $\text{cod}(f \times g) = \text{cod}(f) \times \text{cod}(g)$
- for all $A, B \in \text{Obj}(\underline{\mathcal{C}})$, $\text{id}_A \times \text{id}_B = \text{id}_{A\times B}$
- for all $A, A', A'', B, B', B'' \in \underline{\mathcal{C}}$ and $f \in \underline{\mathcal{C}}[A, A'], f' \in \underline{\mathcal{C}}[A', A''], g \in \underline{\mathcal{C}}[B, B'], g' \in \underline{\mathcal{C}}[B'B'']$, we have $(f' \circ f) \times (g' \circ g) = (f' \times g') \circ (f \circ g)$

The first part follows directly from the definitions. The second part is true by noting that

$$\text{id}_A \times \text{id}_B =< \text{id}_A \circ \text{outl}\ , \text{id}_B \circ \text{outr}\ >=< \text{outl}\ , \text{outr}\ >= \text{id}_{A\times B}$$

the last part of which follows by the reflection law. To show the final part, we note the following proposition

4.5.6 Proposition: The Absorbtion Law

$(f \times g)\circ < p, q >=< f \circ p, g \circ q >$

Proof

$(f \times g) \circ < p, q >=< f \circ \text{outl} , g \circ \text{outr} > \circ < p, q >=< f \circ \text{outl} \circ p, g \circ \text{outr} \circ q >=< f \circ p, g \circ q >$

Thus, our previous proposition follows thus:

$(f' \times g') \circ (f \circ g) = (f' \times g') \circ < f \circ \text{outl} , g \circ \text{outr} >= (f' \circ f \circ \text{outl} , g' \circ g \circ \text{outr}) = (f' \circ f) \times (g' \circ g)$

4.5.7 Proposition

If $\underline{\mathcal{C}}$ is a category with binary products and with a terminal object, then the following natural isomorphisms exist for all $A, B, C \in \mathbb{C}$:

$$\begin{array}{l} \text{unit}_A : A \times 1 \to A \\ \text{swap}_{A,B} : A \times B \to B \times A \\ \text{assoc}_{A,B,C} : (A \times B) \times C \to A \times (B \times C) \end{array}$$

4.5.8 Proposition

A binary product of objects A, B in a category $\underline{\mathcal{C}}$ is a terminal object in the span$[A, B](\underline{\mathcal{C}})$ of spans over A and B:

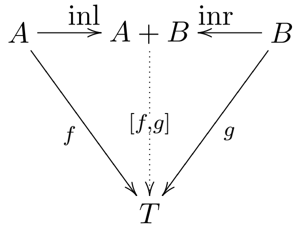
- Objects: pairs of morphisms from $\underline{\mathcal{C}}$ with a common source, i.e. (f, g) where $A \xleftarrow[f]{} T \xrightarrow{g} B$.
- $m : (f, g) \to (f', g')$ where $m : T \to T'$ is a morphism on $\underline{\mathcal{C}}$ such that $f' \circ m = f$ and $g' \circ m = g$.

4.6 Binary Sum

A <u>Binary Sum</u> of objects A, B in a category $\underline{\mathcal{C}}$ is specified by

- an object $A + B$ of $\underline{\mathcal{C}}$ with
- two injective morphisms $\text{inl} : A \to A + B$ and $\text{inr} : B \to A + B$

such that the following diagram commutes:



Note that this is the dualized idea of binary products. This gives us the following propositions.

4.6.1 Proposition: Reflection Law

$[\text{inl}, \text{inr}] = \text{id}_{A+B}$

4.6.2 Proposition: Fusion Law

$h \circ [f, g] = [h \circ f, h \circ g]$

4.7 Examples of Binary Sums

4.7.1 Set

Let A and B be sets.

