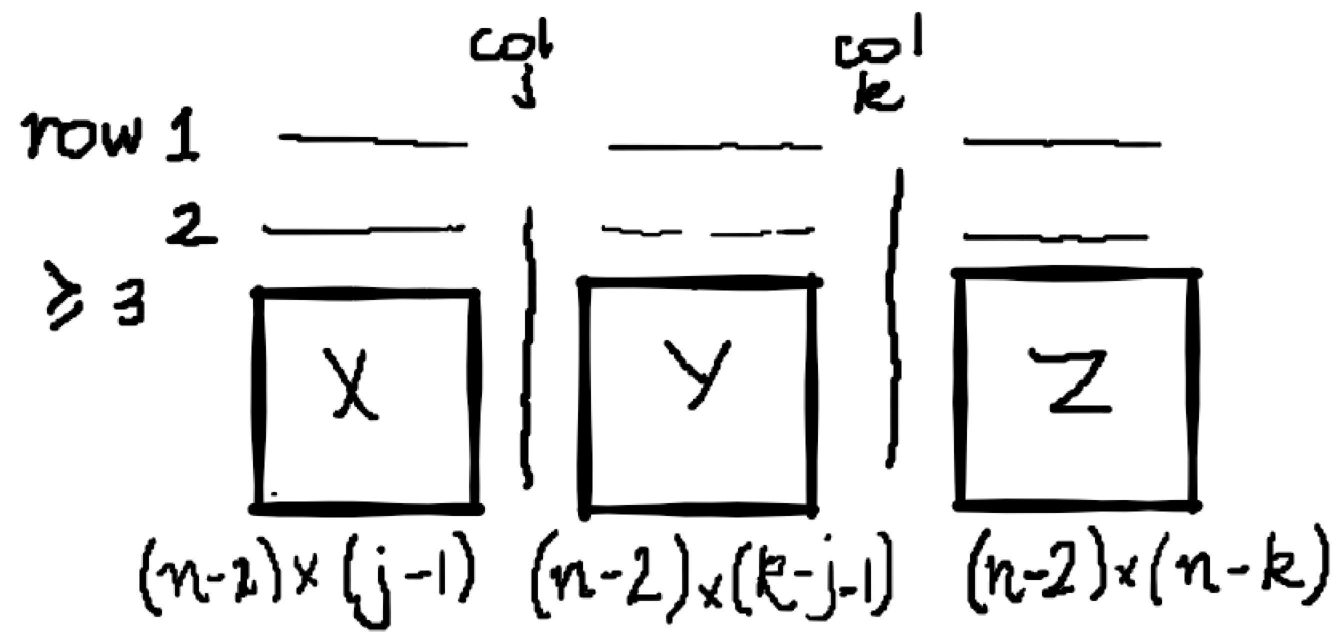
The following result is the source of many facts about determinants

Theorem. Let $A$ be an $n \times n$ matrix where $n > 1$. If the top two rows are swapped, then we get a matrix whose determinant is $(-1) \times \det A$.

Say $1 \le j < k \le n$.

$A$ :

