U11502 lin alg. + stats.

Section 1 notation / outline syllabus

check 1.1 U11501 — prerequisites

1.2 outline

1.3 Notation $\mathbb{N} = \{0, 1, 2 \cdots\}$

$\boxed{\mathbb{R} \text{ (reals)}}$ $\mathbb{C}$ complex $\mathbb{Q}$ rationals

point P in $\mathbb{R}^2$ or $\mathbb{R}^3$ displacement $\overrightarrow{PQ}$ vector $\vec{v}$

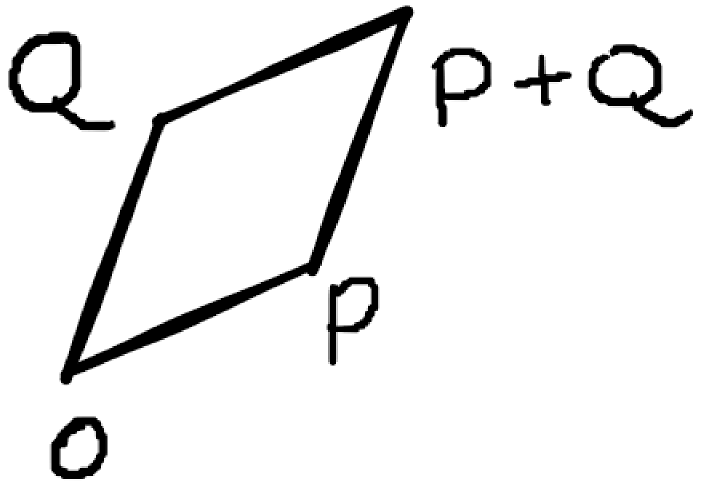
$\overrightarrow{OP} \cdot \overrightarrow{OQ}$ or $P^T Q$ (if column vectors)

or $P \cdot Q$

Triple product $\overrightarrow{OP} \cdot (\overrightarrow{OQ} \times \overrightarrow{OR})$ or $P \cdot (Q \times R)$

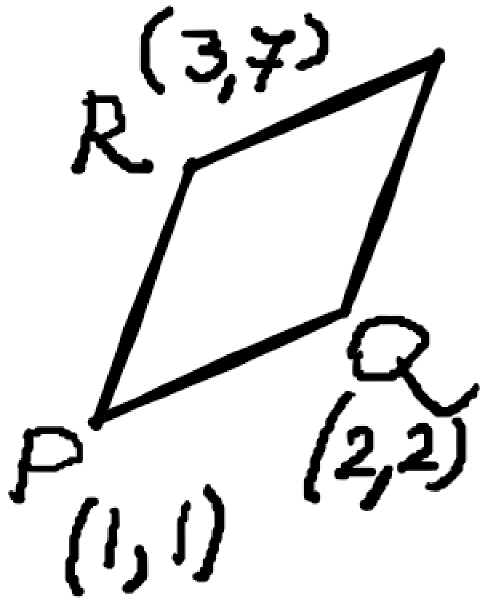
2: determinants in 2 and 3 dimensions

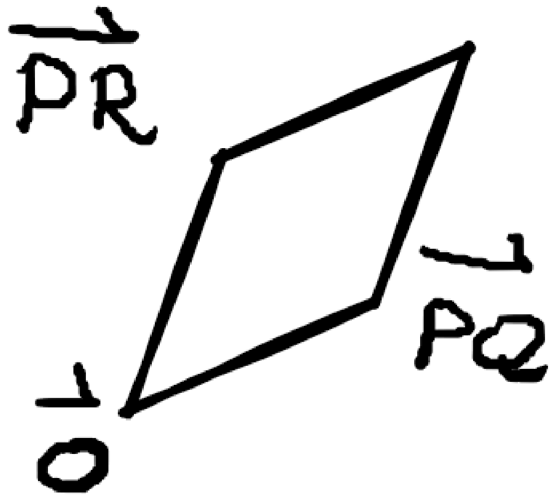
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad \begin{vmatrix} 2 & 7 \\ 1 & 8 \end{vmatrix} = 2 \times 8 - 1 \times 7 = 16 - 7 = 9$$