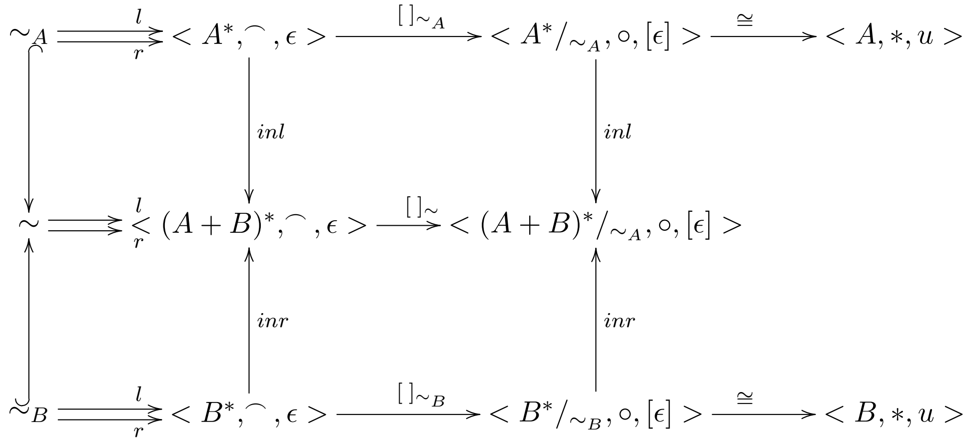
- $A + B := A \uplus B := (\{l\} \times A) \cup (\{r\} \times B)$
- $\mathrm{inl}(a) = (l, a)$
- $\mathrm{inr}(b) = (r, b)$

4.7.2 Mon

Let $M =< M, *, u >$ and $M' =< M', *', u' >$ be monoids, and define $(A + B)^*$ to be the set of finite sequences of elements from the set $A + B$

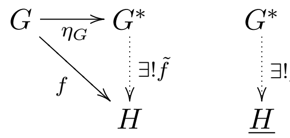
- $M + M' :=< (M + M')^* / \sim, \cdot, [\epsilon] >$, where $\cdot$ is the concatenation operation i.e. $[(x, \dots, y)] \cdot [(x', \dots, y')] = [(x, \dots, y, x', \dots, y')]$, $[\epsilon]$ is the empty word and $\sim$ is the least equivalence relation such that
  - $u \sim \epsilon$ and $\epsilon \sim u'$
  - $(\dots, a, a', \dots) \sim (\dots, a * a', \dots)$ for all $a, a' \in M$,
  - $(\dots, b, b', \dots) \sim (\dots, b *' b', \dots)$ for all $b, b' \in B$
- $\mathrm{inl}(a) = [(a)]$
- $\mathrm{inr}(b) = [(b)]$

in the above, what we means by the least equivalence relation is the equivalence relation $\sim$ given by:



Example: Graph

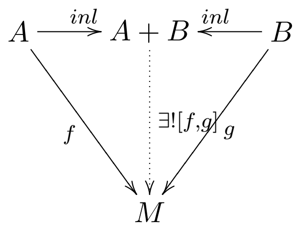
If G is a directed graph, G^* is the underlying graph of the path category over G, $\underline{H}$ $\in$Obj(**Cat**) and H is the underlying graph of the category $\underline{H}$, then we have the following:



If we are given $f : (A, *, u) \to (M, *'', u'')$ and $g : (B, *', u') \to (M, *'', u'')$, then we want to construct a unique $h : ((A+B)^*/_\sim, \cdot, [\epsilon]) \to (M, *'', u'')$ such that $h = h\circ$inl and $g = h\circ$inr.

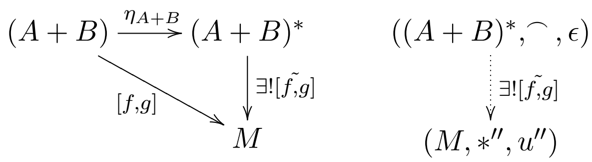
Step 1

In **Set**, we have

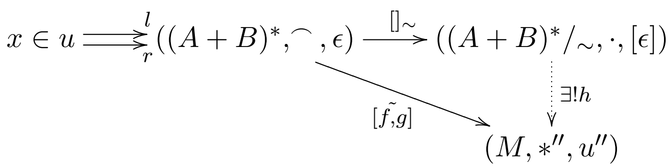


Step 2

In **Set** and then in **Mon**, we have



4.7.3 Step 3



Chapter 5

More on Functors

5.1 Definition: Covariant Hom Functor

Given a fixed object A of a catagory $\underline{\mathcal{C}}$ the COVARIANT HOM FUNCTOR is a mapping $\underline{\mathcal{C}}[A,+] = H_+^A : \underline{\mathcal{C}} \to \mathbf{Set}$ such that

$$\begin{array}{ccc} B & & h \in \underline{\mathcal{C}}[A,B] \\ \Big\downarrow g & \longmapsto & \Big\downarrow \underline{\mathcal{C}}[A,g]=H_g^A \\ B' & & \underline{\mathcal{C}}[A,B'] \end{array}$$

where $\underline{\mathcal{C}}[A,g] = H_g^A$ is given by $H_g^A(h) = g \circ h$.

