Let A be the 3x3 matrix with rows P, Q, R. Its

determinant $|A|$ or $\det A$ is defined as the triple product $P \cdot (Q \times R)$. We get the same result if P, Q, R are the columns of A. Cramer's rule in 3d: to solve $Px+Qy+Rz=S$, let

$a = S \cdot (Q \times R)$, $b = S \cdot (R \times P)$, $c = P \cdot (Q \times S)$,
$d = P \cdot (Q \times R)$. Then $x=a/d$, $y=b/d$, $z=c/d$.

(a,b,c,d are four determinants)

An example, rigged to have integer solution

$$2x+4y+16z=10;\quad 2x+4y+15z=9;\quad 2x+3y+13z=10$$

$$P\begin{bmatrix}2\\2\\2\end{bmatrix}\quad Q\begin{bmatrix}4\\4\\3\end{bmatrix}\quad R\begin{bmatrix}16\\15\\13\end{bmatrix}\quad S\begin{bmatrix}10\\9\\10\end{bmatrix}\qquad P\times Q\quad \begin{matrix}2 & 4\\ 2 & 4\\ 2 & 3\end{matrix}\quad \begin{bmatrix}-2\\2\\0\end{bmatrix}\text{(sign)}$$

$$Q\times R = \begin{bmatrix}7\\-4\\-4\end{bmatrix}\qquad P\cdot(Q\times R) = -2 = d$$

$$S\cdot(Q\times R) = -6 = a$$

$$R\times P = \begin{bmatrix}4\\-6\\2\end{bmatrix}$$

$$\boxed{P\cdot(S\times R) = R\cdot(P\times S) = S\cdot(R\times P) = 6} = b$$

P dot (QxS) = S dot (PxQ) = -2   c

Solution $x = a/d = 3,\ y = b/d = -3,\ z = c/d = 1$

Determinant of a 3x3 matrix with rows (or columns) P,Q,R: P dot (Q x R).
...... also, Q dot (R x P) and R dot (P x Q)
give the same result.

Note Q x P = - (P x Q), P dot (R x Q) =
- P dot (Q x R), etcetera. More of that later.A

A more common notation is, for example, $\begin{vmatrix} 2 & 4 & 16 \\ 2 & 4 & 15 \\ 2 & 3 & 13 \end{vmatrix} = -2$

Crossed diagonals formulae

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \qquad 1 \cdot 4 - 2 \cdot 3 = -2$$

sub · 3   4 · add

add the downward products (black) and subtract the upward products (red)

2	4	16	2	4
2	4	15	2	
2	3	13	2	3

$2 \times 4 \times 13 = 104$ $2 \times 4 \times 16 = 128$

$4 \times 15 \times 2 = 120$ $3 \times 15 \times 2 = 90$

$16 \times 2 \times 3 = 96$ $13 \times 2 \times 4 = 104$

$$320 - 322 = -2$$

Cramer's rule problem repeated different notation $A = \begin{bmatrix} 2 & 4 & 16 \\ 2 & 4 & 15 \\ 2 & 3 & 13 \end{bmatrix}$; $S = \begin{bmatrix} 10 \\ 9 \\ 10 \end{bmatrix}$

$d = \begin{vmatrix} 2 & 4 & 16 \\ 2 & 4 & 15 \\ 2 & 3 & 13 \end{vmatrix} = -2 \qquad a = \begin{vmatrix} 10 & 4 & 16 \\ 9 & 4 & 15 \\ 10 & 3 & 13 \end{vmatrix} = -6$

$b = \begin{vmatrix} 2 & 10 & 16 \\ 2 & 9 & 15 \\ 2 & 10 & 13 \end{vmatrix} = 6 \qquad c = \begin{vmatrix} 2 & 4 & 10 \\ 2 & 4 & 9 \\ 2 & 3 & 10 \end{vmatrix} = -2$

$x = 3, \quad y = -3, \quad z = 1$

EG a: $\begin{matrix} 10 & 4 & 16 & 10 & 4 \\ 9 & 4 & 15 & 9 & \\ 10 & 3 & 13 & 10 & 3 \end{matrix} \qquad \begin{matrix} 520 \\ +600 \\ +432 \end{matrix} \quad - \quad \begin{matrix} +640 \\ +450 \\ 468 \end{matrix}$

$= 1552 - 1558$

$= -6$