Adjoint of a 3x3 matrix A. Slight variant of notes.

$$A = \underbrace{\begin{bmatrix} P & Q & R \end{bmatrix}}_{\text{columns}} \qquad \text{adj } A = \begin{bmatrix} Q \times R \\ R \times P \\ P \times Q \end{bmatrix} \ \underline{\underline{\text{Rows}}}$$

$$\det A = P \cdot (Q \times R)$$

$$(\text{adj } A)\, A = \begin{bmatrix} (Q \times R) \cdot P & (Q \times R) \cdot Q & (Q \times R) \cdot R \\ (R \times P) \cdot P & \text{etcetera} & \end{bmatrix}$$

$$= \begin{bmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{bmatrix} \quad \therefore \ A^{-1} = \frac{1}{\det A} \text{ adj } A \quad (\text{if } \det A \neq 0)$$

* from earlier slides

$$\text{adj} \begin{bmatrix} 2 & 4 & 16 \\ 2 & 4 & 15 \\ 2 & 3 & 13 \end{bmatrix} = \begin{bmatrix} 7 & -4 & -4 \\ 4 & -6 & 2 \\ -2 & 2 & 0 \end{bmatrix}$$

Adjoint of 3x3 matrix

Say the columns are P,Q,R. The adjoint is

$$\begin{bmatrix} (Q\times R)^T \\ (R\times P)^T \\ (P\times Q)^T \end{bmatrix} \leftarrow \text{rows}$$

EG $\begin{bmatrix} 2 & 4 & 16 \\ 2 & 4 & 15 \\ 2 & 3 & 13 \end{bmatrix}$

$Q\times R^T$:
$$\begin{array}{ccc} 4 & 4 & 3 \\ 16 & 15 & 13 \\ \hline 7 & -4 & -4 \end{array}$$

$R\times P^T$
$$\begin{array}{ccc} 16 & 15 & 13 \\ 2 & 2 & 2 \\ \hline 4 & -6 & 2 \end{array}$$

$(P\times Q)^T$
$$\begin{array}{ccc} 2 & 2 & 2 \\ 4 & 4 & 3 \\ \hline -2 & 2 & 0 \end{array}$$

$$A \text{ adj } A \quad \begin{bmatrix} 2 & 4 & 16 \\ 2 & 4 & 15 \\ 2 & 3 & 13 \end{bmatrix}\begin{bmatrix} 7 & -4 & -4 \\ 4 & -6 & 2 \\ -2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = (\det A)\, I$$

$$A \text{ adj } A = (\det A)\, I \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{A^{-1} = \frac{1}{\det A} \text{ adj } A}$$

Adjoint, calculations arranged differently

(1) minors (2) cofactors (3) transpose

(i,j) MINOR: delete row i and column j and take determinant

$$\begin{bmatrix} \cancel{2} & \cancel{4} & \cancel{16} \\ \cancel{2} & 4 & 15 \\ \cancel{2} & 3 & 13 \end{bmatrix} \quad (1,1) : 7$$

$$(1,2) : -4 \quad \begin{bmatrix} \cancel{2} & \cancel{4} & \cancel{16} \\ 2 & \cancel{4} & 15 \\ 2 & \cancel{3} & 13 \end{bmatrix}$$

etc

$$\text{minors} \quad \begin{bmatrix} 7 & -4 & -2 \\ 4 & -6 & -2 \\ -4 & -2 & 0 \end{bmatrix}$$

(i,j) COFACTOR =

(-1)^{i+j} x (i,j) minor

$$
\begin{array}{ccc}
\text{minors} & \text{cofactors} & \text{transpose} \\[6pt]
\begin{bmatrix} 7 & -4 & -2 \\ 4 & -6 & -2 \\ -4 & -2 & 0 \end{bmatrix}
&
\begin{bmatrix} 7 & 4 & -2 \\ -4 & -6 & 2 \\ -4 & 2 & 0 \end{bmatrix}
&
\begin{bmatrix} 7 & -4 & -4 \\ 4 & -6 & 2 \\ -2 & 2 & 0 \end{bmatrix} \\[6pt]
 & & \text{adjoint}
\end{array}
$$

(i) $Q \times P = -P \times Q$

(ii) $P.(Q \times R) = Q.(R \times P) = R.(P \times Q)$
$\det(P,Q,R) = \det(Q,R,P) = \det(R,P,Q)$
$= -\det(Q,P,R) = -\det(R,Q,P) = -\det(P,R,Q)$

(iii) $(xP_1 + yP_2).(Q \times R) =$
$x(P_1.(Q \times R)) + y(P_2.(Q \times R))$
$\det(xP_1 + yP_2, Q, R) =$
$x\det(P_1,Q,R) + y\det(P_2,Q,R)$

(iv) $\det(P,Q,R) = 0$ if and only if
O,P,Q,R coplanar

(v) volume of parallelopiped and
of tetrahedron