5.2 Definition: Covariant Hom Functor

Given a fixed object B of a catagory $\underline{\mathcal{C}}$ the CONTRAVARIANT HOM FUNCTOR is a mapping $\underline{\mathcal{C}}[-,B] = H_B^- : \underline{\mathcal{C}}^{\text{op}} \to \mathbf{Set}$ such that

$$\begin{array}{ccc} A & & h \in \underline{\mathcal{C}}[A,B] \\ \Big\downarrow \mathcal{F}^{\text{op}} & \longmapsto & \Big\downarrow \underline{\mathcal{C}}[\mathcal{F}^{\text{op}},B]=H_B^{\mathcal{F}^{\text{op}}} \\ A' & & \underline{\mathcal{C}}[A',B] \end{array}$$

where $\underline{\mathcal{C}}[\mathcal{F}^{\text{op}},B] = H_B^{\mathcal{F}^{\text{op}}}$ is given by $H_B(h) = h \circ \mathcal{F}^{\text{op}}$.

5.3 Definition: Subcategory

$\underline{\mathcal{C}}$ is called a SUBCATEGORY of $\underline{\mathcal{D}}$ if there exists an inclusion functor $I : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ such that $I(f : A \to B) = (f : A \to B)$.

5.3.1 Definition: Faithful

A functor $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is called FAITHFUL if the morphism mapping of the functor is injective.

5.3.2 Definition: Full

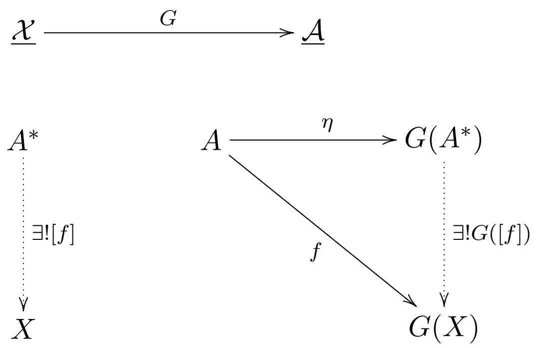
A functor $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is called FULL if the morphism mapping of the functor is surjective.

5.3.3 Definition: Isomorphic

Two categories $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ are said to be isomorphic, denoted $\underline{\mathcal{C}} \cong \underline{\mathcal{D}}$ if there exist mappings $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ and $G : \underline{\mathcal{D}} \to \underline{\mathcal{C}}$ such that $F \circ G = \mathrm{id}_{\underline{\mathcal{C}}}$ and $G \circ F = \mathrm{id}_{\underline{\mathcal{D}}}$.

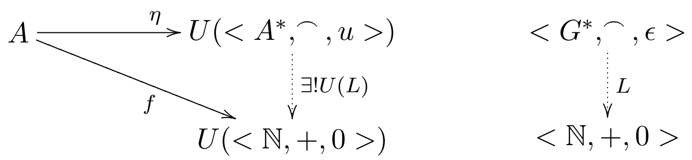
5.4 Universal Morphism

If $G : \underline{\mathcal{X}} \to \underline{\mathcal{A}}$ is a functor and $A \in \underline{\mathcal{A}}$ is an object, then a universal morphism is a pair $< A^*, \eta : A \to G(A^*) >$ consisting of an object $A^* \in \underline{\mathcal{X}}$ and a morphism $\eta : A \to G(A^*)$ of $\underline{\mathcal{A}}$ such that to every pair $< X, f : A \to G(X) >$ with $X \in \underline{\mathcal{X}}$ an object and f a morphism of $\underline{\mathcal{A}}$, there exist a unique mapping $[f] : A^* \to X$ with $G([f]) \circ \eta = f$, i.e.