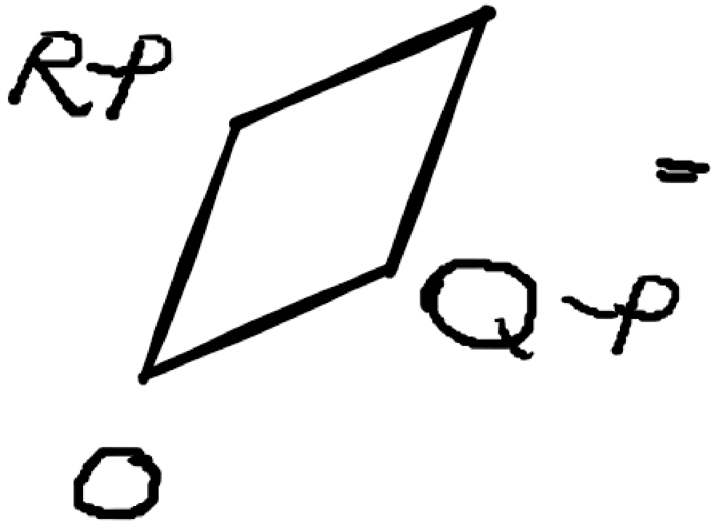


$P = (a,b) \quad Q = (c,d)$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \pm \text{ area of parallelogram.}$$



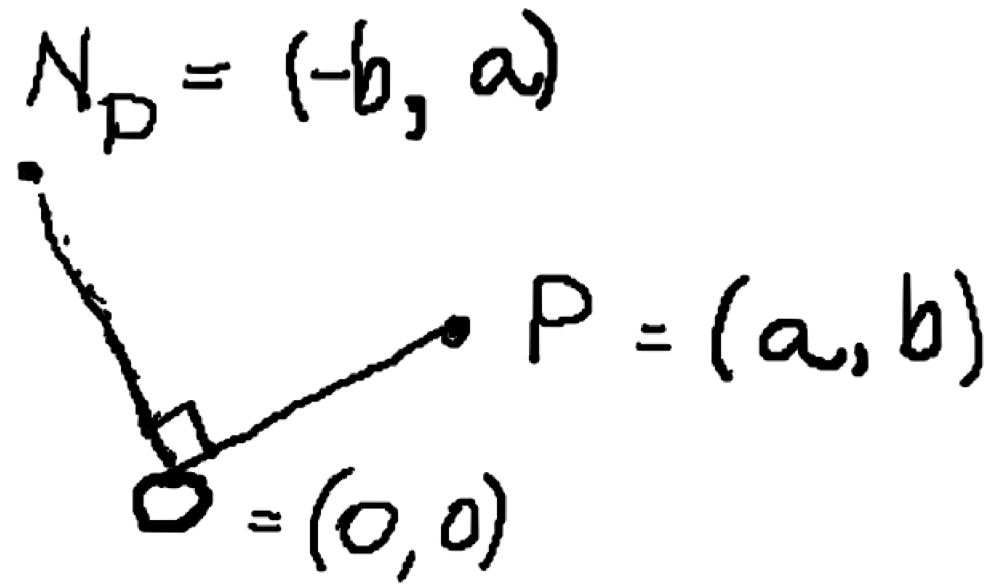




$$\Delta = \frac{1}{2} ||^{lm}$$

$$= \pm \frac{1}{2} \begin{vmatrix} 1 & 1 \\ 2 & 6 \end{vmatrix}$$

$$= \frac{1}{2} \times 4 = 2$$



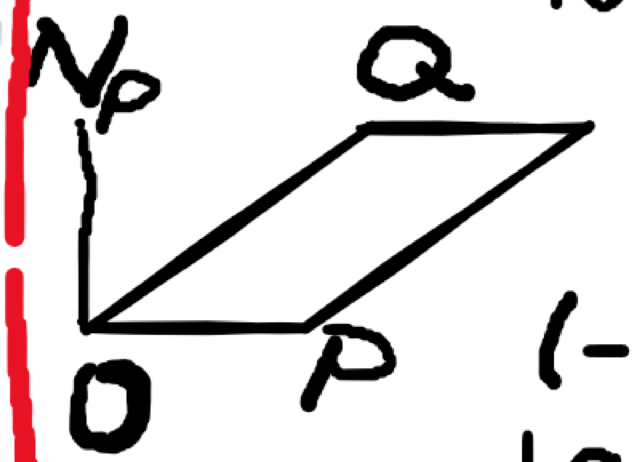
The positive normal N_p
$(P \neq 0)$

- $|N_p| = |P|$
- $\overrightarrow{ON_p} \perp \overrightarrow{OP}$
- 90^0 anticlockwise.