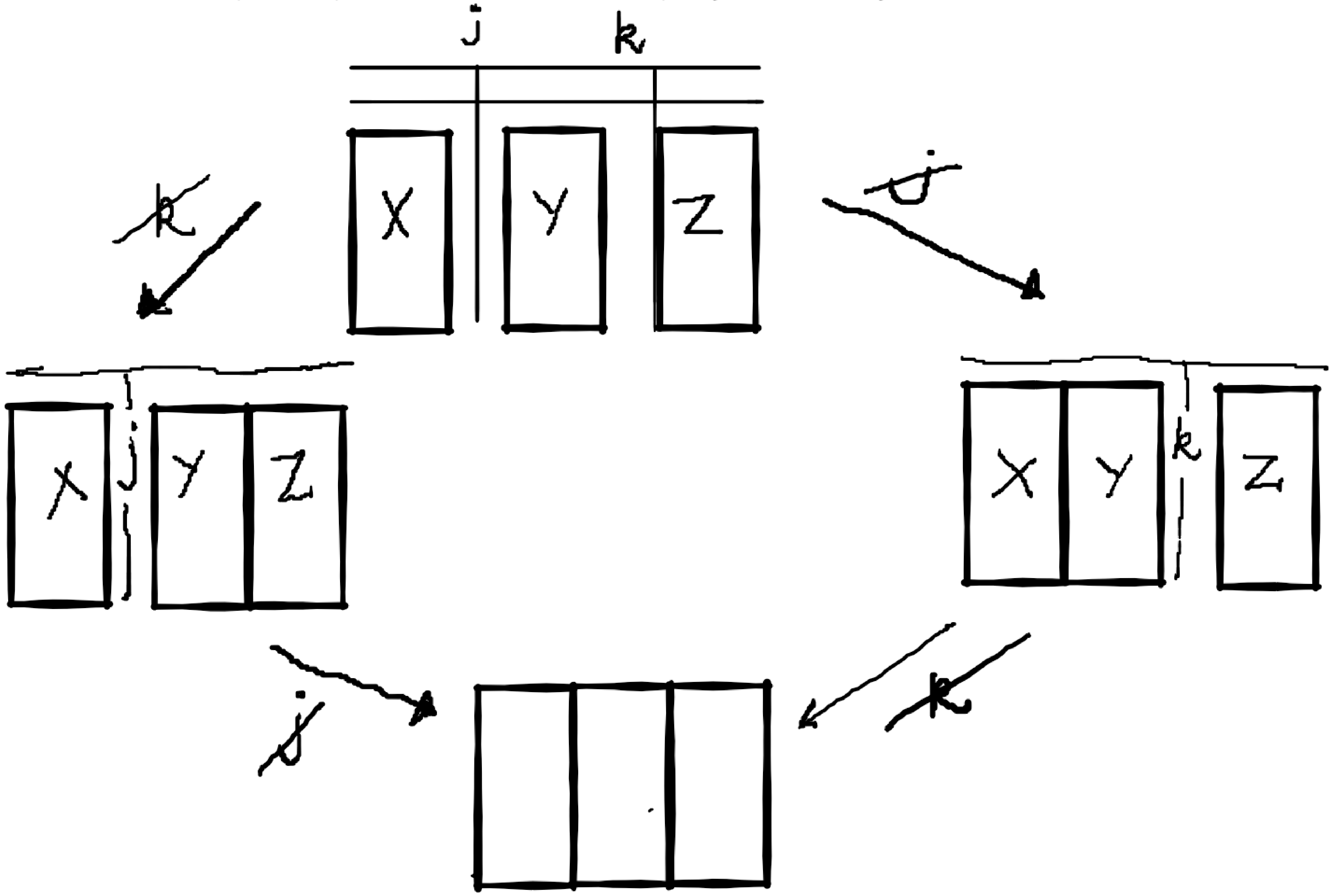



If we delete row 1 and column $j$ from $A$, we get $(X + Y + \text{col } k + Z)_{(n-1)\times(n-1)}$

If we delete row 1 + column $k$ we get $(X + \text{col } j + Y + Z)_{(n-1)\times(n-1)}$

MORE LATER

Swap top rows, multiply det by (-1)



det (XYZ) occurs twice

$a_{1k}(-1)^{k+1}\,\text{minor}_{1k} A$ $\quad$ $\text{minor}_{1k}$: $\boxed{X}\; j \;\overline{\boxed{Y \mid Z}}$

and within minor

$a_{2j}(-1)^{j+1}$ det ..... $\boxed{X \mid Y \mid Z}$

and

~~a~~ $a_{1j}(-1)^{j+1}\,\text{minor}_{1j}(A)$ ..... $\overline{\boxed{X \mid Y}}\; k \;\overline{\boxed{Z}}$

within:

$a_{2k}(-1)^{k}$ det $\boxed{X \mid Y \mid Z}$

So deleting rows 1,2 cols $j, k$ gives $[XYZ]_{(n-2)\times(n-2)}$

and contribution to $\det A$ is

$$(-1)^{j+k+1}(a_{1j}a_{2k} - a_{1k}a_{2j}) \det \text{`}XYZ\text{'}$$

$$= (-1)^{j+k+1} \begin{vmatrix} a_{1j} & a_{1k} \\ a_{2j} & a_{2k} \end{vmatrix} \det \text{`}XYZ\text{'}$$

Swap rows here

reverses sign.

Swap rows in $A$ reverses sign: $-\det A$

QED

Swap top two rows multiplies determinant by
(-1) (done)

Next: swap any two consecutive (adjacent) rows
has same effect

Next: swap any two rows has the same effect