5.4.1 Example: U :Mon $\to$ Set

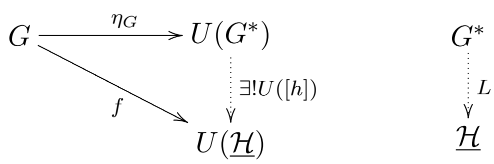
Let A be a set, define $\eta(a) =< a >$ and $f(a) = 1$ for all $a \in A$, we have



where L is the length mapping.

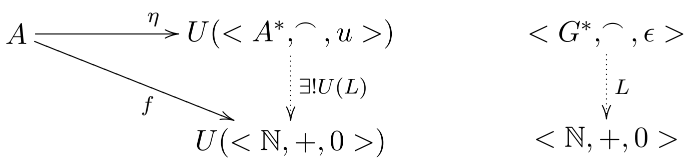
5.4.2 Example: U :Graph $\to$ Cat

Let G be a graph and $\underline{\mathcal{H}}$ be a small category. Then, we have



5.4.3 Example: U :**Mon** $\rightarrow$ **Set**

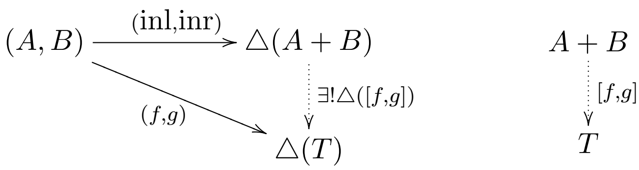
Let A be a set, define $\eta(a) =< a >$ and $f(a) = 1$ for all $a \in A$, we have



where L is the length mapping.

5.4.4 Example: Diagonal Functor

Let $\triangle : \underline{\mathcal{C}} \times \underline{\mathcal{C}} \rightarrow \underline{\mathcal{C}}$ be the functor given by $\triangle(A, A) = A$, called the diagonal functor. Then we have:

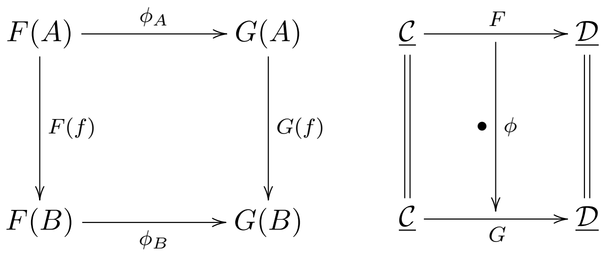


5.5 Natural Transformations

Given categories $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ and two functors $F, G : \underline{\mathcal{C}} \rightarrow \underline{\mathcal{D}}$, we define a Natural Transformation denoted

$$\psi : F \rightarrow^{\bullet} G : \underline{\mathcal{C}} \rightarrow \underline{\mathcal{D}}$$

as a collection $\phi = \{\phi_A : F(A) \rightarrow G(A) \mid A \in \text{obj}(\underline{\mathcal{C}})\}$ of morphisms of $\underline{\mathcal{D}}$ indexed by objects of $\underline{\mathcal{C}}$ such that for all $f(: A \rightarrow B) \in \text{Mor}(\underline{\mathcal{C}})$, we have



The morphism ϕ_A is called the component of ϕ. A natural transformation $\phi : F \rightarrow^{\bullet} G : \underline{\mathcal{C}} \rightarrow \underline{\mathcal{D}}$ is called an Isomorphism if ϕ_A is an isomorphism for all $A \in \text{Obj}(\underline{\mathcal{C}})$.

5.6 Equivalence

We say that two categories $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ are Equivalent if there exists two functors $F, G : \underline{\mathcal{C}} \rightarrow \underline{\mathcal{D}}$ together with two natural isomorphisms $\epsilon : FG \rightarrow^{\bullet}_{\cong} \text{id}_{\underline{\mathcal{D}}}$ and $\eta : \text{id}_{\underline{\mathcal{C}}} \rightarrow^{\bullet}_{\cong} GF$.