$P = (a, b)$ $\quad Q = (c, d)$

$N_p \cdot Q =$

$- P \cdot N_Q$



$N_p \cdot Q =$

$(-b, a) \cdot (c, d)$

$= \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Cramer's Rule

$$\begin{array}{r|cc|} P\rightarrow & a & b \\ Q\rightarrow & c & d \end{array} = \begin{array}{|cc|} a & c \\ b & d \\ \multicolumn{1}{c}{\uparrow} & \multicolumn{1}{c}{\uparrow} \\ \multicolumn{1}{c}{P} & \multicolumn{1}{c}{Q} \end{array} = \begin{array}{|cc|} a & b \\ c & d \\ \multicolumn{1}{c}{\uparrow} & \multicolumn{1}{c}{\uparrow} \\ \multicolumn{1}{c}{P} & \multicolumn{1}{c}{Q} \end{array}$$

$$ax + by = c \qquad P = \begin{bmatrix} a \\ d \end{bmatrix} \quad Q = \begin{bmatrix} b \\ e \end{bmatrix} \quad R = \begin{bmatrix} c \\ f \end{bmatrix}$$

$$dx + ey = f \qquad Px + Qy = R$$

$$N_p \cdot Px + N_p \cdot Qy = N_p \cdot R$$

$$N_p \cdot P = 0 \quad \rightarrow (N_p \cdot Q)\, y = N_p \cdot R$$

$$y = \frac{N_P \cdot R}{N_P \cdot Q} = \begin{vmatrix} a & c \\ d & f \end{vmatrix} \div \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

Similarly $$x = \frac{-R \cdot N_Q}{-P \cdot N_Q} = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}$$

EG $x + 3y = 2$, $x + 7y = 3$

$$x = \begin{vmatrix} 2 & 3 \\ 3 & 7 \end{vmatrix} \div \begin{vmatrix} 1 & 3 \\ 1 & 7 \end{vmatrix} = \frac{5}{4}$$

$$y = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \div \begin{vmatrix} 1 & 3 \\ 1 & 7 \end{vmatrix} = \frac{1}{4}$$

Check

$$\frac{5}{4}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$ ✓

Adjoint of a 2x2 matrix

Suppose P,Q are the rows of a 2x2 matrix. The adjoint of the matrix is defined as

~~N_Q~~

$$\Bigg[\quad\Bigg]$$

$$\underbrace{\overset{\uparrow}{N_Q^T} \quad \overset{\uparrow}{-N_P^T}}_{\text{Transposed to column vectors.}}$$

EG:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

adj A =

$$\begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}$$

$$A \operatorname{adj} A = \begin{bmatrix} \det A & 0 \\ 0 & \det A \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

Adjoint and Cramer's Rule summed up

matrix $A = [P \quad Q]$ (columns)

$|A|$ or det A : determinant $= -P \cdot N_Q = N_P \cdot Q$

or $|P \quad Q|$ ...

$A\begin{bmatrix}x\\y\end{bmatrix} = R$ $\quad x = \dfrac{|R \quad Q|}{|A|} \quad y = \dfrac{|P \quad R|}{|A|}$

adjoint $A = \begin{bmatrix} N_Q^T \\ -N_P^T \end{bmatrix}$ (rows) $\quad A^{-1} = \dfrac{1}{|A|}$ adjoint A

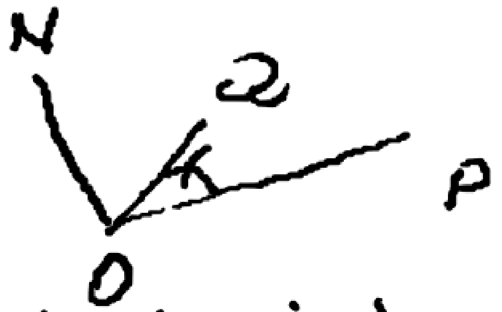
$$\begin{bmatrix}2 & 3\\1 & 4\end{bmatrix}^{-1} = \frac{1}{5}\begin{bmatrix}4 & -3\\-1 & 2\end{bmatrix}$$

Cross product corresponds to the positive normal in 2 dimensions. Given P, Q in $\mathbb{R}^3$, $\vec{OP} \times \vec{OQ} = \vec{ON}$

where * $|\vec{ON}|$ = area of parall^{m} O, P, P+Q, Q

(0 if O, P, Q collinear)

* If not, $\vec{ON}$ is ⊥ this parall^{m}
* if not, P, Q, N right-handed.



(viewed from N, angle $P\hat{O}Q$ anticlockwise).

