8 Subspaces, linear dependence, basis, and dimension

8.1 Subspaces

(8.1) Definition *A* vector subspace X *of* $\mathbb{R}^n$ *is a subset which is*

- *nonempty (a technicality), and*
- *closed under linear combination.*

This means that for any points $x_1, \ldots, x_k$ *in* X *and any real numbers* $\alpha_1, \ldots, \alpha_k$*, the linear combination*

$$\alpha_1 x_1 + \ldots + \alpha_k x_k$$

also belongs to X*.*

The vector subspaces of $\mathbb{R}^3$ are

- $\mathbb{R}^3$ itself,
- Any plane *through the origin*,
- Any straight line through the origin, and
- The origin on its own: $\{O\}$.

The set of linear combinations. It is easy to prove that given any finite set of points (column vectors)

$$\vec{X}_1, \ldots \vec{X}_n$$

all of the same height (m, say), the set of all possible linear combinations of these points is a vector subspace of $\mathbb{R}^m$.

For example, let

$$X_1 = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}.$$

The set of all linear combinations of these points is

$$\left\{ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} : \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\}.$$

The first question is, given any vector X, is X a linear combination of these columns and if so, what? For example, is $[4, 7, 10]^T$ in the subspace?

This is easily answered with Gauss-Jordan elimination.

```
1   2   3   4    =R1
4   5   6   7    -4*R1
7   8   9  10    -7*R1

1   2   3   4    -2*R2
0  -3  -6  -9   *(-1/3)   =R2
0  -6 -12 -18    +6*R2

1   0  -1  -2
0   1   2   3
0   0   0   0   in rref
```

So the answer is yes; there are infinitely many different linear combinations yielding this right-hand side, including $\alpha_1 = -2, \alpha_2 = 3$, and $\alpha_3 = 0$.

Of course, not every point belongs to the subspace, because the matrix is not invertible. For example,

```
1     2     3     3    =R1
4     5     6     7    -4*R1
7     8     9    10    -7*R1

1     2     3     3    -2*R2
0    -3    -6    -5   *(-1/3)   =R2
0    -6   -12   -11    +6*R2

1     0    -1  -1/3    +1/3*R3
0     1     2   5/3    -5/3*R3
0     0     0    -1   *(-1)   =R3

1     0    -1     0
0     1     2     0
0     0     0     1   in rref
```

The given point is not in the space; the equations have no solution.

(8.2) Definition *Let $X_1, \ldots, X_n$ be a set of column vectors in $\mathbb{R}^n$. The* subspace generated by *these vectors, or the* subspace spanned by *these vectors, is the set of all linear combinations of these vectors.*

8.2 Linear independence

A set of points $X_1, \ldots, X_n$ is *linearly independent* if the only solution to

$$\alpha_1 X_1 + \ldots + \alpha_n X_n = O$$

is $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$.

Using Gauss-Jordan elimination, the criterion is that all columns in the RREF are leading columns. For example, testing the 3 column vectors given below in $\mathbb{R}^4$

```
1   2   3   =R1
4   5   6   -4*R1
7   8   9   -7*R1
3   2   1   -3*R1

1   2   3   -2*R2
0  -3  -6  *(-1/3)  =R2
0  -6 -12   +6*R2
0  -4  -8   +4*R2

1   0  -1
0   1   2
0   0   0
0   0   0  in rref
```

Not all columns are leading columns. Indeed:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Longleftrightarrow$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

whence

$$\alpha_1 - \alpha_3 = 0, \quad \alpha_2 + 2\alpha_3 = 0$$

and α_3 and α_4 can be arbitrary.

So for example $\alpha_3 = 1, \alpha_4 = 0, \alpha_1 = 1, \alpha_2 = -2$, will work:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

On the other hand, consider

```
1 2 3
4 5 6
7 8 9
3 1 2
```

```
1   2   3    =R1
4   5   6    -4*R1
7   8   9    -7*R1
3   1   2    -3*R1

1   2   3    -2*R2
0  -3  -6   *(-1/3)  =R2
0  -6 -12    +6*R2
0  -5  -7    +5*R2

1   0  -1
0   1   2
0   0   0   swap
0   0   3   swap

1   0  -1    +1*R3
0   1   2    -2*R3
0   0   3   *(1/3)  =R3
0   0   0

1   0   0
0   1   0
0   0   1
0   0   0   in rref
```

so these 3 column vectors *are* linearly independent.

8.3 Bases and matrix invertibility

Let $X_1, \ldots, X_n$ be column vectors in $\mathbb{R}^m$, so the set of all linear combinations

$$\alpha_1 X_1 + \ldots + \alpha_n X_n$$

is a vector subspace X of $\mathbb{R}^m$.

(8.3) Definition *When the columns X_j are linearly independent, they are said to form a* basis *for X.*

Useful facts.

- Given a vector subspace X of $\mathbb{R}^m$, all bases for X have the same cardinality.
- In particular, all bases for $\mathbb{R}^m$ itself have the same cardinality — m.
- A list of m column vectors forms a basis for $\mathbb{R}^m$ if and only if the square $m \times m$ matrix they form is invertible.
- Given a square matrix A, its columnns (respectively, rows) are linearly independent if and only if $\det(A) \neq 0$.