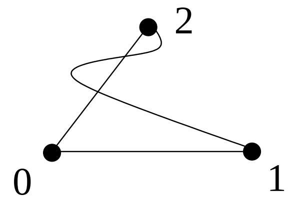

Figure 1: The complete graph K_3 is obviously planar, but this is not a plane embedding.

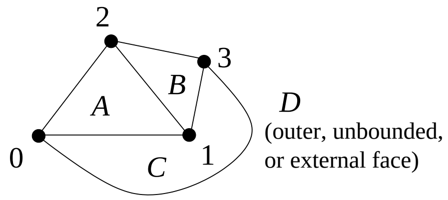

Figure 2: Plane embedding of K_4 and its faces A, B, C, D.

21 Planar graphs

21.1 Plane embeddings and planar graphs

A graph $G = (V, E)$ is *planar* if a plane embedding exists, and a *plane embedding* is a joint map

- $f : V \to \mathbb{R}^2$, injective; an injective map from vertices to points in the plane, and
- $f : E \to$simple curve-segments in $\mathbb{R}^2$, where every edge $\{u, v\}$ is mapped to a simple curve joining $f(u)$ to $f(v)$,
- **and** the curve-segments have disjoint interiors.

A simple curve-segment in the plane is the image of a continuous injective map from the unit interval $[0, 1]$ into $\mathbb{R}^2$.

The Jordan Curve Theorem and its stronger form, the Jordan-Schönflies Theorem, give a theoretical basis for the arguments given here, but that would be too formal.

Given a plane embedding of a graph G, the image of the embedding is a union of simple curve-segments, which by abuse of notation we shall denote as G. Then $\mathbb{R}^2 \backslash G$ is an open subset of $\mathbb{R}^2$.

(21.1) Definition *The* faces *of the plane embedding are the connected components of* $\mathbb{R}^2 \backslash G$. *Exactly one of these faces is unbounded. It is called the unbounded, outer, or external face.*

For any n, the *complete graph* K_n is $(V, V^{(2)})$, where $V = \{0, \ldots, n-1\}$. The distinction between 'planar graph' and 'plane embedding' is illustrated in Figure 1.

(21.2) Proposition *If* $n \geq 3$, *every planar graph with* n *vertices has at most* $3n - 6$ *edges, and every plane embedding of such a graph has at most* $2n - 4$ *faces.* ∎

(21.3) Corollary *The complete graph* K_5 *is not planar.*

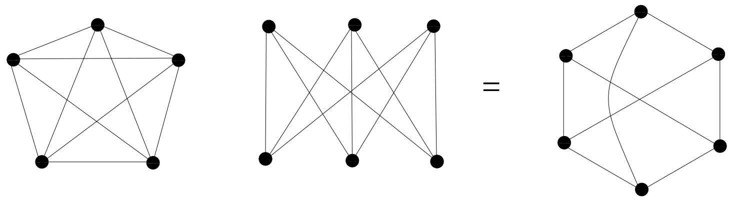

Figure 3: The Kuratowski graphs K_5 and $K_{3,3}$.

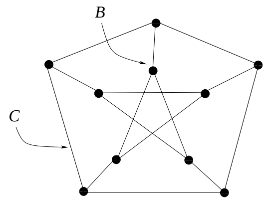

Figure 4: B is planar but $C + B$ is not.

Proof. It has 5 vertices and 10 edges, and $3 \times 5 - 6 = 9 < 10$. ∎

(21.4) Proposition *The complete bipartite graph $K_{3,3}$ is not planar.*

Proof. $K_{3,3}$ is the graph with 6 vertices $0 \ldots 5$, and edges $\{i, j\}$ for $0 \le i \le 2$ and $3 \le j \le 5$.

Another way to describe it is a 6-cycle with three extra edges, 'chords,' as depicted in Figure 3. Suppose two of the chords are placed, one inside and one outside (they can't both be inside or outside). If the third chord is placed inside, it crosses the inside chord; and if it is placed outside, it crosses the outside chord. So no plane embedding is possible. ∎

21.2 Bridges and interlacing

We are about to study the Hopcroft-Tarjan planarity testing algorithm. It works only with biconnected graphs. Put very simply, the Hopcroft-Tarjan algorithm chooses a simple cycle C, and recursively places other pieces of the graph relative to C.

(21.5) Definition *If C is a (simple) cycle, then a subgraph B of C is a* bridge *if either* (i) *B consists of a single edge $\{u_1, u_2\}$, not on C, joining two vertices in C (u_1, u_2 are called* attachments*), or* (ii) *there is a connected subgraph B' of B such that B' is disjoint from C, and there are two or more edges $\{u_1, v_1\}, \ldots, \{u_k, v_k\}$, where $u_j \in C$ and $v_j \in B'$, and B is B' plus these edges $\{u_j, v_j\}$, $1 \le j \le k$.*

The vertices $u_1, \ldots, u_k$ are called attachments.

A bridge which is a single edge is called a singular *bridge; bridges of the other kind are called* nonsingular.

Observation: C is of course planar, and it is possible that B is planar while $C + B$ (the union) is not planar (Figure 4).

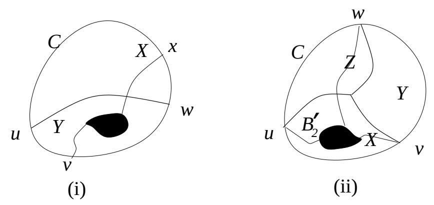

Figure 5: interlacing prevents B_1 and B_2 from being laid out on the same side of C.

(21.6) Definition *Suppose that B_1 and B_2 are different bridges relative to C, where $C + B_1$ and $C + B_2$ are both planar*[1] *Then B_1 and B_2* interlace *if either* (i) *there exist four distinct vertices* in cyclic order u, v, w, x *on C where u, w are attachments to B_1 and v, x are attachments to B_2, or* (iii) *B_1 and B_2 have three or more attachments in common.*

(21.7) Lemma *Suppose that B_1 and B_2 are bridges relative to C, disjoint except at their attachments, and they interlace. Then in any planar layout of $C + B_1 + B_2$, B_1 goes inside C and B_2 outside, or vice-versa.*

Proof. Without loss of generality, B_1 goes inside C. In case (i), u, w are attachments to B_1 and v, x are attachments to B_2. There is a path P in B_1 joining u to w, and it divides the inside of C into two disjoint regions X and Y. A path from v to x inside C crosses between these regions and intersects the path from u to v, which is impossible since B_1 and B_2 are disjoint except for their attachments.

In case (ii), let u, v, w be the common attachments. There are paths within B_1 joining a common vertex x to u, v, w, and they divide the inside of C into three regions X, Y, Z. If B_2' is placed inside C without crossing any of these paths, it must be entirely within one of the regions: X, say. There is a path within B_2 joining w, which is well separated from X, and it would one of the three paths joining B_1 to C, again impossible if B_1 and B_2 are disjoint except at their attachments. ∎

[1]Otherwise G is nonplanar.