5.7 The Functor Category

Let $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ be two categories. The Functor Category usually denoted $[\underline{\mathcal{C}},]$ or $\underline{\mathcal{D}}^{\underline{\mathcal{C}}}$ is given by:

- The objects are functors from $\underline{\mathcal{C}}$ to $\underline{\mathcal{D}}$
- The morphisms are natural transformations between functors
- We define

$$\operatorname{dom}(\alpha : F \to^{\bullet} G : \underline{\mathcal{C}} \to \underline{\mathcal{D}}) = F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$$
$$\operatorname{cod}(\alpha : F \to^{\bullet} G : \underline{\mathcal{C}} \to \underline{\mathcal{D}}) = G : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$$

- We define the identity transformation on a functor F as

$$\mathrm{id}_F = \{\mathrm{id}_{F(A)} : F(A) \to F(A) \mid A \in \mathrm{Obj}(\underline{\mathcal{C}})\}$$

- We define the composition of two natural transformations $\alpha : F \to^{\bullet} G$ and $\beta : G \to^{\bullet} H$ as the mapping $\beta \circ \alpha : F \to^{\bullet} H$, where

$$\beta \circ \alpha = \{(\beta \circ \alpha)_A = \beta_A \circ \alpha_A : F(A) \to H(A) \mid A \in \mathrm{Obj}(\underline{\mathcal{C}})\}$$

Chapter 6

Yoneda Embeddings

Let $\underline{\mathcal{C}}$ be a locally small category, and define $H^- : \underline{\mathcal{C}}^{\text{op}} \to [\underline{\mathcal{C}}, \textbf{Set}\ ]$ as follows:

$$\begin{array}{ccc} A & & \underline{\mathcal{C}}[A,+] = H^A_+ \\ \Big\downarrow{\scriptstyle f^{\text{op}}} & \longmapsto & \Big\downarrow{\scriptstyle H^{f^{\text{op}}}} \\ A' & & \underline{\mathcal{C}}[A',+] = H^{A'}_+ \end{array}$$

where $H^{f^{\text{op}}} : H^A_+ \to^{\bullet} H^{A'}_+ : \underline{\mathcal{C}} \to \textbf{Set}$ is defined for all $(g : B \to B') \in \text{Mor}(\underline{\mathcal{C}})$ as

$$\begin{array}{ccc} H^A_B & \xrightarrow{H^{f^{\text{op}}}_B} & H^{A'}_B \\ \Big\downarrow{\scriptstyle H^A_g} & & \Big\downarrow{\scriptstyle H^{A'}_g} \\ H^A_{B'} & \xrightarrow[H^{f^{\text{op}}}_{B'}]{} & H^{A'}_{B'} \end{array}$$

6.1 The Yoneda Lemma

Let $\underline{\mathcal{C}}$ be a locally small category, $F : \underline{\mathcal{C}} \to \textbf{Set}$ a functor and $A \in \text{Obj}(\underline{\mathcal{C}})$. The collection $\text{nat}[H^A_+, F]$ of natural transformations $\alpha : H^A_+ \to^{\bullet} F$ is a set, so we may define a functor $\text{Nat}[H^-, +] : \underline{\mathcal{C}} \times [\underline{\mathcal{C}}, \textbf{Set}\ ] \to \textbf{Set}$ given by

$$\begin{array}{ccc} < A, F > & & \text{Nat}[H^A_+, F] \\ \Big\downarrow{\scriptstyle <f,\mu>} & \longmapsto & \Big\downarrow{\scriptstyle \text{Nat}[H^f,\mu]} \\ < A', F' > & & \text{Nat}[H^{A'}_+, F'] \end{array}$$

where we define $\text{Nat}[H^f, \mu] = (\text{Nat}[H^-, +]) < f, \mu >$. We can also define the eval functor, namely $\text{ev} : \underline{\mathcal{C}} \times [\underline{\mathcal{C}}, \textbf{Set}\ ] \to \textbf{Set}$ as

$$\begin{array}{ccc} < A, F > & & F(A) \\ \Big\downarrow{\scriptstyle <f,\mu>} & \longmapsto & \Big\downarrow{\scriptstyle \text{ev}<f,\mu>} \\ < A', F' > & & F(A') \end{array}$$

where for all $x \in F(A)$, we have

$$(\text{ev} < f, \mu >)(x) = [F'(f) \circ \mu_A](x) = [\mu_{A'} \circ F(f)](x)$$

