# MAU11S02 sixth Friday quiz, week 9
# Friday 25/3/22 ANSWERS

**Rules and procedures.**
**1.** Attempt 3 questions. Only *your first three answers* will be marked. **2.** Each question carries 20 marks, so the maximum quiz mark is 60. **3.** If a particular method of solution is stipulated, you get no marks if you don't use it. **4.** ***Show all work.*** No marks will be given for answers which do not show the calculations. **5.** Your answers should be scanned and submitted to Blackboard as a 'Friday assignment.'

**Question 1.** Calculate the quadratic function $y = ax^2 + bx + c$ best fitting the following data (least squared error estimate). ***You must use the formula given in lectures, and show the calculations.***

$$(-3, 1)\ (-2, 0)\ (-1, 1)\ (1, 3)$$

Answer.

```
 99     -35      15      13  *(1/99)   =R1
-35      15      -5      -1   +35*R1
 15      -5       4       5   -15*R1

  1  -35/99    5/33   13/99   +35/99*R2
  0  260/99   10/33  356/99  *(99/260)   =R2
  0   10/33   19/11  100/33   -10/33*R2

  1       0    5/26    8/13   -5/26*R3
  0       1    3/26   89/65   -3/26*R3
  0       0   22/13   34/13  *(13/22)   =R3

  1       0       0    7/22
  0       1       0 131/110
  0       0       1   17/11  in rref
```

$y = 7x^2/22 + 131x/110 + 17/11.$

**Question 2.** Find eigenvalues and eigenvectors, and the change-of-basis matrix $S$, for the matrix $A$, where

$$A = \begin{bmatrix} 1 & 2 \\ 5 & -2 \end{bmatrix}$$

Answer.

$$\lambda^2 + \lambda - 2 - 10 = (\lambda - 3)(\lambda + 4)$$

$$\lambda = 3: \quad \begin{matrix} 2 & -2 \\ -5 & 5 \end{matrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \lambda = -4: \quad \begin{matrix} -5 & -2 \\ -5 & -2 \end{matrix} \mapsto \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix}$$

**Question 3.** Calculate $e^{At}$, where $A$ is as in Question 2.
Answer.

$$e^{At} = Se^{A't}S^{-1} =$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-4t} \end{bmatrix} \begin{bmatrix} \frac{5}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} \frac{5}{3}e^{3t} & -\frac{2}{3}e^{3t} \\ -\frac{1}{3}e^{-4t} & \frac{1}{3}e^{-4t} \end{bmatrix} =$$

$$\frac{1}{3} \begin{bmatrix} 5e^{3t} - 2e^{-4t} & 2e^{3t} - 2e^{-4t} \\ 5e^{3t} - 5e^{-4t} & -2e^{3t} + 5e^{-4t} \end{bmatrix}.$$

**Question 4.** Solve in full the differential equations

$$\frac{dx}{dt} = x + 2y; \qquad \frac{dy}{dt} = 5x - 2y$$

with $x = 1$ and $y = 2$ at $t = 0$.
Answer.

$$e^{At}X_0 = Se^{A't}S^{-1}X_0 =$$

$$e^{At} \begin{bmatrix} 1 \\ 2 \end{bmatrix} =$$

$$\frac{1}{3} \begin{bmatrix} e^{3t} + 2e^{-4t} \\ e^{3t} + 5e^{-4t} \end{bmatrix}$$

**Question 5.** There are 4 balls in one jar, labelled $2, 3, 4$ respectively, and 3 balls in another jar, labelled $2, 7, 7, 8$ respectively.

The random variables $X_1, X_2$ represent the labels $x_1, x_2$ on two balls chosen randomly from each jar.

(i) List the *nine* possible outcomes $(x_1, x_2)$ and their probabilities.

(ii) List the *seven* possible values of $x_1 + x_2$ and their probabilities.

Answer.

| | 2 | 7 | 8 |
|---|---|---|---|
| 2 | 1/12 | 1/6 | 1/12 |
| 3 | 1/12 | 1/6 | 1/12 |
| 4 | 1/12 | 1/6 | 1/12 |

| + | 2 | 7 | 8 |
|---|---|---|---|
| 2 | 4 | 9 | 10 |
| 3 | 5 | 10 | 11 |
| 4 | 8 | 11 | 12 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 4 | 5 | 8 | 9 | 10 | 11 | 12 |
| 1/12 | 1/12 | 1/12 | 1/6 | 1/4 | 1/4 | 1/12 |

**Question 6.** The least-squared-error method can be used to find best-fitting curves of many different shapes. Find the best-fitting curve of the form $y = 2^x a + (-1)^x b$ for the data points in Question 1.

Show that

$$A^T A = \begin{bmatrix} 4.328125 & -2.375 \\ -2.375 & 4 \end{bmatrix}, \quad \text{(corrected) and } A^T Y = \begin{bmatrix} 6.625 \\ -5 \end{bmatrix}$$

and hence solve the problem.

Answer.

```
Cramer's Rule
2 3
4.578125  -2.375 6.625
 -2.375  4  -5
x = 14.625000/12.671875 = 1.154131
y = -7.156250/12.671875 = -0.564735
```

Estimated curve $y = 1.154131 \times 2^x - 0.564735 \times (-1)^x$.