Quantum Mechanics

1 Momentum + Position

For a given wave, $\psi(x)$ we have:
Probability of finding particle in the range between x and dx:

$$|\psi(x)|^2 dx \tag{1}$$

Square integrable functions

$$\int_{\infty}^{\infty} dx\psi^*(x)\psi(x) = 1 \tag{2}$$

Mean value of position:

$$\langle x \rangle = \int_{\infty}^{\infty} dx\psi^*(x)x\psi(x) = \int_{\infty}^{\infty} dx|\psi(x)|^2 x \tag{3}$$

Mean square deviation of position:

$$(\Delta x)^2 = \int_{\infty}^{\infty} dx\psi^*(x)(x - \langle x \rangle)^2\psi(x) \tag{4}$$

$$(\Delta x)^2 = \int_{\infty}^{\infty} dx\psi^*(x)[x^2 + \langle x \rangle^2 - 2x\langle x \rangle]\psi(x) \tag{5}$$

Uncertainty in position:

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \tag{6}$$

Mean value of momentum:

$$\langle p \rangle = \int_{\infty}^{\infty} dx\psi^*(x)p\psi(x) \tag{7}$$

Solve using:

$$\psi(x,t) = Ae^{i(kx-\omega t)} \tag{8}$$

$$\frac{\partial\psi(x,t)}{\partial x} = ik\psi(x,t) \tag{9}$$

$$p\psi(x,t) = \frac{\hbar}{i}\frac{\partial\psi(x,t)}{\partial x} \qquad \text{(Using } p = \hbar k\text{)} \tag{10}$$

Gives:

$$\langle p \rangle = \frac{\hbar}{i}\int_{\infty}^{\infty} dx\psi^*(x)\frac{\partial\psi(x)}{\partial x} \tag{11}$$

Similarly:

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \tag{12}$$

Check that:

$$\Delta x\Delta p \geq \frac{\hbar}{2} \tag{13}$$

2 Operators

$$\langle O \rangle = \int_{\infty}^{\infty} dx \psi^*(x) \hat{O} \psi(x) \tag{14}$$

Kinetic Energy:

$$\hat{T} = \frac{\hat{P}^2}{2m} = \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \tag{15}$$

Angular Momentum:

$$\vec{\hat{L}} = \vec{\hat{R}} \times \vec{\hat{P}} \tag{16}$$

Commutator:

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \tag{17}$$

3 Schroedinger Equation

Time evolution:

$$i\hbar \frac{\partial}{\partial t} \Psi(\vec{r}, t) = \hat{H} \Psi(\vec{r}, t) \tag{18}$$

Hamiltonian:

$$\hat{H} = \frac{\hat{P}^2}{2m} + \hat{V}(\vec{r}, t) = -\frac{\hbar^2}{2m} \nabla^2 + \hat{V}(\vec{r}, t) \tag{19}$$

Solving Schroedinger Equation:

$$i\hbar \frac{\partial}{\partial t} \Psi(x, t) = -\frac{\hbar^2}{2m} \nabla^2 \Psi(x, t) + V(x) \Psi(x, t) \tag{20}$$

Use separation of variables gives two equations:

$$i\hbar \frac{d\phi(t)}{dt} = E\phi(t) \rightarrow \phi(t) = e^{-\frac{i}{\hbar} Et} \tag{21}$$

Time independent Schroedinger equation:

$$\vec{H}(x) \psi(x) = E\psi(x) \tag{22}$$

Full time independend Schroedinger equation:

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + V(x) \psi(x) = E\psi(x) \tag{23}$$

4 Infinite Square Well

Square well:

$$V(x) = \{0 \to 0 \leq x \leq a | \infty \to \text{otherwise}\} \tag{24}$$

Outside the well

$$\psi(x) = 0 \tag{25}$$

Inside the well:

$$\frac{d^2\psi(x)}{dx^2} = -\frac{2mE}{\hbar^2}\psi(x) = -k^2\psi(x) \tag{26}$$

$$k = \frac{\sqrt{2mE}}{\hbar}, (E > 0) \tag{27}$$

General Solution:

$$\psi(x) = Asin(kx) + Bcos(kx) \tag{28}$$

Boundary conditions, continuity of Ψ:

$$\psi(0) = \psi(a) = 0 \tag{29}$$

$$\psi(0) = B \to B = 0 \tag{30}$$

$$\psi(x) = Asin(kx) \tag{31}$$

$$\psi(a) = Asin(ka) \to k_n = \frac{n\pi}{a}, n = 1, 2, 3 \dots \tag{32}$$

Energy:

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2ma^2} \tag{33}$$

4.1 Time Dependence

$$\Psi_n(x,t) = \psi_n(x)e^{-\frac{i}{\hbar}E_n t} \tag{34}$$

Linear combination:

$$f(x) = \sum_{n=1}^{\infty} C_n\psi_n(x) \tag{35}$$

In this case:

$$\psi_n(x) = \sqrt{\frac{2}{a}}sin(k_n x) \tag{36}$$

Fourier Series:

$$C_n = \int dx \psi_n^*(x) f(x) \tag{37}$$

So time dependence is:

$$\Psi_n(x,t) = \sqrt{\frac{2}{a}}sin(\frac{nx\pi}{a})e^{-\frac{i}{\hbar}(\frac{n^2\pi^2\hbar^2}{2ma^2})t} \tag{38}$$

5 Harmonic Oscillator