There exists natural ismorphisms $\Phi : \text{Nat}[H^-, +] \leftrightarrow^\bullet \text{ev} : \Psi$ such that for all $A \in \text{Obj}(\underline{\mathcal{C}})$ and $F : \underline{\mathcal{C}} \to \textbf{Set}$, we have

$$\Phi_{<A,F>} : \text{Nat}[H^A, F] \leftrightarrow^{\bullet}_{\cong} F(A) : \Psi_{<A,F>}$$

6.1.1 Proof

Part A

We'll show that for all $A \in \text{Obj}(\underline{\mathcal{C}}), F : \underline{\mathcal{C}} \to \textbf{Set}$, we have $\text{Nat}[H^A_+, F] \in \textbf{Set}$ by showing that there exists a bijection from $\text{Nat}[H^A_+, F] \to F(A)$. For all $\alpha : H^A_+ \to^\bullet F$, we have

$$\Phi_{<A,F>} = \alpha_A(\text{id}_A)$$

Also, for all $a \in F(A), B \in \text{Obj}(\underline{\mathcal{C}}), f \in H^A_B$, we have

$$\Psi_{<A,F>}(a) = [F(f)](a)$$

We need to show that $\Psi_{<A,F>}(a) \in \text{Nat}[H^A_+, F]$, namely for all $(g : B \to B') \in \text{Mor}(\underline{\mathcal{C}})$ we have

$$\begin{array}{ccc} H^A_B & \xrightarrow{\Psi_{<A,F>}(a)_B} & F(B) \\ \Big\downarrow{\scriptstyle H^A_g} & & \Big\downarrow{\scriptstyle F(g)} \\ H^A_{B'} & \xrightarrow[\Psi_{<A,F>}(a)_{B'}]{} & F(B') \end{array}$$

We note the following:

$$\begin{aligned} (F(g) \circ \Psi_{<A,F>}(a)_B)(f) &= F(g)(\Psi_{<A,F>}(a)_B(f)) \\ &= F(g)(F(f(a)) \\ &= [F(g) \circ F(f)](a) \\ &= [F(g \circ f)](a) \\ &= [F(H^A_g(f))](a) \\ &= \Psi_{<A,F>}(a)_{B'}(H^A_g(f)) \\ &= (\Psi_{<A,F>}(a)_{B'} \circ H^A_g)(f) \\ &\in \text{Nat}[H^A_+, F] \end{aligned}$$

We now wish to show that

$$\Psi_{<A,F>} \circ \Phi_{<A,F>} = \mathrm{id}_{\mathrm{Nat}[H_+^A, F]} \qquad \Phi_{<A,F>} \circ \Psi_{<A,F>} = \mathrm{id}_{F(A)}$$

For all $\alpha \in \mathrm{Nat}[H_+^A, F], B \in \underline{\mathcal{C}}, f \in H_B^A$, we have the following:

$$\begin{aligned}
(\Psi_{<A,F>} \circ \Phi_{<A,F>})(\alpha) &= \Psi_{<A,F>}(\Phi_{<A,F>}(\alpha)) \\
&= \Psi_{<A,F>}(\alpha_A(\mathrm{id}_A)) \\
&= F(f)(\alpha_A(\mathrm{id}_A)) \\
&= (F(f) \circ \alpha_A)(\mathrm{id}_A) \\
&= (\alpha_B \circ H_f^A)(\mathrm{id}_A) \\
&= \alpha_B(H_f^A(\mathrm{id}_A)) \\
&= \alpha_B(f \circ \mathrm{id}_A)) \\
&= \alpha_B(f) \\
&= \alpha
\end{aligned}$$

Also, note that for all $a \in F(A)$, we have

$$\begin{aligned}
(\Phi_< A, F > \circ \Psi_{<A,F>})(a) &= \Phi_< A, F > (\Psi_{<A,F>}(a)) \\
&= \Psi_< A, F > (a)_A(\mathrm{id}_A) \\
&= F(\mathrm{id}_A)(a) \\
&= \mathrm{id}_{F(A)}(a) \\
&= a
\end{aligned}$$

And so, we have that there exists an isomorphism between $F(A)$ and $\mathrm{Nat}[H_+^A]$ for all $A \in \mathrm{Obj}(\underline{\mathcal{C}})$.