FACT (write P×Q etcetera)

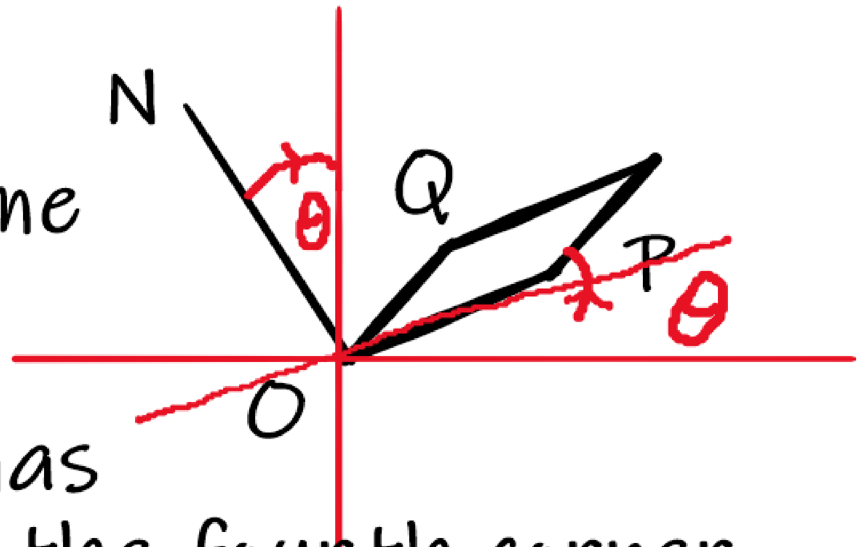
$$(a, b, c) \times (d, e, f) =$$

$$\left(\begin{vmatrix} b & c \\ e & f \end{vmatrix}, -\begin{vmatrix} a & c \\ d & f \end{vmatrix}, \begin{vmatrix} a & b \\ d & e \end{vmatrix} \right)$$

SIGN!!

It is possible to explain the formula. Given P,Q, and N = PxQ (assume nonzero), project N horizontally onto the z-axis and the parallelogram vertically onto the xy-plane.

The length of ON, and the area of the parallelogram, are both multiplied by the same factor cos theta



Say P=(a,b,c)
Q = (d,e,f)

The projected parallelogram has corners O, (a,b,0), (d,e,0), and the fourth corner. Its area is +/- (ae-bd), ... as given. The projection of N is the third component of PxQ.

Purely for interest, not for examination.

Let A be the 3x3 matrix with rows P, Q, R. Its

determinant $|A|$ or $\det A$ is defined as the triple product $P \cdot (Q \times R)$. We get the same result if P, Q, R are the columns of A. Cramer's rule in 3d: to solve $Px+Qy+Rz=S$, let

$a = S \cdot (Q \times R)$, $b = S \cdot (R \times P)$, $c = P \cdot (Q \times S)$,
$d = P \cdot (Q \times R)$. Then $x=a/d$, $y=b/d$, $z=c/d$.

(a,b,c,d are four determinants)

An example, rigged to have integer solution

$2x+4y+16z=10;\quad 2x+4y+15z=9;\quad 2x+3y+13z=10$

$P\begin{bmatrix}2\\2\\2\end{bmatrix}\quad Q\begin{bmatrix}4\\4\\3\end{bmatrix}\quad R\begin{bmatrix}16\\15\\13\end{bmatrix}\quad S\begin{bmatrix}10\\9\\10\end{bmatrix}\qquad P\times Q\quad \begin{matrix}2&4\\2&4\\2&3\end{matrix}\quad \begin{bmatrix}-2\\2\\0\end{bmatrix}$ (sign n)

$Q\times R=\begin{bmatrix}7\\-4\\-4\end{bmatrix}\qquad P\cdot(Q\times R)=-2=$ ⓓ

$S\cdot(Q\times R)=-6=$ ⓐ

$R\times P=\begin{bmatrix}4\\-6\\2\end{bmatrix}\checkmark$

$\boxed{P\cdot(S\times R)=R\cdot(P\times S)=S\cdot(R\times P)=6}$ $=$ ⓑ

P dot (QxS) = S dot (PxQ) = -2 ⓒ

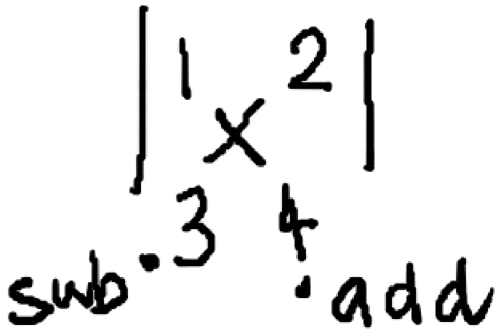
Solution $\quad x=a/d=3,\ y=b/d=-3,\ z=c/d=1$

Determinant of a 3x3 matrix with rows (or columns) P,Q,R: P dot (Q x R). ...... also, Q dot (R x P) and R dot (P x Q) give the same result.

