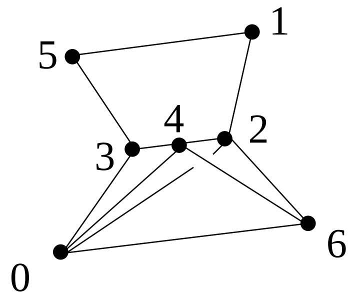

Figure 1: graph for questions 1, 2, 3, and 4

MA U34605 Quiz 04 w/e 27/11/20 ANSWER

Answer any 3 questions. Submit them using the `submit-work` program as pdfs, either handwritten and scanned, or typeset. They should be submitted before 1pm on Tuesday 1 December. All questions carry 20 marks.

(1) Simulate the bcc algorithm on the graph in Figure 1, beginning at vertex 0. Show the order in which the calls to dfs begin and end, the preorder rank, and the highpt function.

Answer

```
7 22
0 4 2 3 4 6
1 2 2 5
2 4 1 4 0 6
3 3 4 5 0
4 4 0 6 2 3
5 2 1 3
6 3 0 2 4
dfs 0 begins
   dfs 2 begins
      dfs 1 begins
         dfs 5 begins
            dfs 3 begins
               dfs 4 begins
                  dfs 6 begins
                  dfs (6) ends
               dfs (4) ends
            dfs (3) ends
         dfs (5) ends
      dfs (1) ends
   dfs (2) ends
BCC no. 1
 3  0
```

```
 4  2
 6  2
 6  0
 4  6
 4  0
 3  4
 5  3
 1  5
 2  1
 0  2
dfs (0) ends
   graph extras, 7 vertices
   vertex    parent  pre_rank post_rank highpt
        0        -1         0         6      0
        1         2         2         4      0
        2         0         1         5      0
        3         5         4         2      0
        4         3         5         1      0
        5         1         3         3      0
        6         4         6         0      0
```

---

(2) Apply the DFS from the Hopcroft-Tarjan algorithm to the graph in Figure 1. Begin at vertex 0. Show the order in which calls to dfs begin and end, and calculate parent, preorder rank, highpt1 and highpt2.

Then relabel the vertices, the parent function values, and the two highpt function values by the preorder ranks and tabulate them (in the order of preorder ranks).

Answer

The DFS sequence is the same as in question 1.

```
7 22
0 4 2 3 4 6
1 2 2 5
2 4 1 4 0 6
3 3 4 5 0
4 4 0 6 2 3
5 2 1 3
6 3 0 2 4
tables
   vertex    parent  pre_rank  highpt_1 highpt_2
        0        -1         0         0        0
        1         2         2         0        2
        2         0         1         0        2
```

```
       3         5         4         0        2
       4         3         5         0        2
       5         1         3         0        2
       6         4         6         0        2
tables, preorder labels
vertex  parent  pre_rank  highpt_1  highpt_2
     0      -1         0         0        0
     1       0         1         0        1
     2       1         2         0        1
     3       2         3         0        1
     4       3         4         0        1
     5       4         5         0        1
     6       5         6         0        1
```

---

(3) Continuing (2): Tabulate the ϕ- function, bearing in mind that vertices are labelled by their preorder ranks.

Then construct the palm tree version of the graph, with the out-edges ordered according to the ϕ function.

Answer

```
phi function
 u  v  phi(u,v) u  v  phi(u,v) u  v  phi(u,v) u  v  phi(u,v)
 0  2      0    1  5      1    2  1      1    3  0      0
 3  4      1    4  0      0    4  6      1    4  2      2
 5  3      1    6  0      0    6  2      2
palmtree
7 11
0 1 2
1 1 5
2 1 1
3 2 0 4
4 3 0 6 2
5 1 3
6 2 0 2
```

---

(4) Continuing (3): Extract the paths recursively in the same order as they would be extracted in the recursive attempt-layout, and for each path say whether it is tied to its parent path (except for the first path).

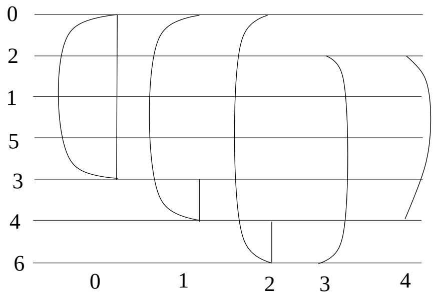

Figure 2: the paths extracted in Question 4. Vertices are arranged in downward preorder.

Answer

```
path index 0 0:0 2:1 1:2 5:3 3:4 0:0  initial cycle
path index 1 3:4 4:5 0:0    basepath = parent = 0 not tied
path index 2 4:5 6:6 0:0    parent 1 base 0 tied
path index 3 6:6 2:1        parent 2 base 0 tied
path index 4 4:5 2:1        parent 1 base 0 tied
```

(5) Give a partial proof that a plane embedded graph G with n vertices, assuming $n \geq 3$, has at most $3n - 6$ edges and $2n - 4$ faces. Note without proof:

- If G has at least 3 edges, then every face is incident to at least 3 edges.
- Every edge is incident to 1 or 2 faces.
- Euler formula: $v - e + f = c + 1$ where G has v vertices, e edges, f faces, and c components.

Answer

Proof. Count the pairs (f, e) where f is a face and e an edge and f is incident to e.

First count

$$\sum_f |\{e : \ e \text{ incident to } f\}| \geq 3f$$

since every face is incident to at least 3 edges.

Then count

$$\sum_e |\{f : \ e \text{ incident to } f| \leq 2e$$

since every edge is incident to at most 2 edges. Therefore

$$2f \leq 3e$$

By Euler's formula

$$v - e + f \geq 2$$

Since $f \leq 2e/3$,

$$\begin{aligned} v - e + 2e/3 &\geq 2 \\ v - e/3 &\geq 2 \\ v - 2 &\geq e/3 \\ e &\leq 3v - 6 \end{aligned}$$

Since $f \leq 2e/3$,

$$f \leq 2v - 4.$$