6.1.2 Part B

We now wish to show that above-defined morphism actually defined a natural morphism, i.e. we want the following diagram to commute:

$$\begin{array}{c}
< A, F > \\
\Big\downarrow {\scriptstyle <f,\mu>} \\
< A', F' >
\end{array}
\quad \longmapsto \quad
\begin{array}{ccc}
\mathrm{Nat}[H_+^A, F] & \xrightarrow{\Phi_{<A,f>}} & \mathrm{ev} < A, F >= F(A) \\
\Big\downarrow {\scriptstyle \mathrm{Nat}[H^{f^{\mathrm{op}}}, \mu]} & & \Big\downarrow {\scriptstyle \mathrm{ev}<f,\mu>} \\
\mathrm{Nat}[H_+^{A'}, F'] & \xrightarrow{\Phi_{<A,f>}} & \mathrm{ev} < A, F >= F(A)
\end{array}$$

For all $\alpha \in \mathrm{Nat}[H_+^A, F]$, we have the following:

$$\begin{aligned}
(\Psi_{<A',F'>} \circ \mathrm{Nat}[H_+^{f^{\mathrm{op}}}, \mu])(\alpha) &= \Psi_{<A',F'>}(\mathrm{Nat}[H_+^{f^{\mathrm{op}}}, \mu](\alpha)) \\
&= \Psi_{<A',F'>}(\mu \circ \alpha \circ H_+^{f^{\mathrm{op}}}) \\
&= (\mu \circ \alpha \circ H_+^{f^{\mathrm{op}}})_{A'}(\mathrm{id}_{A'}) \\
&= (\mu_{A'} \circ \alpha_{A'} \circ H_{A'}^{f^{\mathrm{op}}})(\mathrm{id}_{A'}) \\
&= (\mu_{A'} \circ \alpha_{A'})(H_{A'}^{f^{\mathrm{op}}}(\mathrm{id}_{A'})) \\
&= (\mu_{A'} \circ \alpha_{A'})(\mathrm{id}_{A'} \circ f) \\
&= (\mu_{A'} \circ \alpha_{A'})(f) \\
&= (\mu_{A'} \circ \alpha_{A'})(f \circ \mathrm{id}_A) \\
&= (\mu_{A'} \circ \alpha_{A'})(H_f^A(\mathrm{id}_A)) \\
&= (\mu_{A'} \circ \alpha_{A'} \circ H_f^A)(\mathrm{id}_A) \\
&= (\mu_{A'} \circ F(f) \circ \alpha'_A)(\mathrm{id}_A) \\
&= (F'(f) \circ \mu_A \circ \alpha'_A)(\mathrm{id}_A) \\
&= (F'(f) \circ \mu_A)(\alpha'_A(\mathrm{id}_A)) \\
&= \mathrm{ev} < f, \mu > (\alpha'_A(\mathrm{id}_A)) \\
&= \mathrm{ev} < f, \mu > (\Phi_{<A,F>}(\alpha)) \\
&= (\mathrm{ev} < f, \mu > \circ \Phi_{<A,F>})(\alpha)
\end{aligned}$$

Thus, we have shown that there exists a natural transformation from $\mathrm{Nat}[H^-, +]$ to ev whose components are isomorphisms. Hence, result. ■

We note that the Yoneda embedding $y : \underline{\mathcal{C}}^{\mathrm{op}} \to [\underline{\mathcal{C}}, \mathbf{Set}\ ]$ is both full and faithful, by noting that for all $A, A' \in \mathrm{Obj}(\underline{\mathcal{C}})$, we have

$$\begin{aligned}
[\underline{\mathcal{C}}, \mathrm{Set}][y(A), y(A')] &= \mathrm{Nat}[H_+^A, H_+^{A'}] \\
&\cong \mathrm{ev} < A, H_+^{A'} > \\
&= H_+^{A'}(A) \\
&= H_A^{A'} \\
&= \underline{\mathcal{C}}[A, A'] \\
&\cong \underline{\mathcal{C}}^{\mathrm{op}}[A, A']
\end{aligned}$$

as required.