Note Q x P = - (P x Q), P dot (R x Q) = - P dot (Q x R), etcetera. More of that later.A

A more common notation is, for example, $\begin{vmatrix} 2 & 4 & 16 \\ 2 & 4 & 15 \\ 2 & 3 & 13 \end{vmatrix} = -2$

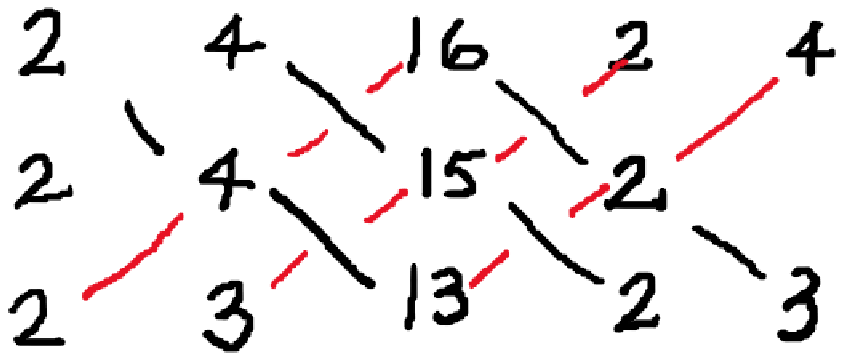
Crossed diagonals formulae



$1 \times 4 - 2 \times 3$

$= -2$

add the downward products (black) and subtract the upward products (red)



$2 \times 4 \times 13 = 104$ $2 \times 4 \times 16 = 128$

$4 \times 15 \times 2 = 120$ $3 \times 15 \times 2 = 90$

$16 \times 2 \times 3 = 96$ $13 \times 2 \times 4 = 104$

$320 - 322 = -2$

Cramer's rule problem repeated different notation $A = \begin{bmatrix} 2 & 4 & 16 \\ 2 & 4 & 15 \\ 2 & 3 & 13 \end{bmatrix}$; $S = \begin{bmatrix} 10 \\ 9 \\ 10 \end{bmatrix}$

$d = \begin{vmatrix} 2 & 4 & 16 \\ 2 & 4 & 15 \\ 2 & 3 & 13 \end{vmatrix} = -2$ $\quad a = \begin{vmatrix} 10 & 4 & 16 \\ 9 & 4 & 15 \\ 10 & 3 & 13 \end{vmatrix} = -6$

$b = \begin{vmatrix} 2 & 10 & 16 \\ 2 & 9 & 15 \\ 2 & 10 & 13 \end{vmatrix} = 6$ $\quad c = \begin{vmatrix} 2 & 4 & 10 \\ 2 & 4 & 9 \\ 2 & 3 & 10 \end{vmatrix} = -2$

$x = 3, \quad y = -3, \quad z = 1$

EG a: $\begin{matrix} 10 & 4 & 16 & 10 & 4 \\ 9 & 4 & 15 & 9 & \\ 10 & 3 & 13 & 10 & 3 \end{matrix} \quad \begin{matrix} 520 \\ +600 \\ +432 \end{matrix} = \begin{matrix} +640 \\ +450 \\ +468 \end{matrix}$ $=1552-1558$

$= -6$

Adjoint of a 3x3 matrix A. Slight variant of notes.

$$A = \underbrace{\begin{bmatrix} P & Q & R \end{bmatrix}}_{\text{Columns}} \qquad \operatorname{adj} A = \begin{bmatrix} Q \times R \\ R \times P \\ P \times Q \end{bmatrix} \ \underline{\underline{\text{Rows}}}$$

$$\det A = P \cdot (Q \times R)$$

$$(\operatorname{adj} A)\, A = \begin{bmatrix} (Q \times R) \cdot P & (Q \times R) \cdot Q & (Q \times R) \cdot R \\ (R \times P) \cdot P & \text{etcetera} & \end{bmatrix}$$

$$= \begin{bmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{bmatrix} \quad \therefore A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

(if $\det A \neq 0$)

from earlier slides

$$\operatorname{adj} \begin{bmatrix} 2 & 4 & 16 \\ 2 & 4 & 15 \\ 2 & 3 & 13 \end{bmatrix} = \begin{bmatrix} 7 & -4 & -4 \\ 4 & -6 & 2 \\ -2 & 2 & 0 \end{bmatrix}$$

Adjoint of 3x3 matrix

Say the columns are P,Q,R. The adjoint is

$$\begin{bmatrix}(Q\times R)^T\\(R\times P)^T\\(P\times Q)^T\end{bmatrix} \leftarrow \text{rows}$$

EG $\begin{bmatrix}2&4&16\\2&4&15\\2&3&13\end{bmatrix}$

$Q\times R^T$: $\begin{array}{ccc}4&4&3\\16&15&13\\\hline 7&-4&-4\end{array}$

$R\times P^T$ $\begin{array}{ccc}16&15&13\\2&2&2\\\hline 4&-6&2\end{array}$

$(P\times Q)^T$ $\begin{array}{ccc}2&2&2\\4&4&3\\\hline -2&2&0\end{array}$

$$A \text{ adj } A \quad \begin{bmatrix}2&4&16\\2&4&15\\2&3&13\end{bmatrix}\begin{bmatrix}7&-4&-4\\4&-6&2\\-2&2&0\end{bmatrix} = \begin{bmatrix}-2&0&0\\0&-2&0\\0&0&-2\end{bmatrix} = (\det A)\, I$$

$$I = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$A \text{ adj } A = (\det A)\, I$$

$$\boxed{A^{-1} = \frac{1}{\det A} \text{ adj } A}$$

Adjoint, calculations arranged differently

(1) minors (2) cofactors (3) transpose

(i,j) MINOR: delete row i and column j and take determinant

$$\begin{bmatrix} 2 & 4 & 16 \\ 2 & 4 & 15 \\ 2 & 3 & 13 \end{bmatrix} \quad (1,1) : 7$$

$$(1,2): -4 \quad \begin{bmatrix} 2 & 4 & 16 \\ 2 & 4 & 15 \\ 2 & 3 & 13 \end{bmatrix}$$

etc

$$\text{minors} \quad \begin{bmatrix} 7 & -4 & -2 \\ 4 & -6 & -2 \\ -4 & -2 & 0 \end{bmatrix}$$

(i,j)COFACTOR =
(-1)^{i+j} x (i,j) minor

minors

$$\begin{bmatrix} 7 & -4 & -2 \\ 4 & -6 & -2 \\ -4 & -2 & 0 \end{bmatrix}$$

cofactors

$$\begin{bmatrix} 7 & 4 & -2 \\ -4 & -6 & 2 \\ -4 & 2 & 0 \end{bmatrix}$$

transpose

$$\begin{bmatrix} 7 & -4 & -4 \\ 4 & -6 & 2 \\ -2 & 2 & 0 \end{bmatrix}$$

adjoint

(i) $QxP= -PxQ$

(ii) $P.(QxR)=Q.(RxP)=R.(PxQ)$
$\det(P,Q,R) = \det(Q,R,P) = \det(R,P,Q)$
$= - \det(Q,P,R) = -\det(R,Q,P) = -\det(P,R,Q)$

(iii) $(xP_1+yP_2).(QxR) =$
$x(P_1.(QxR)) + y(P_2.(QxR))$
$\det(xP_1+yP_2,Q,R) =$
$x\det(P_1,Q,R)+y\det(P_2,Q,R)$

(iv) $\det(P,Q,R)=0$ if and only if
O,P,Q,R coplanar

(v) volume of parallelopiped and
of tetrahedron

nxn determinants There is a recusive formula

Let A = [a_ij] _nxn be a square matrix. We use double indexing in the usual way.

$n=1$ $[a_{11}]$ $n=2$ $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $n=3$ $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

if $A_{1\times 1} = [a_{11}]$ then

$$\det A = a_{11}$$

if $n > 1$, $A_{n\times n} = [a_{ij}]$ then

$$\det A = \sum_{j=1}^{n} a_{1j}(-1)^{j+1}\,\text{minor}_{1j}(A) = \sum_{j=1}^{n} a_{1j}\,\text{cofactor}_{1j}(A)$$

We can handle $n=2$

$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ 1,1 minor a_{22} 1,1 cofactor a_{22}

1,2 minor a_{21} 1 2 cofactor $-a_{21}$

$a_{11}a_{22}-a_{12}a_{21}$ $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$

same as before.

$n=3$ $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ or: $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

minors 1,1 $\begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ \cancel{d} & e & f \\ \cancel{g} & h & i \end{vmatrix}$ 1,2 $\begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ d & \cancel{e} & f \\ g & \cancel{h} & i \end{vmatrix}$ 1,3 $\begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ d & e & \cancel{f} \\ g & h & \cancel{i} \end{vmatrix}$

$\det = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$ same as before $P\cdot(Q\times R)$

4x4 determinants are a long calculation

- a_11 x minor_11 - a_12 x minor_22 + a_13 x minor_13 - a_14 x minor_14

$$\begin{vmatrix} 1 & -2 & 6 & 3 \\ -3 & 7 & -20 & 1 \\ -1 & 3 & -7 & 4 \\ -2 & 3 & -13 & 5 \end{vmatrix}$$

1,1 minor $\begin{vmatrix} 7 & -20 & 1 \\ 3 & -7 & 4 \\ 3 & -13 & 5 \end{vmatrix} = (7, -20, 1) \cdot (17, -3, -18) = 161$

$(1,2)$: $\begin{vmatrix} -3 & -20 & 1 \\ -1 & -7 & 4 \\ -2 & -13 & 5 \end{vmatrix}$

$= (-3, -20, 1) \cdot (17, -3, -1) = 8$

$(1,3)$: $\begin{vmatrix} -3 & 7 & 1 \\ -1 & 3 & 4 \\ -2 & 3 & 5 \end{vmatrix}$

$= (-3, 7, 1) \cdot (3, -3, 3) = -27$

$(1,4)$: $\begin{vmatrix} -3 & 7 & -20 \\ -1 & 3 & -7 \\ -2 & 3 & -13 \end{vmatrix}$

$= (-3, 7, -20) \cdot (-18, 1, 3) = 1$

1×161
$+ 2 \times 8$
$+ 6(-27)$
$- 3 \times 1$
$= 177 - 165$
$= \boxed{12}$

determinant is 12

The following result is the source of many facts about determinants

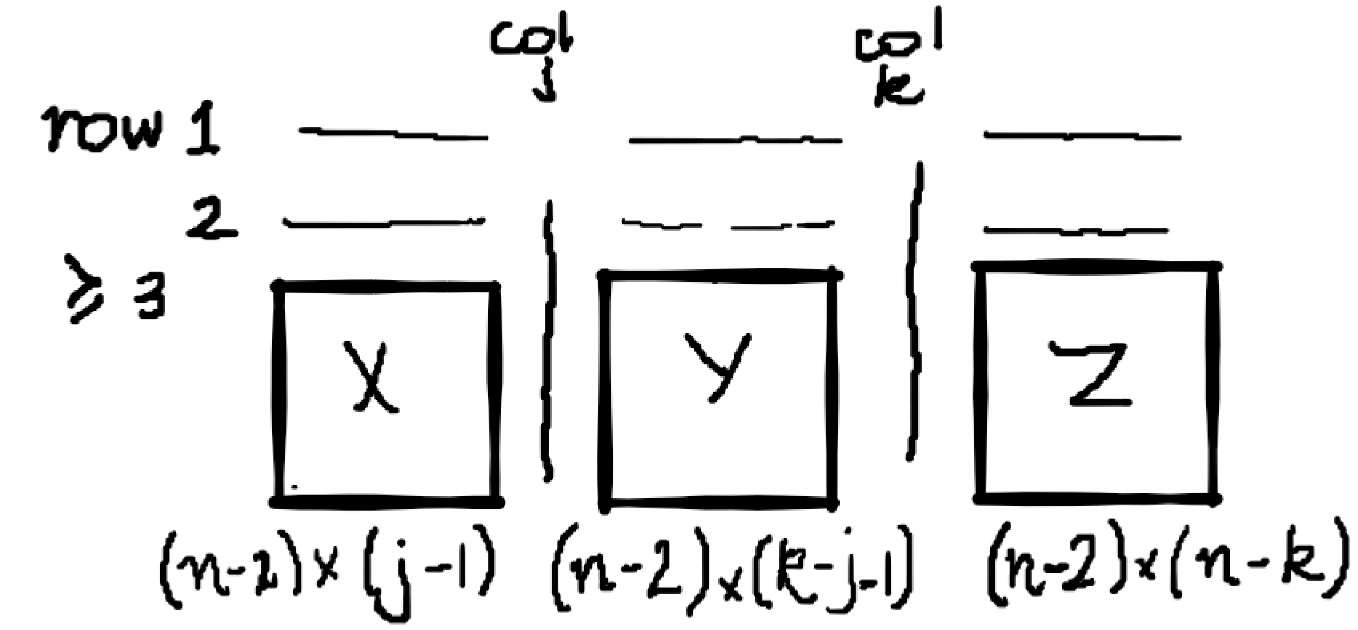
Theorem. Let A be an $n \times n$ matrix where $n > 1$. If the top two rows are swapped, then we get a matrix whose determinant is $(-1) \times \det A$.

Say $1 \leq j < k \leq n$.

If we delete row 1 and column j from A, we get $(X + Y + \text{col } k + Z)_{(n-1) \times (n-1)}$

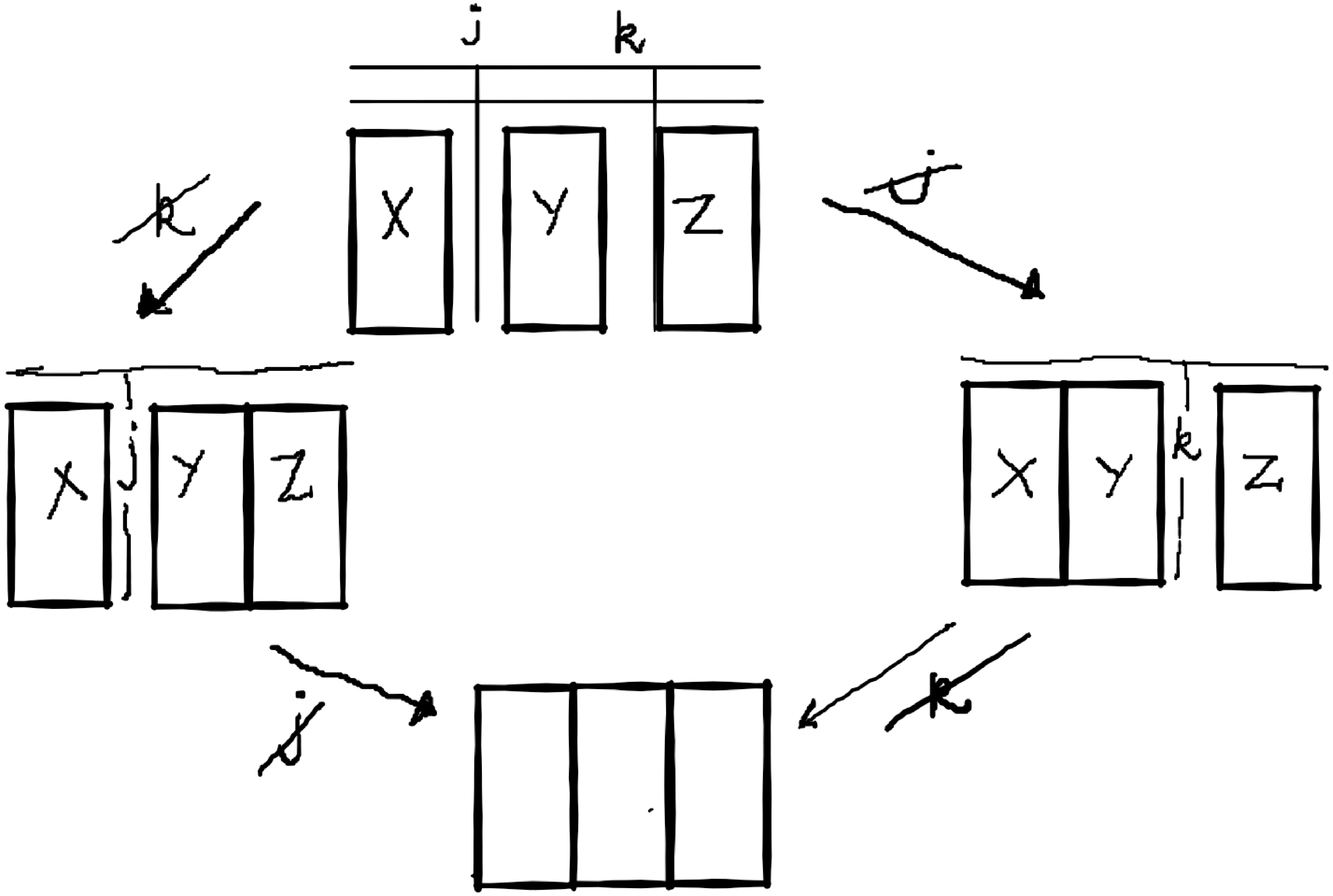
If we delete row 1 + column k we get $(X + \text{col } j + Y + Z)_{(n-1) \times (n-1)}$

A :



MORE LATER

Swap top rows, multiply det by (-1)



det (XYZ) occurs twice

$a_{1k}(-1)^{k+1}$ minor$_{1k}$ A   minor$_{1k}$: $\overline{\boxed{X}}\; j \;\overline{\boxed{Y \mid Z}}$

and within minor

$a_{2j}(-1)^{j+1}$ det . . . . . $\boxed{X \mid Y \mid Z}$

and

~~a~~ $a_{1j}(-1)^{j+1}$ minor$_{1j}$ (A) . . . . . $\overline{\boxed{X \mid Y}}\; k \;\overline{\boxed{Z}}$

within:

$a_{2k}(-1)^{k}$ det $\boxed{X \mid Y \mid Z}$

So deleting rows 1,2 cols j, k gives $[XYZ]_{(n-2)\times(n-2)}$

and contribution to det A is

$$(-1)^{j+k+1}\left(a_{1j}a_{2k} - a_{1k}a_{2j}\right) \det \text{‘}XYZ\text{’}$$

$$= (-1)^{j+k+1}\begin{vmatrix} a_{1j} & a_{1k} \\ a_{2j} & a_{2k} \end{vmatrix} \det \text{‘}XYZ\text{’}$$

Swap rows here reverses sign.

Swap rows in A reverses sign: $-\det A$

QED

Swap top two rows multiplies determinant by (-1) (done)

Next: swap any two consecutive (adjacent) rows has same effect

Next: swap any two rows has the same effect