5 Determinant of upper triangular matrices

5.1 Determinant of an upper triangular matrix

We begin with a seemingly irrelevant lemma.

(5.1) Lemma *Let A be an $n \times n$ matrix containing a column of zeroes. Then* $\det(A) = 0$.

Proof. If $n = 1$ then $\det(A) = a_{11} = 0$.
For the induction,

$$\det A = \sum_{s=1}^{n} a_{1s}(-1)^{1+s}\text{minor}_{1,s}A$$

and suppose that the k-th column of A is zero.

In the above sum, if $s = k$ then $a_{1s} = 0$, and if $s \neq k$, then $\text{minor}_{1,s}$ contains a column of zeroes, so by induction, $\text{minor}_{1,s} = 0$. Hence $\det A = 0$. ∎

(5.2) Corollary *Let A be an upper triangular matrix (i.e., $a_{ij} = 0$ if $i > j$). Then*

$$\det A = a_{11}a_{22}\cdots a_{nn}.$$

Proof. Induction on n. If $n = 1$ the formula is satisfied. For the induction,

$$\det A = \sum_{s=1}^{n} a_{1,s}(-1)^{1+s}\text{minor}_{1,s}A.$$

Note that if $s > 1$ then $\text{minor}_{1,s}A$ contains a column of zeroes and is 0. Hence

$$\det A = a_{11}\text{minor}_{1,1}A$$

and the result follows by induction. ∎

5.2 Converting to UTF

Given a square matrix A, use *elementary row operations* **swap, subtract** to bring A to upper triangular form.

- Swapping rows changes the sign of the determinant.
- Subtracting (from one row a multiple of another) leaves the determinant unchanged.

There is no reason to use scaling. If scaling is used, it must be remembered that it scales the determinant by the same factor.

5.3 Examples

Example. Use EROs to reduce

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 8 \end{bmatrix}$$

to upper triangular form, and hence calculate its determinant.

```
1 2 3   = R1       1  2   3             1  2   3
4 5 6   - 4 * R1   0 -3  -6  = R2          -3  -6
7 8 8   - 7 * R1   0 -6 -13  - 2 * R2          -1
```

There was no swapping nor scaling, so the determinant is the determinant of the last matrix, $(1)(-3)(-1) = 3$.

Example. Use the same method to calculate $\det(A)$ where

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 6 & 6 & 2 \\ 1 & 4 & 5 & 4 \\ 1 & 1 & 4 & 0 \end{bmatrix}.$$

```
0 1 2 3 swap 2 6 6 2  = R1     2  6 6  2        2 6 6 2
2 6 6 2 swap 0 1 2 3           0  1 2  3 = R2   0 1 2 3
1 4 5 4      1 4 5 4  - R1/2   0  1 2  3 - R2   0 0 0 0
1 1 4 0      1 1 4 0  - R1/2   0 -2 1 -1 + 2R2  0 0 3 5
```

We can stop here. The matrix has a row of zeroes and its determinant is zero: the determinant of the orginal matrix is also zero.

If we change the matrix slightly — say the third row is replaced by 1 4 6 6, the determinant should be nonzero.

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 6 & 6 & 2 \\ 1 & 4 & 6 & 6 \\ 1 & 1 & 4 & 0 \end{bmatrix}.$$

```
0 1 2 3 swap 2 6 6 2  = R1     2  6 6  2        2 6 6  2
2 6 6 2 swap 0 1 2 3           0  1 2  3 = R2   0 1 2  3
1 4 6 6      1 4 6 6  - R1/2   0  1 3  5 - R2   0 0 1  2 = R3
1 1 4 0      1 1 4 0  - R1/2   0 -2 1 -1 + 2R2  0 0 5  5 - 5 R3

2 6 6  2      Det of UTF      so det of
0 1 2  3      is -10.         original
0 0 1  2      One swap,       matrix is
0 0 0 -5      no scaling,     10.
```