Appendix A

Supplementary Definitions

A.1 Function and Classes

A.1.1 Single Valued

Given a function $f : A \rightarrow B$ we let R be the set of all pairs (a, b) such that $f(a) = b$ for $a \in A, b \in B$. We say that f is SINGLE VALUED, if $(a, b), (a', b') \in R$ and $a = a'$, then $b = b'$.

A.1.2 Totally Defined

Given a function $f : A \rightarrow B$ we let R be the set of all pairs (a, b) such that $f(a) = b$ for $a \in A, b \in B$. W say that f is TOTALLY DEFINED, if $\{a \in A \mid \exists\, b \in Bs.t.(a, b) \in R\} = A$.

A.1.3 Mapping

A MAPPING is a single-valued, totally defined function.

A.1.4 Partial Mapping

A PARTIAL MAPPING is a single-valued function.

A.1.5 Relation

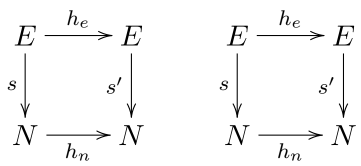
A RELATION is a function between sets.

A.2 Structures

A.2.1 Directed Graph

A GRAPH G is a quadruple $< N, E, s, t >$ where E is a set of edges of the graph, N is a set of nodes of the graph and $s, t : E \rightarrow N$ are the source and target mappings.

A GRAPH MORPHISM $h : G \rightarrow G'$ is a quadruple $< G, h_n, h_e, G' >$ where $h_n : N \rightarrow N'$ and $h_e : E \rightarrow E'$ are mappings such that $h_n \circ s = s' \circ h_e$ and $h_n \circ t = t' \circ h_e$. In other words, the following diagrams commute:



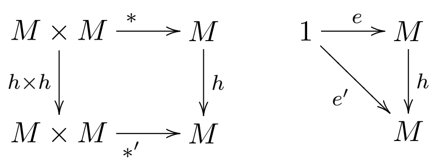
A.2.2 Monoid

A <u>Monoid</u> is a triple $< M, *, u >$ where M is a set, $* : M \times M \to M$ is an associative binary operation and u is a unit (identity) for the operation.

Examples of monoids are:

- $< \mathbb{N}, +, 0 >$
- $< \mathbb{N}_0, \times, 1 >$ where $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$
- $< \mathcal{P}(A), \cup, \varnothing >$
- $< \mathcal{P}(A), \cap, A >$
- $< \mathbf{Set}[A, A], \circ, \mathrm{id}_A >$

Note that a morphism between monoids $h : M \to M'$ is an operation preserving mapping which maps the unit to the unit. In other words, the following diagrams commute:



A.2.3 Preorder

A <u>Preorder</u> is a pair $(P, \preceq)$ where P is a set and $\preceq$ is a reflexive and transitive binary relation.

A.2.4 Poset

A <u>Poset</u> (partially ordered set) is a preorder whose binary relation is also anti-symmetric.

A.2.5 Bounds

Let $< P, \preceq >$ be a poset and $S \subseteq P$. Then an element $z \in P$ is called a

- <u>Lower Bound</u> of S if for all $s \in S$, $z \preceq s$.
- <u>Greatest Lower Bound</u> of S if for all lower bounds y of S, $y \preceq z$. This is sometimes denoted $\sqcap S$
- <u>Upper Bound</u> of S if for all $s \in S$, $s \preceq z$.
- <u>Least Upper Bound</u> of S if for all lower bounds y of S, $z \preceq y$. This is sometimes denoted $\sqcup S$

A.2.6 M-Action

An <u>M-action</u> on a fixed monoid $M =< M, *, u >$ is a pair $< S, \delta >$ where S is a set of states and $\delta : M \times S \to S$ is a mapping such that for all $x, y \in M$ and $s \in S$, we have $\delta(x * y, s) = \delta(x, \delta(y, s))$ and $\delta(u, s) = s$. In other words, the following diagrams commute: