MA U34605 Quiz 03 w/e 6/11/20

Answer any 3 questions. Submit them using the `submit-work` program as pdfs, either handwritten and scanned, or typeset. They should be submitted before 1pm on Tuesday 17 November. There is an extra week because of Reading Week. All questions carry 20 marks.

(1) Show it is possible to sort n integers, all in the range $1 \ldots n^7$, in $O(n)$ steps.

(2) Show that every nonempty acyclic digraph G has a source.

(3) Show that if u is a source of G and G is *not* acyclic, then neither is the deleted graph $G \backslash u$.

Definition: given $G = (V, E)$, $u \in V$,

$$G \backslash u = (V \backslash u, \{(x, y) \in E : \ x \neq u \wedge y \neq u\}),$$

where $G = (V, E)$. It is obtained by deleting u and all edges incident to u.

More generally, given $X \subseteq V$, $G \backslash X$ is obtained by deleting all vertices in X and all edges incident to vertices in X.

(4) Let $G = (V, E)$ be a digraph. Define a relation $\sim$ on V by:

$u \sim v$ if and only if there exists a directed cycle containing both u and v; or, equivalently, there exist paths from u to v and from v to u.

Definition. A *path from* u to v in G is a sequence $u_1, \ldots, u_k$ where $u_1 = u$, $u_k = v$, and, for $1 \leq j < k$, (u_j, u_{j+1}) is an edge of G.

Show that $\sim$ is an equivalence relation on V, i.e., reflexive, symmetric, and transitive.

(5) MSD radix sort is more natural and more cumbersome than LSD. Show that, with k digits, its runtime is $O(kn)$. Here is a rough description of the algorithm.

First, it seems easiest to begin by copying the keys into a linked list, sorting the list, and copying back to an array. So we assume we are sorting a linked list.

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To sort a list S of keys, sorting on k digits;
assume the radix is 10 for simplicity.

  Sort on the (k-1)-th digit, distributing
  the keys into 10 buckets (linked lists)
  according to their k-th digit. The 0-th
  digit is the low-order or ‘units’ digit.

  If k > 0, _recursively_ sort each bucket
  on k-1 digits.

  recombine the sorted buckets into a single
  list.
```

The question is to show that the cost of sorting n keys on k digits is $O(nk)$, assuming that the cost of distributing n keys into 10 buckets is $O(n)$ and the cost of recombining the buckets is $O(1)$ (there are only 10 buckets to be combined).